The use of shells in steel structures

Autor(en): **Laffaille, R.**

Objekttyp: **Article**

Zeitschrift: **IABSE congress report = Rapport du congrès AIPC = IVBH Kongressbericht**

Band (Jahr): **2 (1936)**

PDF erstellt am: **24.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-3212

V 11

The Use of Shells in Steel Structures.

Anwendung von Schalen im Stahlbau.

Application des voiles minces en construction métallique.

B. Laffaille,
Ingénieur des Arts et Manufactures, Paris.

A. General Considerations.

In a paper dealing with the general study of thin skew surface shells, which appeared in the third volume of the publications of the I.A.B.S.E. in 1935, the author has explained what principles it is possible to deduce from consideration of curves of thin surface shells as functions of geometric equilibria, and what applications it is possible to make of these in the domain of reinforced concrete. It was further suggested, at the end of that paper, that the same principles would apply to metal structures, and it was pointed out that in this new field the true problem which presents itself is that of the buckling of curved surfaces. This is the line of thought which it is now proposed to develop.

Before, however, embarking on this recapitulation of our work, we feel it a duty to acknowledge some very valuable collaboration, which has led to the problem being attacked in a team spirit that accords well with its interest to metallurgical industry in general and transcends the rather narrowly personal approach which there might have been some temptation to use towards what might seem a mere problem in the strength of materials.

In the first place, the firm of Rouzand in Paris has assisted us in all matters relating to preliminary tests, up to the stage of the first actual constructions which were made by the firm of Delattre et Trouard. Mr. *L. Beschkine*, Ingénieur des Arts et Manufactures, with whom we have been studying these general questions for a period of several years, has made a personal contribution to the working out of the special calculations that relate to these curved surfaces. [1]

When the question arose of giving consideration to buckling effects, linked with matters of curvature and of local distribution of strain, we found a distinguished collaborator in Mr. *F. Vasilesco*, Docteur ès Sciences, whose mathematical knowledge has made it possible, from the broadest possible angle, to reach solutions of the new problems arising.

Finally — and this is one of the aspects with which the author, personally, is more impressed than with any, for it is here that the possible repercussions in the human plane are greatest — we were able also to secure the interest, in these engi-

[1] See Volume 4 of the "Publications" of I.A.B.S.E. 1936.

neering problems, of Mr. *R. Camelot*, Grand Prix de Rome, an architect of the service of Monuments Civils et Palais Nationaux. The building art confers on its practitioners a classical and profoundly humanist training, and this appeared to us an essential factor to import into this quest after novel forms applied to a material itself relatively new.

Let us now return to the point of view of the engineer.

Firstly let us take a close lock at the arrangement of structural members in a typical steel roof construction. These elements at once fall into two groups: those which have to withstand the loads, such as trusses, purlins, king-posts, struts, and those which constitute the actual covering, such as, for instance, corrugated sheet. In considering the composition and arrangement of these members one is constantly struck with the complication and variety of the pieces bearing upon one another: a lath causes bending stresses in a purlin, a purlin does the same in a secondary truss which rests on purlin-beams; these, finally, bear upon the principal trusses, which transfer the load through stancheons to the ground.

The problem involved in a construction is that of balancing the forces which arise, while making use only of a minimum of material and having due regard to what are the safe stresses. The complicated sequence of load-bearing members which we have just described does not appear to offer scope for a simple solution; the reason it is so widely used must be sought in the fact that the dimensions of the steel units turned out by the steelworks are such as almost to compel the adoption of certain methods of connection, and even of certain complete parts of structures, stereotyped once and for all in well-known, easily applied forms. If considerations of weight and transport are taken into account also, the upshot is that these complicated structural schemes may nevertheless lay claim to be practical, in the ordinary sense of the word.

Already the introduction of welding as a method of connection has led to simpler forms of design. If welding can be done not in the shop but actually on the job, the dimensions of the pieces to be transported cease to be a factor in construction, and roofing schemes can approximate to the simple, purely geometrical forms that are dictated by the equilibria of the forces involved.

It is possible, however, even while benefiting by this simplicity of design, to look upon the whole problem of construction in an even more general way and to describe it in some such terms as these:

Here is an area to be covered: here is the minimum volume to be enclosed, as defined by the clear dimensions. The conditions under which the external forces act are known. Let our aim be simply to deal with these forces by guiding the reactions on to the supports through the medium of the envelope which is needed to enclose the volume.

This is the general problem; but in the last analysis it is the only problem that matters. The same idea that has led to simple yet novel solutions in the field of reinforced concrete as applied to thin curved shells opens up a promising path in the field of steel construction.

What conditions are needed to confer stability on a thin curved surface under a given system of loading?

In the first place the conditions for geometrical equilibrium must be satisfied;

secondly it is necessary that local deformations due to phenomena outside of geometry should not be such that these equilibria are affected by elastic variations.

For the moment we may view the problem in this way: a thin sheet surface is very deformable in the direction of the tangential dimensions; it is like a tent, and the first idea that comes to mind is to consider the stability of such a "tent" when stretched on a supporting frame. This, indeed, is the conception first carried into effect — see Fig. 1 — and in practice it amounts to doing away with the purlins of ordinary construction and carrying the forces which occur directly on to the

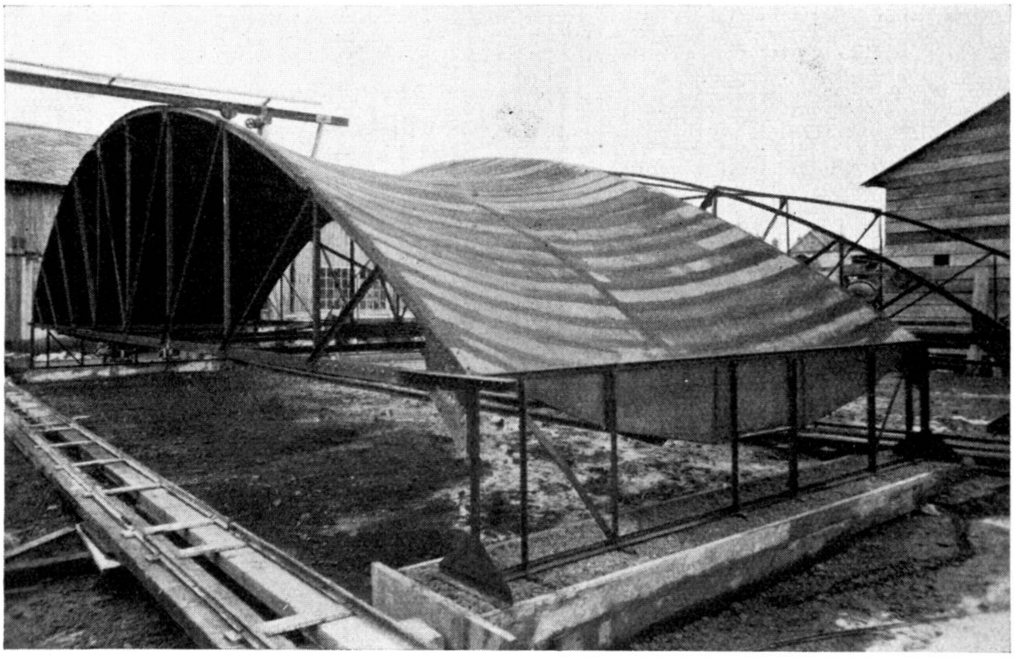

Fig. 1.

Plated shell. Rotary hyperboloid. Span 14 m. Plates 10—12 mm thick. Dreux (France).

trusses, through the medium of the covering surface. This is at once a source of economy.

Here some observations are of interest. We had the idea, first of all, of making greater use of surfaces of reversed curvature, so as to obtain the advantage of stressing the metal in tension, even where there is a reversal of sign — a contingency that may arise in light coverings when the wind causes sometimes a pressure and sometimes a suction effect.

Secondly, if only surfaces in tension (such as cylindrical shapes) have been used — not surfaces with reversed curvature — then under many conditions of loading a "rolling" action will be produced along the curve when the "funicular" corresponding to the loads in question deviates from the mean axis of the resisting element.

Where the surface offers some local resistance to buckling this "rolling" may be withstood by shear stresses; in the present case, however, its consequence is simply a risk of deformations and folds.

Reference may also be made to a note by Mr. *F. Aimond*, published in "Le Génie Civil" in 1933, drawing attention to the advantages of hyperbolic paraboloid surfaces formed from sheet metal; the author did not, however, explain how such a surface was to be applied in relation to the edge reactions. Actually a hyperbolic paraboloid surface is of little practical use in structural steelwork on account of the difficulties involved of setting it out and fixing it in place, and from the earliest of the attempts here described we chose a surface made up of inverse curves of "revolution". In this way the same elementary constructional part is repeated over and over again many times to build up the surface as a whole.

In this way we arrived at what we call a semi-self-supporting roof surface wherein the applied loads are carried to the supports through the medium of simple forms of truss. What is gained is the elimination of purlins.

