Zeitschrift: Archives des sciences [1948-1980]

Herausgeber: Société de Physique et d'Histoire Naturelle de Genève

Band: 29 (1976)

Heft: 2

Artikel: Details of hydrogen-burning thermonuclear reactions

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DOI: https://doi.org/10.5169/seals-739677

DETAILS OF HYDROGEN-BURNING THERMONUCLEAR REACTIONS

BY

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ABSTRACT

In connection with the triple *pp* chain, we study here the temperature dependance of the rate of energy production in the range where the *PP II* chain is predominant. Moreover, we examine the approach towards equilibrium of the complete *CNO* cycle, composed of the *CN* and the two *NO* cycles, showing that the presence of the second *NO* cycle entails a quasi-equilibrium for the *CN* cycle.

RÉSUMÉ

Relativement à la triple chaîne *pp*, on examine ici la dépendance du débit d'énergie à l'égard de la température, dans l'intervalle où la chaîne *PP II* domine. Par ailleurs, nous abordons l'étude de l'approche à l'équilibre du cycle complet *CNO* formé du cycle *CN* et des deux cycles *NO*, en montrant que la présence du deuxième cycle *NO* entraîne un quasi-équilibre pour le cycle *CN*.

1. Energy production in the proton-proton chain

The two most important ways by which hydrogen is converted into helium whitin stars are the so called proton proton chain and the *CNO* cycle.

Now the *pp* chain operates over the temperature range $8 \leqslant T_6 \leqslant 30$ (where $T_6 = T \times 10^{-6}\ °K$), yielding 26.21 *MeV* in form of radiation energy; it consists in fact in a three-fold chain of reactions; the 1^{st} being

$$\begin{aligned} &a_1)\ H^1 + H^1 \rightarrow D^2 + \beta^+ + \nu \\ &a_2)\ D^2 + H^1 \rightarrow He^3 + \gamma \qquad (pp\,I) \\ &a_3)\ He^3 + He^3 \rightarrow He^4 + 2\,H^1 \end{aligned}$$

When He^4 becomes sufficiently abundant, we may have instead of a_3),

$$a_4)\ He^3 + He^4 \rightarrow Be^7 + \gamma$$

and whether Be^7 will capture either a free electron or a proton, we will finally obtain He^4 by the completion of the *pp-II* or the *pp-III* chain respectively, *ViZ.*

$$a_5)\ Be^7 + \beta^- \rightarrow Li^7 + \nu$$
$$a_6)\ Li^7 + H^1 \rightarrow 2He^4$$

or

$$a_7)\ Be^7 + H^1 \rightarrow B^8 + \gamma$$
$$a_8)\ B^8 \rightarrow Be^8 + \beta^+ + \nu$$
$$a^9)\ Be^8 \rightarrow 2He^4$$

When the complete *pp* chain proceeds in equilibrium, the rate of helium formation is given by the expression (see Clayton, 1968)

$$\frac{dHe^4}{dt} = \left(\frac{dHe^4}{dt}\right)_I \varphi(\alpha_1)$$

where $(dHe^4/dt)_I$ is the rate of formation by the *ppI* chain only and

$$\varphi(\alpha_1) = 1 - \alpha_1 + \alpha_1 \left(1 + \frac{2}{\alpha_1}\right)^{\frac{1}{2}}$$

where

$$\alpha_1\left(T, \frac{He^4}{H}\right) = \frac{\lambda_{34}^2}{\lambda_{33}\,\lambda_{pp}} \left(\frac{He^4}{H}\right)^2$$

the λ's being the reaction rates per pair of interacting nuclei. [$\text{cm}^3\ \text{sec}^{-1}$].

The rate of energy liberation per unit volume, $\rho\epsilon$ [$\text{erg sec}^{-1}\ \text{cm}^{-3}$], will not include the energy carried off by the neutrinos which amounts respectively, for each of the neutrinos appearing in a_1), a_5), a_8), to 1.9, 4.0 and 28 per cent of the total energy connected to the mass defect $4\,m_H - m_{He}$.

Denoting by F_{ppi} the fraction of α particles produced by the chain i ($i = I, II, III$), we may write

$$\rho\epsilon = \frac{dHe^4}{dt}(4m_H - m_{He})\,c^2\,(0.981\,F_{ppII} + 0.960\,F_{ppII} + 0.720\,F_{ppIII})$$

whence, if $\epsilon = \epsilon_I$ when $F_{ppI} = 1$ (*ppI* acting alone),

$$\epsilon = \frac{\epsilon_I}{0.981}\,\varphi(\alpha_1)\,[0.981\,F_{ppI} + 0.960\,F_{ppIi} + 0.720\,F_{ppIII})$$

There is no difficulty in showing that, under equilibrium conditions (Clayton, 1968)

$$\frac{F_{ppI}}{F_{ppII} + F_{ppIII}} \equiv \frac{F_{ppI}}{1 - F_{ppI}} = \frac{1}{4}\left[\left(1 + \frac{2}{\alpha^{\mathrm{I}}}\right)^{\frac{1}{2}} - 1\right]$$

whence

$$F_{ppI} = \left[\left(1 + \frac{2}{\alpha_1}\right)^{\frac{1}{2}} - 1\right]\left[\left(1 + \frac{2}{\alpha_1}\right)^{\frac{1}{2}} + 3\right]^{-1}$$

Furthermore, in terms of the branching ratio

$$\alpha_2 = \frac{F_{ppIII}}{F_{ppII} + F_{ppIII}}$$

of the *ppIII* chain at Be^7, we have

$$F_{ppIII} = \alpha_2 (1 - F_{ppI}) \qquad F_{ppII} = (1 - \alpha_2)(1 - F_{ppI})$$

so that, after some easy algebraic manipulation, we can write ϵ in the form

$$\epsilon = \epsilon_I \psi(\alpha_1, \alpha_2)$$

where

$$\psi(\alpha_1, \alpha_2) = 1 + (0.962 - 0.5\,\alpha_2)[\varphi(\alpha_1) - 1]$$

which is (improved value) the expression given by Reeves (1965), obtained here in a slightly different way, through the explicit introduction of the F_{ppi}.

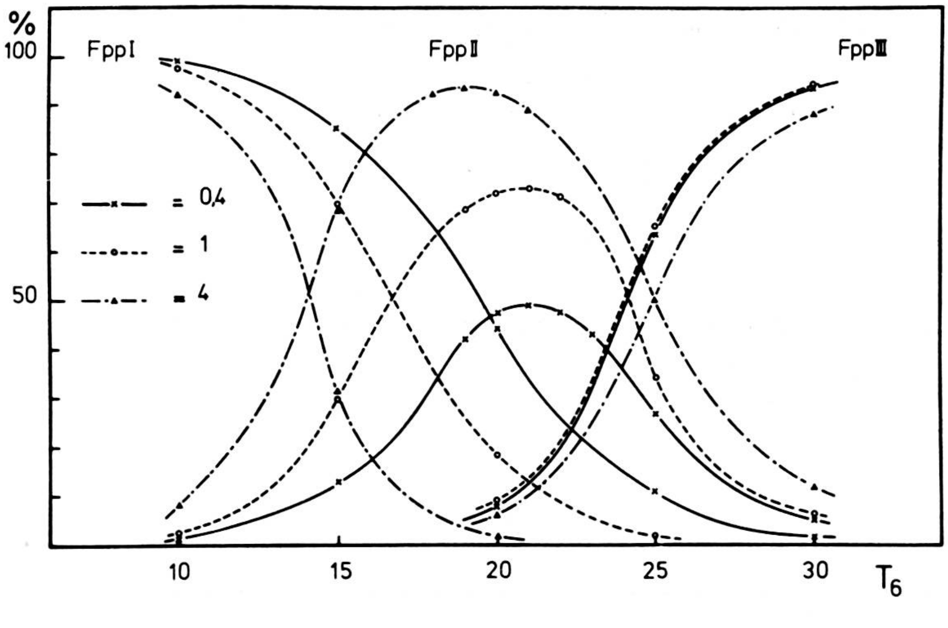

FIG. 1. — Illustrates the dependence of the fractions F_{ppi} on temperature, at given $Y/X = 4$ He^4/H ratio.