The idea next occurred to us that given suitably chosen curves it should be possible to make use of the thin sheet without restriction as to the direction of local strains. Here we were confronted straight away with the phenomena of buckling: these called for special study, and by playing upon the possible variations in curvature along the profile of the compressed section it should be possible to achieve conditions of stable equilibrium.

We made numerous experiments on models, and these showed the importance of the part played by the cross stiffeners, which, together with edges, constitute frames (Fig. 2 and Fig. 3). A thin shell of otherwise similar form made up of inverse

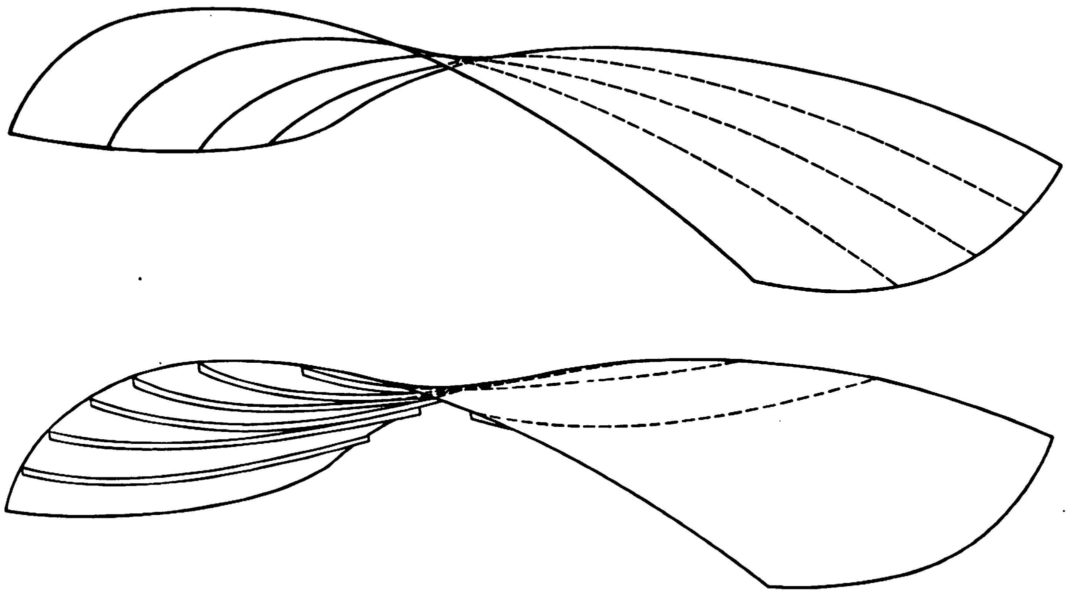

Fig. 2 and 3.

Shell of double curvature and stiffeners. Rotary hyperboloid.

curves, but acting as a true vault without any stiffeners, is "whippy" under load: the slightest shifting of forces is enough to deform it. But if transverse stiffening ribs are provided the rigidity of the whole thing is improved beyond comparison. (The size of spacing between the stiffeners is of no great consequence.)

Through these experiments we were able to find out how to make thin sheet metal shells entirely self-supporting, and we at once carried into effect a first construction of this kind repeating the same dimensions and conditions as in the case of the semi-self-supporting surface stressed only in tension. In the photographs, Fig. 4 and Fig. 5, it is easy to distinguish the transverse stiffener-frames, which are simply strips cut from sheet metal. Such a surface, of 14 m. (45.9 ft.) span, weighed 13 kg. per sq. m. (2.66 lbs. per sq. ft.) of plan, including both the sheet and the stiffeners. In the course of the tests it was loaded to 70 kg. per sq. m. (14.34 lbs. per sq. ft.) without any noticeable deformation.

It may be noted that for relatively small spans like that mentioned above the unit stresses which arise are relatively very low indeed, being of the order of 3 or

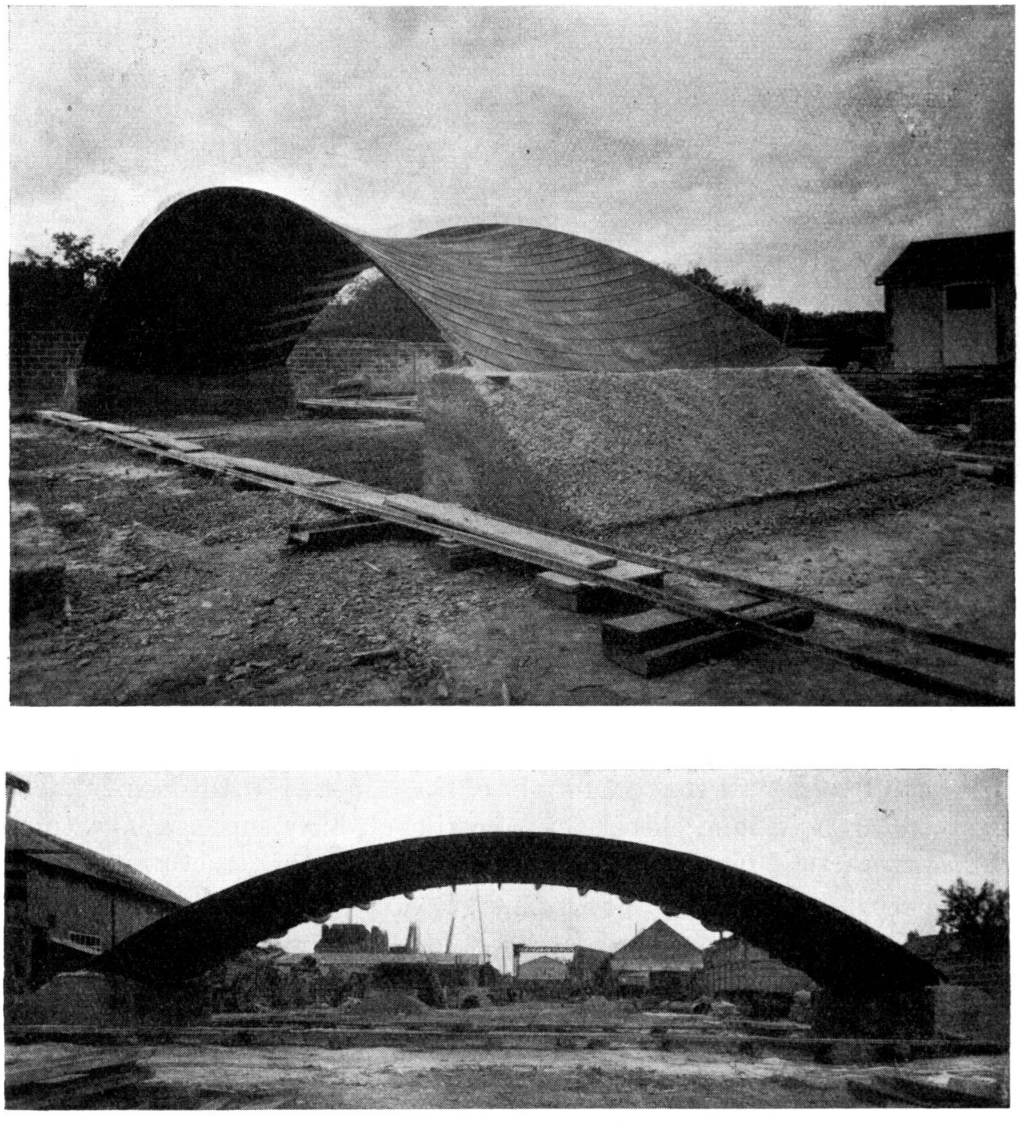

Fig. 4 and 5.

Plated shell (composed of rectangular plates). Rotary hyperboloid. Span 14 m. Plates 10—12 mm thick. Dreux (France).

4 kg. per sp. mm. (say 2 or 2½ tons per sq. in.). In the larger structures — towards which we are working — there would be need to make use of higher stress values if full economic use is to be made of the possibilities of the sheet.

Recently we have been called upon to study the question of local buckling in a curved surface. We had tests carried out in the workshops of the Aciéries de la Marine at St. Chaumond, parts of circular cylinders of varying diameter being subjected to compression in the direction of the generatrix. The experiments were followed both by Mr. *L. Beschkine* and ourselves (Fig. 6).

Subsequently we resumed the experiments on the buckling of sections of thin shells, no longer in pure compression but subjected to bending — which more often than not is the way that the maximum stresses arise in a resisting section.

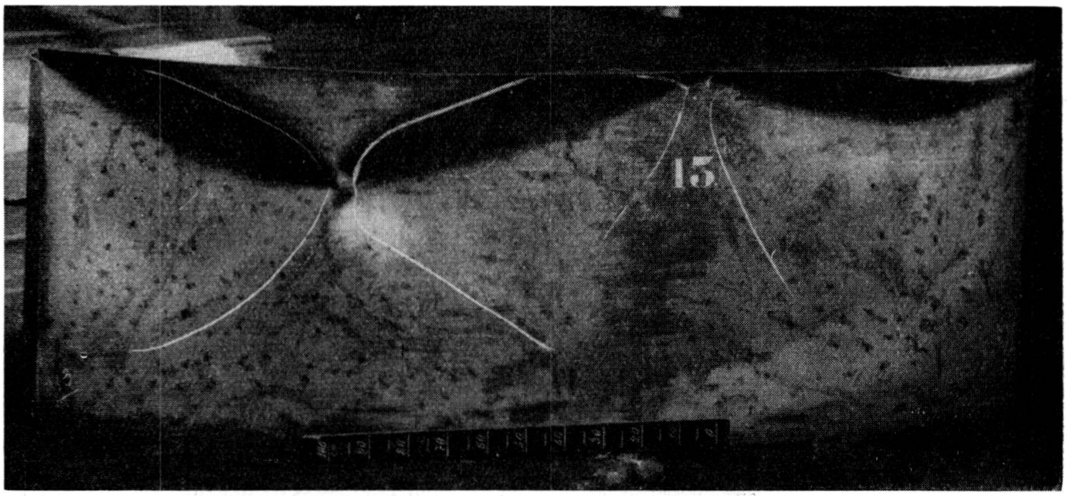

Fig. 6.

Buckling test of plate 30/10 mm by simple compression. St. Chamond.

While these experiments were proceeding we were able, through assistance of Mr. *T. Vasilesco*, to obtain some general mathematical data for solving problems of buckling in curved shells. This mathematical work, which we deem to be essential if full use is to be made of the "material" of sheet metal, is still in hand. The first results thrown up by it, and checked by experiment, have supplied us with enough facts to be of practical use and to enable various types of structure to be built.

First of all we had built for us, in the shops of Delattre and Frouard at Dannemarie-les-Lys, a tubular girder of 32 m. (105 ft.) span formed of 3.0 mm. (0.118 in.) sheeting, and also an arch of 75 m. (246 ft.) span from sheeting of the same thickness. These were framed systems, arranged with a view to resistance against buckling, and the application of loads served to demonstrate their complete adequacy (Figs. 7, 8 and 9). In the pictures of the 75 m. arch in course of erection, the positions of the stiffening ribs may be noticed, these being very light lattice work formed from 30 × 30 mm. angles.

Fig. 9 shows the inside of this great arch. The latticing visible in the upper portion is a temporary erection enabling the arch to resist torsion while its sections were being assembled.

In a series of drawings we show some forms of application of these self-supporting shells of thin sheeting, most of them being for possible future construction.

Fig. 10 shows a double shed formed of two cells along a central girder which is supported inside the shed on a single column behind the front. The depth of superstructure is very small. The following are the characteristics of such a building:

Width. . . . . . . 67.5 m. (221.5 ft.)
Depth . . . . . 67.5 m. (221.5 ft.)
Height . . . . . 8.0 m. (26.2 ft.)
Internal supports: one
Covered area: 4560 sq. m. (49,083 sq. ft.)
Weight per sq. m. of covered area
including the long sides: 62 kg. (12.7 lbs. per sq. ft.)

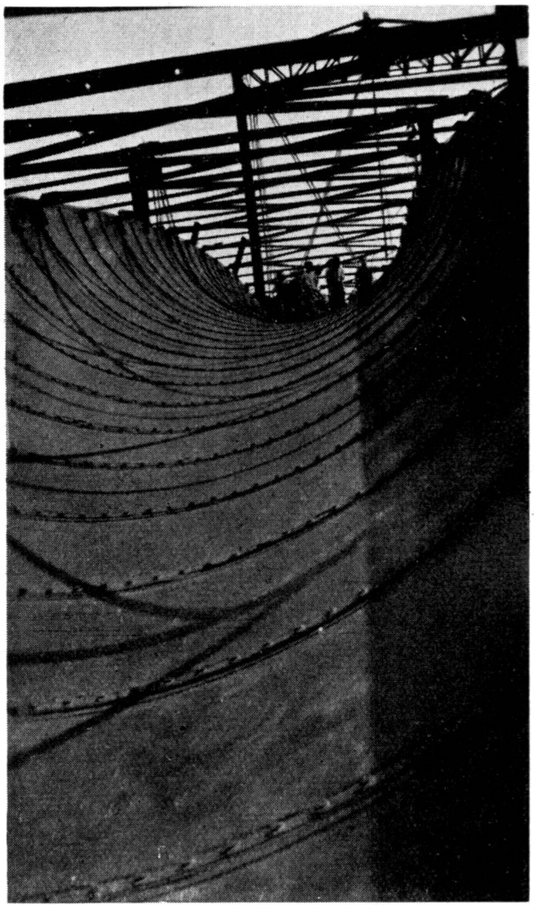

Fig. 7.
View of extrados, in course of erection.

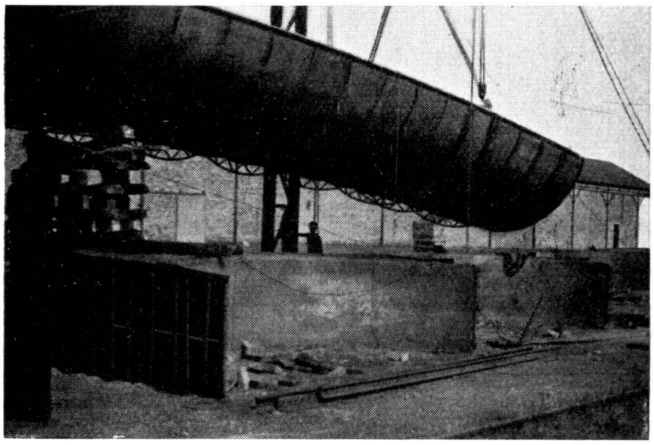

Fig. 8.
Arch of 75 m span, plate 30/10 mm thick. Circular barrel of rectangular plates. Assemblage of elements piece by piece. Dannemarie-les-Lys.

This type of building has been constructed by Delattre et Frouard to the order of the French Air Ministry, both at Dijon and at Casaux.

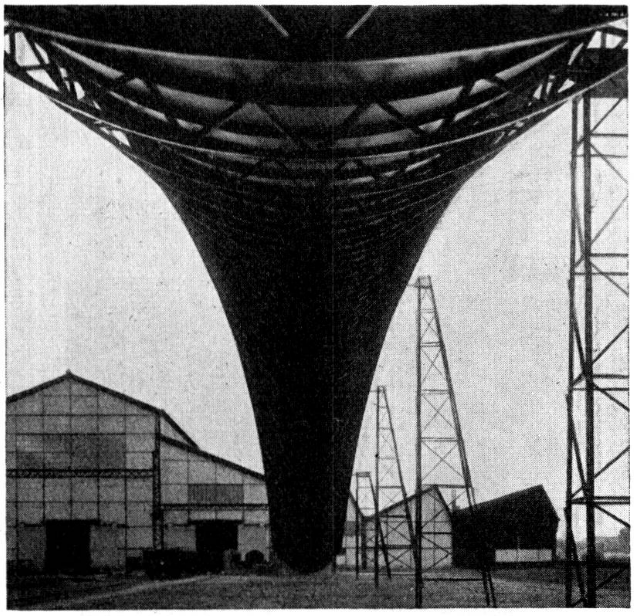

Fig. 9.
Arch of 75 m span. View of intrados with stiffeners (Yard of Delattre et Frouard réunis).

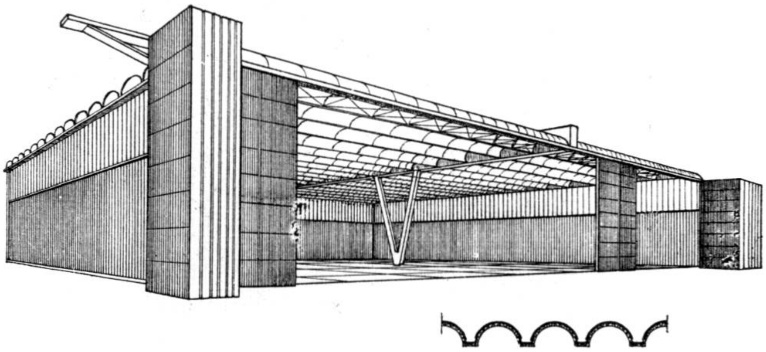

Fig. 10.
Twin airplane hangar 67.5 × 67.5 m. Thickness of plates 30/10 mm. Weight per m² 62 kg. Structures in execution at Dijon and Casaux.

Fig. 11. shows an arched hangar, which is one of the most economical types of roofing. Such an arch element is similar to that built at Dannemaire-les-Lys. The following are the characteristics:

Width.	90.0 m. (295.3 ft.)
Depth	90.0 m. (295.3 ft.) — or as desired
Height	20.0 m. (65.6 ft.)
Internal supports:	none
Covered area:	8100 sq. m. (86,111 sq. ft.)
Weight per sq. m.	of covered area, including
dropped panels:	80 kg. (16.4 lbs. per sq. ft.)