2. PREDOMINANCE OF *pp II* CHAIN

Fig. 1 reveals that, when enough He^4 is present, there is a temperature interval in which the *pp II* chain prevails distinctly over the 2 other chains ($17 \lesssim T_6 \lesssim 22$); since *pp III* has not started significantly yet, we have $\alpha_2 = O$ and consequently,

$$\epsilon \approx \epsilon_I \varphi(\alpha_1)$$

ϵ_l is proportional to the function $\tau^2 e^{-\tau}$ characteristic of a non resonant reaction, where $\tau = BT_6^{-1/3}$, B being a constant depending on the chemical parameters (A, Z) of the interacting nuclei.

On the other hand, it is customary to express the $\epsilon(T)$ dependence as a power law, so that

$$\epsilon \div T^{\nu} \div \tau^{-3\nu} \div \tau^2 e^{-\tau} \varphi(\alpha_1)$$

By logarithmic derivation, we obtain

$$3\nu = \left(1 - \frac{\varphi'_\tau}{\varphi}\right)\tau - 2$$

The explicit form for α_1 is, to a high degree of accuracy,

$$\alpha_1 = A \exp(-100\, T_6^{-1/3})$$

where A is practically independent of T; consequently,

$$\varphi'_\tau = \varphi'_{\alpha_1} \frac{\partial \alpha_1}{\partial \tau} - = \frac{100}{B} \alpha_1 \varphi'_{\alpha_1} \simeq - 3\alpha_1 \varphi'_{\alpha_1}$$

since the value of B in the present pp reaction amounts to 33.7.

Therefore

$$\nu = \frac{\tau - 2}{3} + \alpha_1 \frac{\varphi'_{\alpha_1}}{\varphi} \tau$$

the first term pertains to the *pp I* chain only, the second term being the contribution of the *pp II* chain. Table 1 summarizes the values obtained for ν at different Y/X and T_6 values.

TABLE 1

Y/X \ T_6	18	20	22
0.4	3.62 + 1.03	3.50 + 1.32	3.34 + 1.32
0.63	3.62 + 1.32	3.50 + 1.37	3.34 + 1.08
1	3.62 + 1.42	3.50 + 1.21	3.34 + 0.73
2	3.62 + 1.23	3.50 + 0.65	3.34 + 0.27
4	3.62 + 0.65	3.50 + 0.23	3.34 + 0.08

ν is increased by the *pp II* influence, but not considerably; *pp II* is less sensitive to the temperature as the helium abundance is high (Y/X large).

3. The *CNO* tricycle

When T_6 reaches 12, hydrogen may also burn by the *CNO* cycle, which around $T_6 = 18$, for standard compositions of population *I* stars, soon becomes the dominant reaction as T_6 goes on rising. The *CNO* tricycle, sketched in fig. 2, is made up of the basic *CN* cycle (hexagon) and of two additional *NO* cycles, the first of which, connected to the *CN* cycle joins N^{15} to N^{14} by passing O^{16} whereas the second *NO* cycle, added to the first one, joins O^{17} to N^{15} through O^{18}.