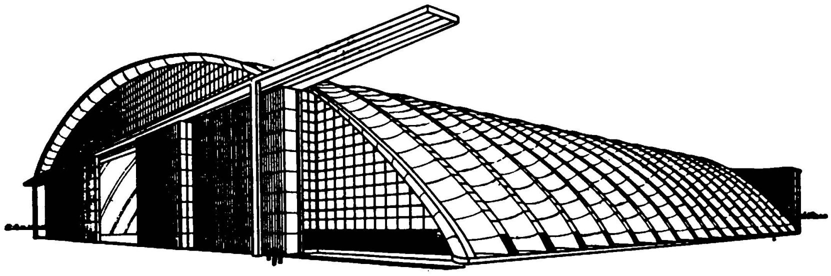

Fig. 11.
Arch airplane hangar rotary surface in panels. Thickness of plates 30/10 mm. Span 90 m.

In Fig. 12 the roof of the building is a "stretched arch". The diagrammatic section in the drawing shows the stiffening frame members which confer stability and prevent buckling. The weight of this form of construction is of the order of 60 kg. per sq. m. (12.3 lbs. per sq. ft.).

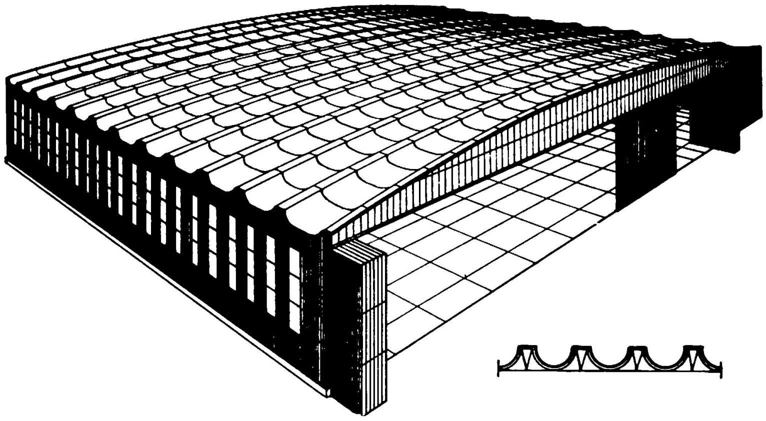

Fig. 12.
Arched shell, rotary surface in panels, flat arch span 40 m.
Weight: 60 kg per m² of covered area.

Figs. 13, 14, 15 and 16: —
Where the stresses are distibuted along the whole length of the section in accordance with alaw of bending, it may be advisable to form that section of varying curves, with the smallest radius of curvature in the zone of greatest compression. This is illustrated diagrammatically in Fig. 15.

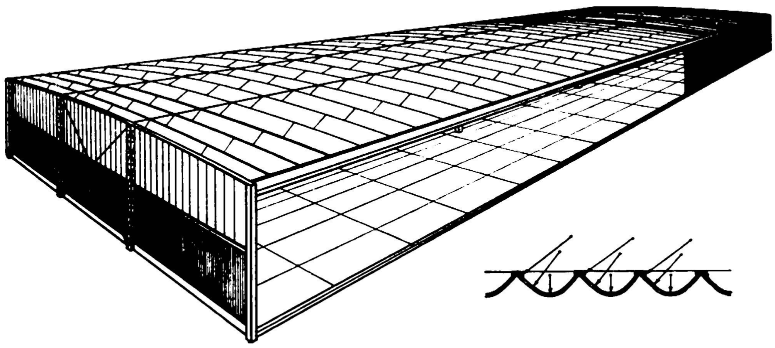

Fig. 13.
Hangar on two sides open, elements of shell 12/10 mm thick. Sections of elements of variable curvature, free cantilever span to open side: 10 m. Weight per m² 45 kg. View from above.

Different arrangements of thin sheet metal shells may be combined in order to leave large free openings on one face of a building, or for the roofing of workshops with lantern lighting.

These various types of building have been designed for work in Italy and have already been the subject of conclusive trials.

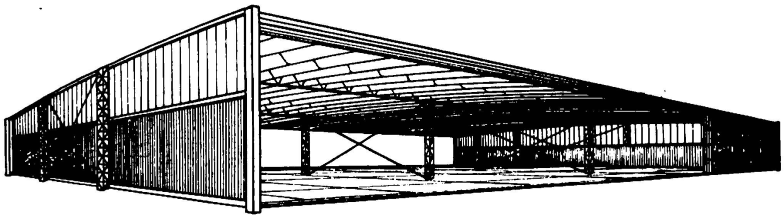

Fig. 14.

Hangar on two sides open, elements of shell 12/10 mm thick. Sections of elements of variable curvature, free cantilever span to open side: 10 m. Weight per m² 45 kg. View from below.

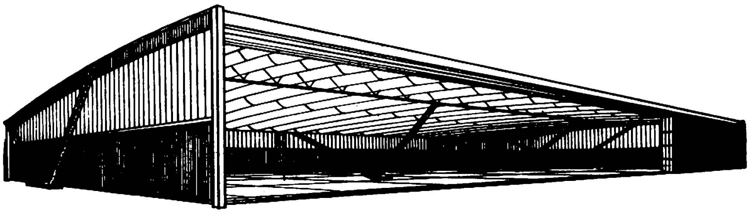

Fig. 15.

Hangar on one side open, elements of shell 14/10 mm thick. Sections of elements of variable curvature. Free cantilever span to open side: 12 m. Weight per m² 48 kg. View from below.

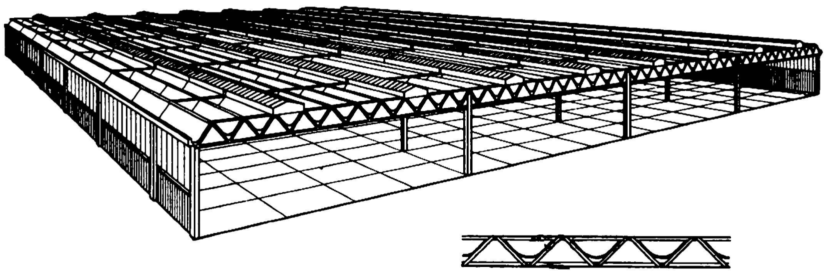

Fig. 16.

Roof shell construction with latern lights. Sections of elements of variable curvature. Plates 12/10 mm thick. Span of elements 12 m. Weight per m² 40 kg. View from above.

Figs. 17 and 21: —

A rather curious way to make use of thin sheet coverings as load bearing and space filling members, is to cover the building with circular cylindrical tubes. Such a scheme involves the fabrication of only a very small number of constructional elements and allows very rapid transport and erection. For spans of the order of 25 m. (82 ft.) this construction weighs 55 kg. per sq. m. (11.3 lbs. per sq. ft.) including the gates and the long sides.

To conclude this very rapid summary of an attempt to make systematic use of thin sheeting in constructional work, it may be useful to formulate our own line of thought:

Unless the whole question is to be regarded in the most strictly utilitarian sense, as being merely one of creating new commercial and industrial outlets for a material such as sheeting, it has another aspect which calls for special attention: all these attempts, all these examples of construction, resting in the main on facts and conside-

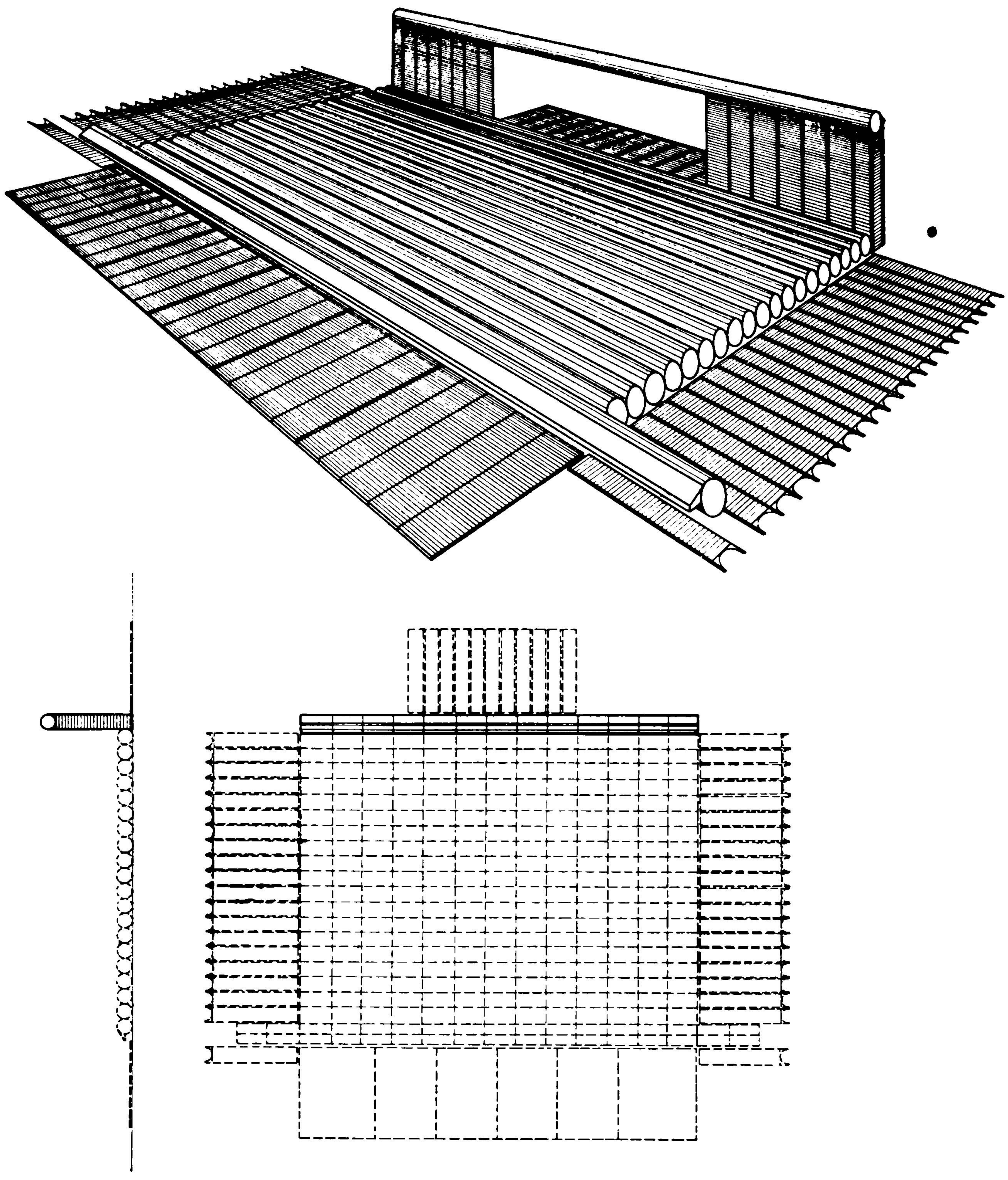

Fig. 17.

Fig. 17 to 21. Collapsible hangar. Erection method and different stages of erection.

rations which are solely technical and mathematical, nevertheless form the origin of new forms of structure, new lines of architecture of unusual types, which in this way are insinuated under the heading of constructive art.

In the face of these new problems I would wish to awaken both architects and engineers to the need that exists for them earnestly to search for solutions that shall be perfect aesthetically as well as technically.

On this plane there is more and more to be done. Mathematically ingenious

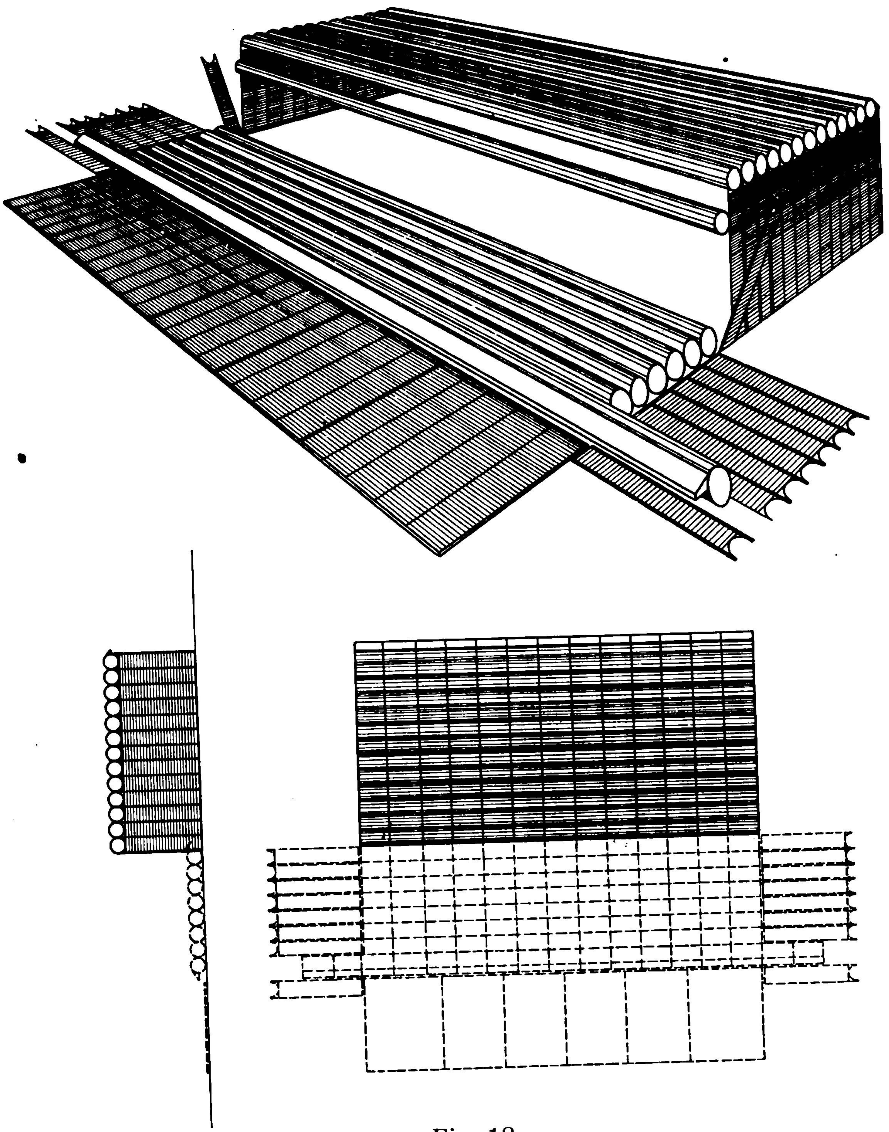

Fig. 18.

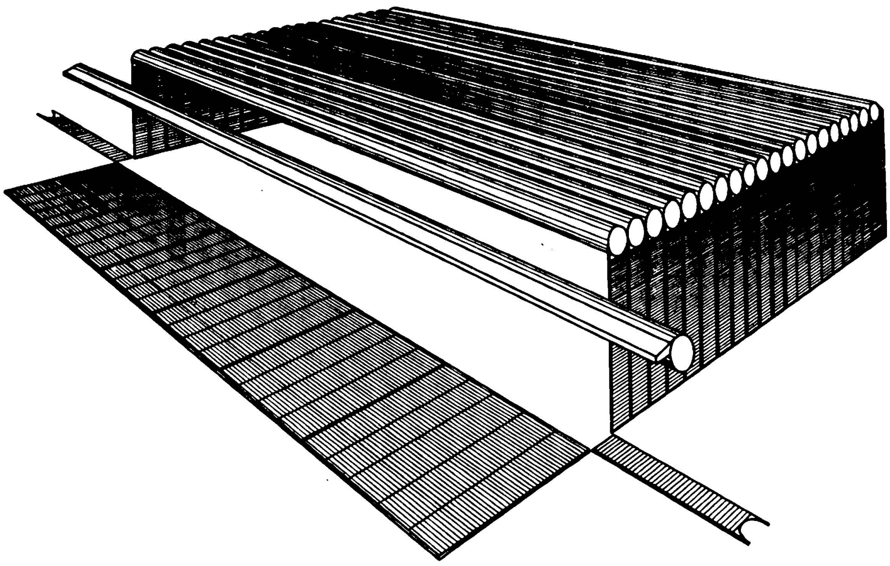

Fig. 19.

solutions are apt to be presented in a rather rigid form and there is a risk that this same rigidity may be imposed on arrangement and design. Confronted with a differential equation, the man who judges construction from relationships of volume and surface and who studies proportions in order to express construction on the

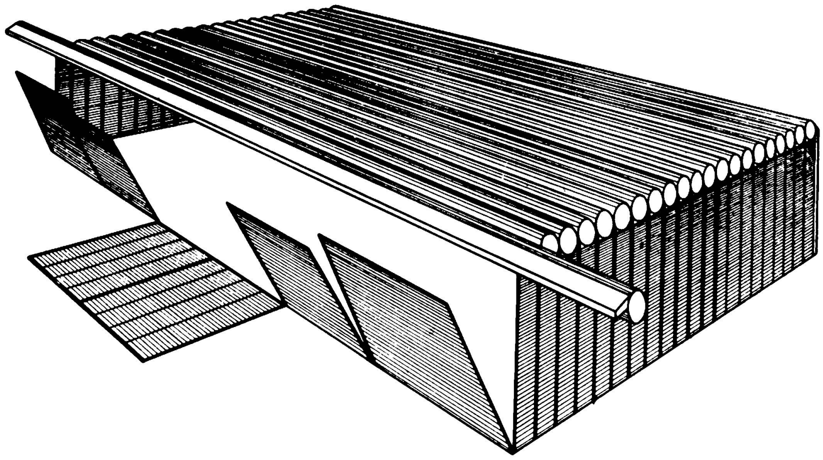

Fig. 20.