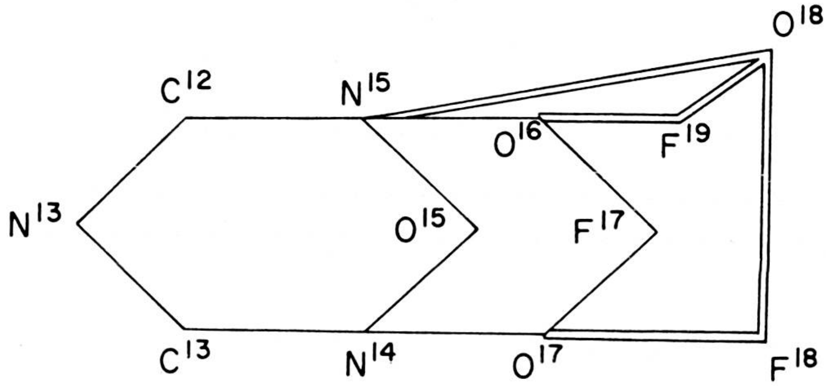

Fig. 2.

All reactions considered here are induced by protons; we neglect any α-induced reaction, corresponding to simultaneous hydrogen and helium burning (Caughlan and Fowler, 1962).

The branching ratio of the N^{15} (p,γ) O^{16} reaction is approximately $\gamma_1 = 8 \times 10^{-4}$ according to Fowler et al. (1975; quoted as *FCZ II*), so that it takes about 10^3 complete *CN* cycles to go through before a significant number of *CN* nuclei will have switched over to the *NO* cycles. In general, the lifetimes of the different nuclear species are fairly time-independent during a steady hydrogen burning phase, so that we may regard these lifetimes as constant at a given temperature; the branching ratios will then also be constant and moreover weakly dependent on temperature changes.

In the physical conditions $T_6 = 25$, $\rho X_H = 28$ which we had adopted in a preliminary study to test the assumption of constant nuclear lifetimes during a particular nuclear burning stage, the lifetimes of the relevant nuclei are, in years,

$$\tau_{12} = 8.98 \times 10^2 \qquad \tau_{16} = 4.51 \times 10^6$$
$$\tau_{13} = 2.58 \times 10^2 \qquad \tau_{19} = 1.98 \times 10^5$$
$$\tau_{14} = 8.38 \times 10^4$$

Beyond F^{19}, *p*-induced reactions could lead either to O^{16} or to Ne^{20}; both such reactions are extremely slow, the second being still slower than the first which could close the chain at O^{16}, thus completing a *CNOF* quadricycle.

We further label $\tau_{A,B}$ the lifetimes of nucleus A with respect to the transmutation $A \to B$ through p-capture; always according to the *FCZ II* reaction rates, we have, in years,

$$\tau_{15,12} = 3.3 \quad \text{and} \quad \tau_{15,16} = 4.1 \times 10^3$$

For the isotopes O^{17}, O^{18}, the values are sensitive to experimental uncertainties; according to whether a resonance in the compound nucleus contributes by a vanishing, partial or dominant amount to the p-capture cross-section, *FCZ II* give three possible values (denoted L, I, H respectively). Retaining here only the intermediate values, viz. in years,

$$\tau_{17,14} = 2.78 \times 10^6 \qquad \tau_{18,15} = 12{,}4$$
$$\tau_{17,18} = 3.71 \times 10^6 \qquad \tau_{18,19} = 7.4 \times 10^5$$

whence

$$\gamma_2 = \frac{\tau_{17,14}}{\tau_{17,14} + \tau_{17,18}} = 0.43 \qquad \gamma_3 = \frac{\tau_{18,15}}{\tau_{18,15} + \tau_{18,19}} = 1.68 \cdot 10^{-5}$$

In spite of the uncertainty in these values, they do show us that the frequency of occurrence of both *ON* cycles is about the same, and that of the complete quadricycle is comperatively much less.

4. Approach toward equilibrium in the *NO* cycles

According to the τ_{12} value given above, we see that the *CN* cycle achieves equilibrium in about 10^3 years and only later will the *NO* cycles become significantly operative. The stellar environment adopted here, namely $\rho\ X_H = 28$, $T_6 = 25$ corresponds closely to the central conditions of a 3 $M_\odot$ main sequence star (Iben, 1965); the fact that T_6 lies well below 100 prevents us from having to consider α-induced reactions and the complications of the rather extensive reaction network for the hot *CNO-Ne* cycle (Audouze et al., 1973).

The time variations of chemical abundances during the approach to equilibrium of the *NO* cycles are the following (see fig. 2):

$$\begin{aligned}
\frac{dN^{14}}{dt} &= (1-\gamma_2)\frac{O^{17}}{\tau_{17}} + (1-\gamma_1)\frac{N^{15}}{\tau_{15}} - \frac{N^{14}}{\tau_{14}} \\
\frac{dN^{15}}{dt} &= \frac{N^{14}}{\tau_{14}} - \frac{N^{15}}{\tau_{15}} + (1-\gamma_3)\frac{O^{18}}{\tau_{18}} \\
\frac{dO^{16}}{dt} &= \gamma_1\frac{N^{15}}{\tau_{15}} - \frac{O^{16}}{\tau_{16}} + \frac{F^{19}}{\tau_{19}} \\
\frac{dO^{17}}{dt} &= \frac{O^{16}}{\tau_{16}} - \frac{O^{17}}{\tau_{17}}
\end{aligned} \tag{1}$$

where we have neglected the short-lived positron emitters. The running of the NO cycles alters somewhat the CN equilibrium abundances reached after 10^3 years; we now have

$$\frac{C_e^{12}}{\tau_{12}} = \frac{C_e^{13}}{\tau_{13}} = \frac{N_e^{15}}{\tau_{15}} = \frac{N^{14}}{\tau_{14}} + \gamma_2 \frac{O^{17}}{\tau_{17}}$$

where N^{14} and O^{17} will change on a longer time scale ($\sim 10^5\, y$); the CN cycle is therefore in a quasistatic equilibrium.

Dropping further the F^{19} channel ($F^{19} = O, \gamma_3 = O$) and noticing that O^{18} reaches its equilibrium value in less than 20 years:

$$\frac{O_e^{18}}{\tau_{18}} = \frac{O_e^{18}}{\tau_{18,\,15}} = \gamma_2 \frac{O^{17}}{\tau_{17}}$$

we obtain

$$\frac{dN^{15}}{dt} = \frac{N^{14}}{\tau_{14}} - \frac{N^{15}}{\tau_{15}} + \gamma_2 \frac{O^{17}}{\tau_{17}}$$

Since $\frac{1}{\tau_{15}} \approx \frac{1}{\tau_{15,\,12}}$, and $\tau_{15,\,12}$ is of order 3½ years, while τ_{14} and τ_{17} are much larger, we may consider that in a few years time, N^{15} has attained its equilibrium value and we are left with the three equations

$$\begin{aligned}
\frac{dN^{14}}{dt} &= (1-\gamma_2\gamma_1)\frac{O^{17}}{\tau_{17}} - \gamma_1 \frac{N^{14}}{\tau_{14}} \\
\frac{dO^{16}}{dt} &= \gamma_1 \frac{N^{14}}{\tau_{14}} + \gamma_1\gamma_2 \frac{O^{17}}{\tau_{17}} - \frac{O^{16}}{\tau_{16}} \\
\frac{dO^{17}}{dt} &= \frac{O^{16}}{\tau_{16}} - \frac{O^{17}}{\tau_{17}}
\end{aligned} \qquad (2)$$

which entail $N^{14} + O^{16} + O^{17} = N_{CNO}$ (const.). Letting $\gamma_2 = O$ brings us back to the case of the CNO-bicycle.