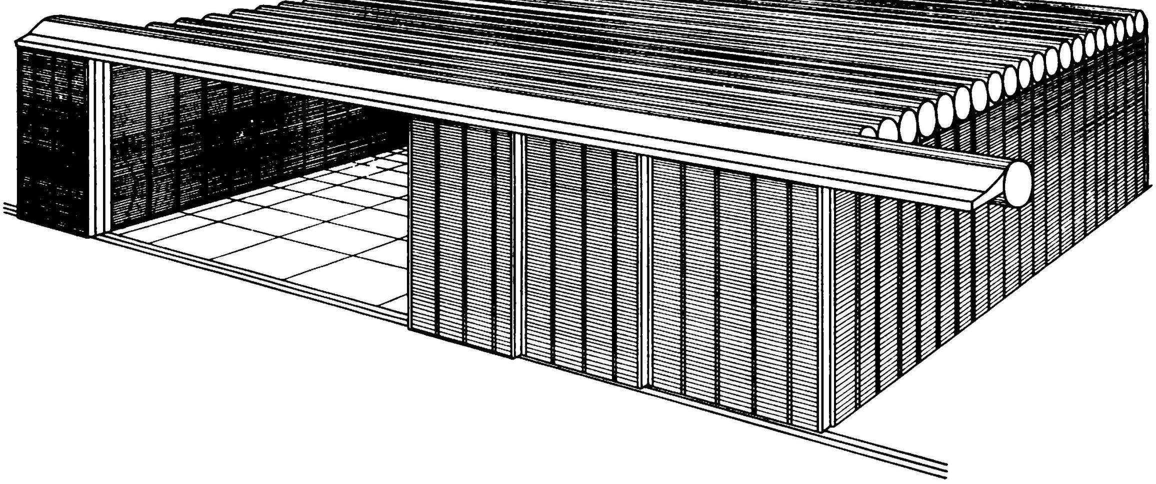

Fig. 21.

scale of human feeling — the man who is an architect — may falter, and faltering may abandon to the engineer this forbidding field wherein nothing counts but the strength of materials, forced to a pitch of analysis that is speculative in the extreme.

This divorce would be regrettable — for mathematical science by itself yields a fruit which is bitter and not always palatable. In the field of construction, examples must needs find their reflection in the collective spirit of man: it is incumbent that in doing so they shall serve not to prostitute, but to elevate.

B. Theoretical study about beams and arches made of stiffened steel shells.

We have shown in the first part of this treatise the considerations which led to the practical realisation and execution of stiffened steel-shell constructions.

In the following a number of theoretical and mathematical deductions are added to these considerations.

In the first instance the importance which is played by shell structures, and which has found its justification by numerous trials and tests shall be elucidated. Secondly the problem of buckling will be treated.

We wish to mention that we found in Mr. *L. Beschkine* a valuable assistant for the mathematical investigations of frames and as regards the problem of buckling we had the useful help of Mr. *Vasilesco.*

If according to fig. 3 an arch is built of thin sheeting its deformations can be studied with the following arrangement. The arch in this case if of thin cardboard and the abutments of wood, the latter will be screwed down on a table. Along the axis of the arch 11 holes are drilled at equal distances. Through each of these holes is put a vertical rule. In this manner it was possible to measure the relative deformation of the axis of the arch for a given load. The experimenting table served as plane of reference.

A few characteristic test results are given herewith: Fig. 22 shows in full line the unloaded axis of an arch (exagerated ordinates). The dash-dotted line was produced by putting a load between point 5 and 6.

If the load would have been applied between 3 and 4 the deformations produced would be according to fig. 23.

After removing the loads the arch went back to its original shape.

The investigation of the arch showed that the deformation of the cross section of the doubly curved shell were very great at the moment where maximum deformation occurred; if the arch in its unloaded state would have an invariable hyperbolic cross section, it would "open" and "close" in the case of loading. This was the reason which led to the idea of establishing the whole surface more rigid against bending and to provide along the arch such elements which are capable to withstand bending. This elements are called frame elements.

If we examine the illustrations of the experiments above mentioned more closely we can see the influence exerted by the two framing elements, which are placed symmetrically at point 4 and 8 of the arch axis.

In Fig. 24 is shown in dotted line the deformation of the arch axis for a load acting between point 5 and 6, while Fig. 25 shows the deformation due to a load placed between 3 and 4.

In both cases the important influence of the framing element on the behaviour of the thin shell is evident.

We have carried out repeated tests and trials by letting the force P act at various position, in every case the deformation were carefully noted, even the position of the framing elements were changed and the shape of the generatrix was altered, by employing the hyperbola, circle, ellipse, cycloid and parabola as basic curves.

The results can be mathematically summarised as follows:

The beams and arches made of thin shells are solid bodies produced by a given profile (section), or body of small thickness compared to its length. The profile follows a translatory movement in space along a curve of given law, all points of the profile come to lie therefore in parallel planes.

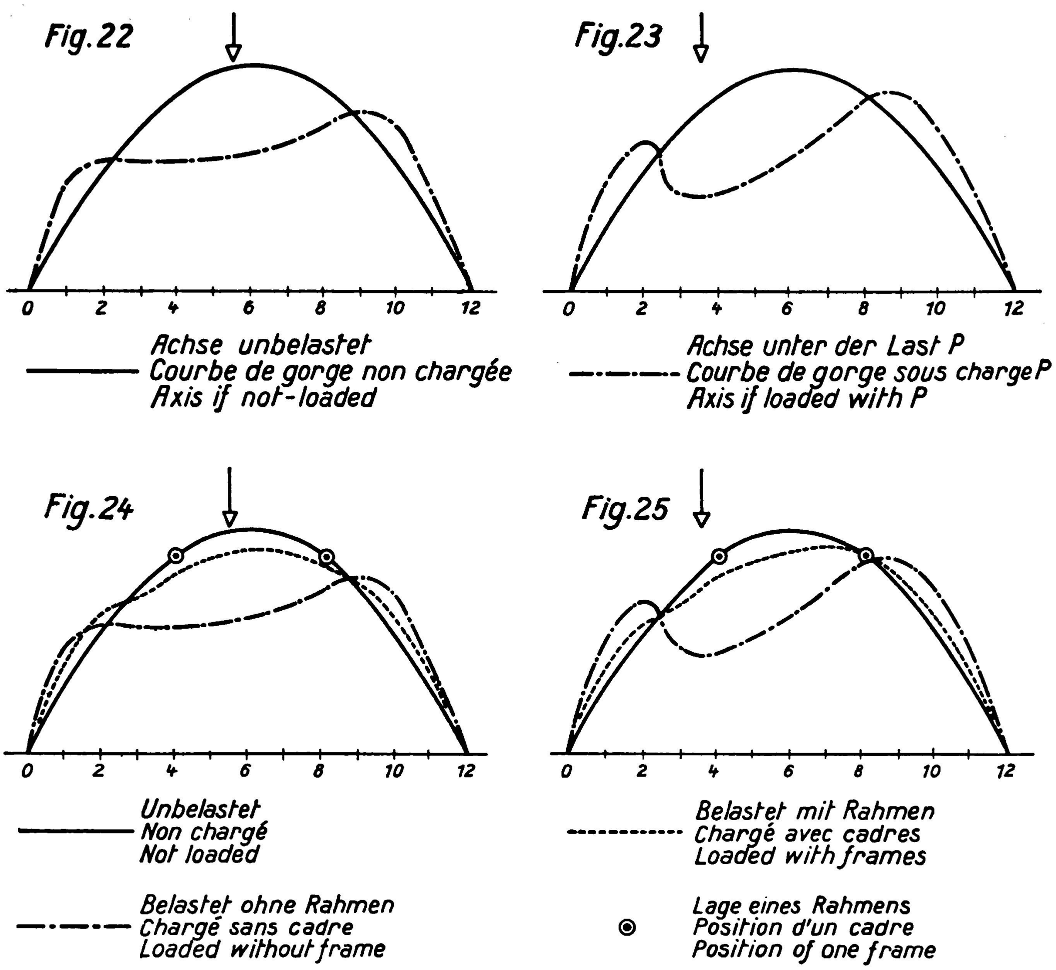

Fig. 22 to 25.

Since a skin (shell) formed in this way does not offer any resistance against bending and possesses only little strength against compression (buckling) it is necessary to provide for special stiffening systems located at suitable positions of the body.

These stiffening systems can be devided up in two groups.

1) Stiffeners resisting to buckling only.
2) Stiffening frames of sufficient inertia to be strong enough to transmit bending effects, apart from acting as stiffeners, and to enable the whole body to act according to the laws of the strength of materials.

We examine now a body which is subjected to the action of any forces.

By neglecting the internal forces we find according to the rules of internal energy a stress distribution which harmonises fairly well with the results of the theory of strength of materials (linear distribution of forces for straight-line bodies, hyperbolic distribution for curved bodies) for a plain section through the body. These forces, except the case of point loading, can only be in equilibrium with the external forces if the sections are assumed to act in bending, this however, is not possible on account of the inferior thickness.

We now consider a panel of the shell situated between two stiffeners. Since the shell on account of its thinness is not apt to stand bending, three equilibrium equations suffice to express the internal forces of the shell (normal and shear forces). These equations have the form of arbitrary functions received by integration from differential equations. Since these differential equations are linear, their solution consists of a superposition of a partial solution (integration) with constants which are not zero, and a general solution whose constants are nil.

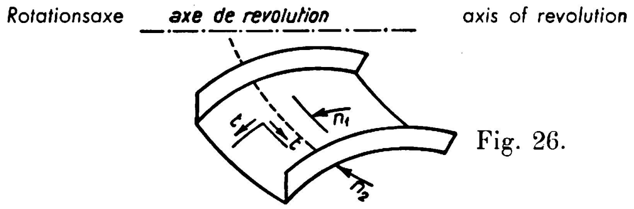

Fig. 26.

A panel as above mentioned can be regarded as being formed of rotary shells, since the movement of the cross section can be regarded as a sequence of rotary movements.

Based on these assumption and the following of denominations:

1) n_1 and n_2 denotes the normal stress in a meridian and latitude circle respectively,
2) r the radius of the parallel circle,
3) t the shear stress,
4) z (r) the position in relation to the axis,
5) α the angle between tangent to the meridian and the axis of rotation,
6) ϑ the angle of rotation,

we receive:

$$\sigma_n = n_1 \cdot \sin \alpha = \frac{n_1}{\sqrt{1 + z'^2}}$$

$$\sigma_\vartheta = \frac{n_2}{\sin \alpha} = n_2 \sqrt{1 + z'^2}$$

$$\tau_{\vartheta r} = t.$$

and the differential equation have the following forms:

$$\frac{\partial}{\partial r}(r z' \sigma_r) + \frac{\partial}{\partial \vartheta}(z' \tau_{\vartheta r}) = Z r = \frac{V r}{\sin \alpha}$$

$$\frac{\partial}{\partial \vartheta}(\sigma_\vartheta r) + \frac{\partial}{\partial r}(r^2 \tau_{\vartheta r}) = T r^2$$

$$\sigma_r \frac{\partial^2 z}{\partial r^2} + \sigma_\vartheta \frac{z'}{r} = \frac{N}{\sin \alpha}.$$

These equations can be simplified and reduced to two in the case of coubly curved shells and particularly in the case of straight line surfaces (Regelflächen).

Z and T are the components parallel to the axis and parallel to the tangents of the latitude circles of the forces acting on a unit surface, projected on a plane at right angles to the axis; V and N are the normal components of the forces acting on the surface.

When considering an element of sufficient smallness, such that the force between the stiffening elements can be regarded as independant from ϑ, we receive the following equation which are free from ϑ:

$$\frac{d}{dr}(r z' \sigma_r) = Z r$$

$$\frac{d}{dr}(r^2 \tau_{\vartheta r}) = T r^2$$

$$\sigma_r \frac{d^2 z}{dr^2} + \sigma_\vartheta \frac{z'}{r} = \frac{N}{\sin \alpha}.$$

These equation which are of the first order can be integrated without difficulties.

For a symmetrical section in respect to a parallel plane, where r_o represents the radius of this plane we can write:

$$1) \qquad t = \tau_{\vartheta r} = \frac{1}{r^2}\int_{ro}^{r} T r^2 dr$$

$$2) \qquad \sigma_r = \frac{\int_{r_o}^{r} Z r\, dr}{r z'}$$

out of which follows:

$$n_1 = \frac{\int_{r_o}^{r} \frac{V r\, dr}{\sin \alpha}}{r \cos \alpha}$$

$$3) \qquad \frac{n_2}{R_2} = N - \frac{n_1}{R_1}$$

herein R_2 and R_1 are the main radii of curvature

$$R_1 = \frac{(1 + z'^2)^{3/2}}{z''} = \frac{1}{z'' \sin 2\alpha}$$

$$R_2 = \frac{r}{\cos \alpha}$$

Hence we receive:

$$n_2 = r \left[\frac{N}{\cos \alpha} - n_1 z'' \operatorname{tg} \alpha \cdot \sin^2 \alpha\right|$$

To solve the problem completely it is necessary to introduce arbitrary terms for pure bending and bending with shear.

a) Pure bending:

we put

$$\sigma_\vartheta = \sum u_n \cos n\,\vartheta$$

if σ_ϑ corresponds to the laws of strength of materials for long bodies for $\vartheta = 0$. If so, we have:

$$\sigma_r = -\frac{z'}{r\,z''}\sum u_n \cos n\,\vartheta$$

By putting

$$\tau_{\vartheta r} = \sum Vn \cdot \frac{\sin n\,\vartheta}{n}$$

we receive

$$Vn = \frac{1}{z'}\frac{\partial}{\partial r}\left(\frac{z'^2 u_n}{z''}\right)$$

hence

$$\frac{\partial}{\partial r}\left[\frac{r^2}{z'}\frac{\partial}{\partial r}\left(\frac{z' u_n}{z''}\right)\right] = n^2 \cdot r u_n$$

and by introducing

$$\sum u_n = (\sigma_\vartheta)_{\vartheta = o} = \sigma_o$$

we receive finally

$$\sum n^2 u_n = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r^2}{z'}\frac{\partial}{\partial r}\left(\frac{z'^2 \sigma_o}{z''}\right)\right] = \lambda_o$$

By considering only the terms of the first order we see that n does not occur in any of the stresses σ_r, σ_ϑ and $\tau_{r\vartheta}$ from which follow that, for a small value of ϑ (small distance between stiffening frames), it is not necessary to determine $n_1 \ldots\ldots n_n$ separately. It could be salved by employing the theory of least work, applied to a shell area bounded by edges, where the effects would be expressed as functions of n:

b) Bending with shear:

we put again

$$\sigma_\vartheta = \sum S_n \frac{\sin n\,\vartheta}{n} + \sum u_n \cos n\,\vartheta$$

We examine the first term so the second has been scrutinesed above

$$\sigma_\vartheta = \sum S_n \frac{\sin n\,\vartheta}{n}$$

$$\sigma_r = -\frac{z'}{r\,z''}\sum S_n \frac{\sin n\,\vartheta}{n}$$

we put

$$\sigma_{r\vartheta} = \sum t_n \cos n\,\vartheta$$

and receive

$$t_n = \frac{1}{r^2}\int S_n\, r\, dr$$

from which follows

$$n^2 \frac{1}{r^2} \int S_n r\, dr = \frac{\partial}{\partial r} \left(\frac{z'^2}{z''} S r \right).$$

The same considerations as made for pure bending can be made.

In the first instance we have

$$\tau_{r\vartheta} = \sum t_n = \frac{1}{r^2} \int (\sigma_\vartheta)_0\, r\, dr$$

In this equation ϑ does not appear. The terms of higher order are determined by the theory of least work.

The problem can be regarded as fully solved since we know

1) the local distribution of stresses,

2) the distribution of the stresses in general, provided the distribution of the stresses n_2 (or of $\sigma \vartheta$) would be known.

The construction of the shells usual to-day is this, that all forces of the first order, the forces n of the second and the stresses in the stiffening frames are of small influence to the internal potential in relation to the total stresses n_2 and t.

From this follows .that the results of elasticity calculation for two-dimensional bodies (beams and arches of inferior thickness) can be applied to such cases. Hence it follows that the forces n_2, with an approximation, which is considerably smaller than the calculated value $\frac{n'_2}{n_2}$ (n'_2 = local forces), can be regarded as such which follow the classic laws of the strength of materials. The values n_2 are used to determine the total stresses n_1 and t, from which result, under application of the above mentioned equations, the forces acting on the stiffeners and the edges.

The problem of stress distribution is completely solved by this. It further follows that the investigation of beams and frames composed of shells can be regarded as being in direct connection with the classic hypothesis, and that the secondary forces are used to calculate the stiffening elements.

This portion of the report refers only to the theory, but the structures referred to in the first section are based on this theory.

We shall show in the publication which will be issued in course of 1937 some numerical examples to illustrate the application of the equation systems. We have mentioned that some of the stiffeners added to the shell serve only to withstand buckling.

Buckling plays an important part in the study of shells. The fact, that the shells are thin and therefore having only a small inertia, indicates the tendency of these shells to buckle and to bulge (warp).

In one case, before executing a shell structure, buckling tests in particular were carried out with full size stiffening frame specimens. These tests were carried out with punching presses in workshops of the marine steel works at St-Chamond in July 1935.