5. Temporal behaviour oe the chemical composition

Labeling N_i ($i = 1, 2, 3$) the fractional abundances of the 3 isotopes involved in equations (2), it is easy to find, by suitable elimination, a 2^d-order differential equation for each of the N_i, of the form

$$\frac{d^2Ni}{dt^2} + A\frac{dN_i}{dt} + BN_i - C_i = O \qquad (3)$$

where

$$N_1 = N^{14}, \quad N_2 = O^{16}, \quad N_3 = O^{17},$$

$$C_1 = \frac{1 - \gamma_1 \gamma_2}{\tau_{16}\tau_{17}} N_{CNO}, \quad C_2 = \frac{\gamma_1}{\tau_{14}\tau_{17}} N_{CNO}, \quad C_3 = \frac{\gamma_1}{\tau_{14}\tau_{16}} N_{CNO},$$

$$A = \frac{\gamma_1}{\tau_{14}} + \frac{1}{\tau_{16}} + \frac{1}{\tau_{17}} \qquad N_{CNO} B = \sum_{i=1}^{3} C_i$$

The general solution of (3) has the form

$$N_i(t) = K_i e^{r_1 t} + L_i e^{r_2 t} + \frac{C_i}{B} \tag{4}$$

where K_i, L_i are constants, $\frac{Ci}{B}$ is the equilibrium value of N_i and

$$r_1 = -\frac{A}{2} + \frac{1}{2}\sqrt{A^2 - 4B}, \qquad r_2 = -\frac{A}{2} - \frac{1}{2}\sqrt{A^2 - 4B}$$

The initial conditions to which K_i, L_i are connected are taken at a time $t = O$ chosen as the starting time for the *NO* cycles display, well after the *CN* cycle has attained equilibrium. In the particular stellar environment adopted here and also for other temperatures between 20 and 80 million degrees, it turns out that $A^2 - 4B$ is always positive (sometimes only weakly); therefore r_1, r_2 and consequently K_i, L_i are real and we get a superposition of exponential variations for $N_i(t)$. Some of the abundances will increase from $N_i(0)$ to C_i/B while others shall decrease, since at all times we have

$$\sum_{i+1}^{3} N_i(t) = N_{CNO}$$

But the accuracy of the τ-values is often poor and, for other similar situations, we could possibly be faced with the case $A^2 - 4B < O$; this happens, for example, when the lifetimes τ_{14}/γ, τ_{16}, τ_{17} become very close to each other and the approach to equilibrium would then have an oscillatory character.

Letting $\frac{1}{2}\sqrt{4B - A^2} = \omega$, the solution (4) becomes explicitly

$$N_i(t) = \left(N_i(o) - \frac{C_i}{B}\right) \exp\left(-\frac{A}{2}t\right)\left[\cos \omega t + \frac{A}{2\omega} \sin \omega t\right] + \frac{C_i}{B}$$

having assumed $\left(\frac{dN_i}{dt}\right)_{t=0} = 0$.

The condition that $N_i(t)$ must always remain positive is warranted by the fact that the period $2\pi/\omega$ is distinctly larger than the damping time $2/A$. After a quarter of period, for instance, when $t = \pi/2\omega$, the amplitude of the oscillation is already reduced by $e^{-\pi u/2} . u$ (where $u = A/2\omega$) and this factor is much less than unity because, from the definitions of A and B, $A^2 - 2B$ is always $> O$, so

$$u = \frac{A}{\sqrt{4B - A^2}} > 1 .$$

We conclude that $N_i(t)$ is indeed always $\geqslant O$, whatever the sign of $N_i(O) - Ci/B$.

We are indebted to Professor W. A. Fowler for having sent us in advance of publication, a preprint of his revised thermonuclear reaction rates.

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BIBLIOGRAPHY

Audouze, J., J. W. Truran and B. A. Zimmermann (1973). Ap. J. *184*, 492.
Coughlan, G. R. and W. A. Powler (1962). Ap. J. *136*, 453.
Clayton, D. D. (1968). Principles of stellar evolution and nucleosynthesis, McGraw-Hill.
Fowler, W. A. et al. (1975). Thermonuclear reaction rates II Annual review of astronomy and astrophysics. Vol. 13.
Iben, I. Jr. (1965). Ap. J. *142*, 1447.
Reeves, H. (1965). Stellar structure, ed. L. H. Aller and D. B. McLaughlin. (Stars and Stellar Systems)