A number of cylindrical plates of different thickness and different radii of curvature were manufactured and equipped with stiffeners composed of bent angle irons.

The Fig. 27 and 28 show one of these specimens before and after the buckling test.

A number of tests were only pressure tests for a portion of the section. The purpose of these tests was to establish a stress distribution n_2, corresponding to the most common case of stress distribution, which has already been described above. The results were satisfactory and could be regarded, apart from the factor of safety, as being within acceptable limits.

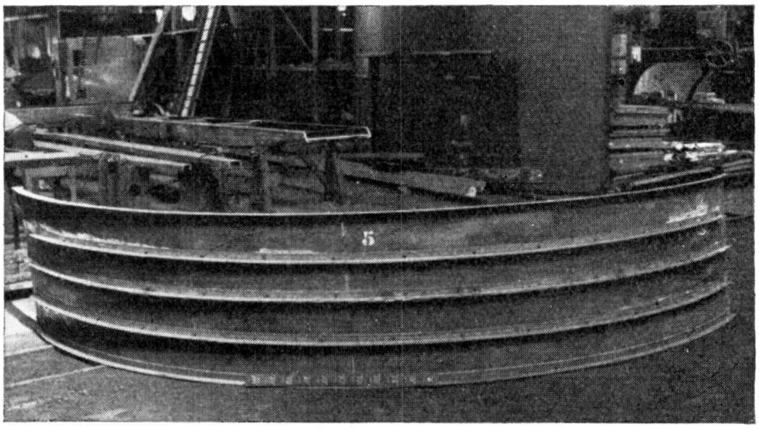

Fig. 27.

Curved plate with stiffeners previous to buckling test, St. Chamond (France) July 1935.

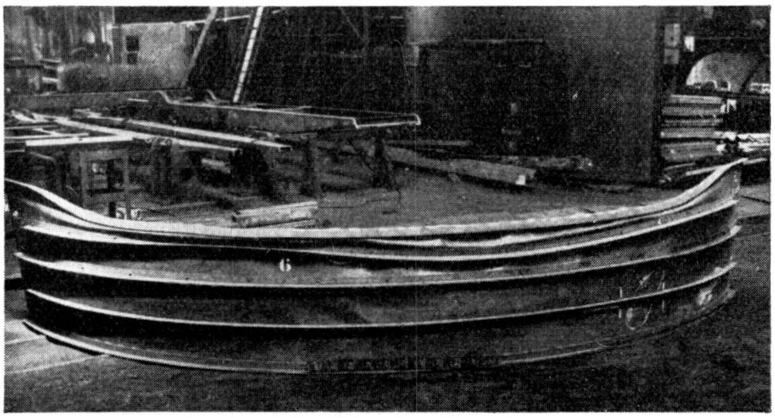

Fig. 28.

Curved plate with stiffeners after buckling test, St. Chamond (France), July 1935.

After these tests were made we started to examine the various phenomena from a theoretical point of view. The results of these investigations are given herewith:

Circular cylindrical pieces were examined. The calculations were also extended to non-circular cylindrical pieces; this made it possible to introduce an alteration of the radius of curvature along the cross section. We found it advisable to approach such cases as do really occur in practice. It was of interest to increase locally the resistance to buckling in function to the stresses n_2 at right angles to a curved section.

We studied the case of a shell of circular cylindrical shape with an opening Φ, a hight l, a radius r and a thickness $t = 2h$.

This shell may be considered as being subjected to compression acting on the circular edges in the direction of the axis.

The problem is to find the critical load, i. e. the smallest value of compression for which elastic equilibrium exists of the deformed shell.

To find this the method of *Ritz-Rayleigh* has to be employed. It consists in assuming a priori a function with free arbitrarily chosen constants for the shape of deformation. Further this function shall express the total internal and external energy of the forces, and for determining the constants this function shall be made to assume extreme values. The conditions for the constant will then allow to determine the pressure required to produce the critical loading.

Mr. *Redshau* in his paper "Elastic instability of a thin curved panel", Aeronautical Research Committee R.J.M. Nr. 1565, London, has already treated this question and uses for the displacements the following terms:

$$u = A \cdot \sin \frac{\pi \vartheta}{q} \sin \frac{kx}{r}$$

$$v = r \cdot \eta = o$$

$$w = C \cdot \sin \frac{\pi \vartheta}{\varphi} \cos \frac{kx}{r}$$

His investigation is based on a loading uniformly distributed along the whole section.

We have tried to express the displacements in a more general way and to consider the pressure by the following very general term

$$p \cdot f\ (\vartheta)$$

Accordingly we receive for the displacements

$$u = u\,(\vartheta, x)$$

$$v = r\eta = o$$

$$w = C \cdot \sin \frac{a \pi \vartheta}{\varphi} \cdot \sin \frac{q \pi x}{l}$$

wherein a and q remain permanently positive. We shall now study the function $u\ (\vartheta,\ x)$.

As regards this function, it is periodical, with a period φ, and for $u\ (\vartheta,\ o) = -\,u\ (\vartheta, l)$ we find that in relation to x, $u\ (\vartheta,\ x)$ this periodical function has a period 2 l.

From this we deduce that u must be of a general form, which can be developed into Fourier series.

$$u = F\,(\vartheta, x) = \frac{1}{2} \sum_{s=1}^{\infty} a^{o}_{2s-1} \cos (2s - 1) \frac{\pi x}{l} + \ldots$$

$$+ \sum_{n=1}^{\infty} \left[\cos n \frac{2\pi\vartheta}{\varphi} \sum_{s=1}^{\infty} a^{n}_{2s-1} \cos (2s - 1) \frac{\pi x}{l} \right.$$

$$\left. + \sin n \frac{2\pi\vartheta}{\varphi} \sum_{s=1}^{\infty} b^{n}_{2s-1} \cos (2s - 1) \frac{\pi x}{l} \right].$$

With this calculation we can determine the terms a^o_{2s-1}, a^n_{2s-1}, b^n_{2s-1}, hence u will be known.

The compression of the shell due to deformation is given by the expression:

$$\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx \quad {}^{2}$$

The work done by the pressure p f (ϑ) can be written:

$$w = 2\,hrp\int_0^{\varphi}\int_0^{l}\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] f(\vartheta)\,dx\,d\vartheta$$

After integration being done we receive:

$$w = 2\,hpr\left\{\int_0^{\varphi} f(\vartheta)\,[u(\vartheta, l) - u(\vartheta, o)]\,d\vartheta + \frac{1}{2}\int_0^{\varphi} f(\vartheta)\,d(\vartheta)\int_0^{l}\left(\frac{\partial w}{\partial x}\right)^2 dx\right\}$$

$$= 2\,hpr\left\{-2\int_0^{\varphi}\left[\frac{1}{2}\sum_1^{\infty} a^o_{2s-1}\,f(\vartheta) + \sum_1^{\infty} f(\vartheta)\cos n\,\frac{2\pi\vartheta}{\varphi}\sum_1^{\infty} a^n_{s-1}\right.\right.$$

$$\left.\left. + \sum_1^{\infty} f(\vartheta)\sin n\,\frac{2\pi\vartheta}{\varphi}\sum_1^{\infty} b^n_{2s-1}\right] d\vartheta + \frac{c^2}{2}\left(\frac{q\cdot\pi}{l}\right)^2\int_0^{\varphi} f(\vartheta)\sin^2\frac{a\pi\vartheta}{\varphi}\int_0^{l}\cos^2\frac{q\pi x}{l}\,dx\right\}$$

The function f (ϑ) is of the period φ if we put:

$$\frac{2\pi\vartheta}{\varphi} = \xi$$

with which we receive

$$f(\vartheta) = f\left(\frac{\varphi}{2\pi}\cdot\xi\right)$$

This latter function has a period 2π, if A_n and B_n represent its Fourier coefficients.

We receive

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{\varphi}{2\pi}\,\xi\right)\cos n\,\xi\,d\xi$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{\varphi}{2\pi}\,\xi\right)\sin n\,\xi\,d\xi$$

hence:

$$A_n = \frac{2}{\varphi}\int_0^{\varphi} f(\vartheta)\cos n\,\frac{2\pi\vartheta}{\varphi}\,d\vartheta$$

$$B_n = \frac{2}{\varphi}\int_0^{\varphi} f(\vartheta)\sin n\,\frac{2\pi\vartheta}{\varphi}\,d\vartheta$$

[2] Applied Elasticity I. Prescott p. 562.

From this we deduce that

$$w = -4\,h\,p\,r\,\frac{\varphi}{2}\left\{\frac{1}{2}A_o \sum_1^{\infty} a^o_{2s-1} + \sum_{n=1}^{\infty} A_n \sum_1^{\infty} a^n_{2s-1} + \sum_{n=1}^{\infty} B_n \sum_1^{\infty} b^n_{2s-1}\right\}$$

$$+\,h\,p\,r\left(\frac{q\,\pi}{l}\right)^2 c^2 \frac{1}{2}\,\frac{\varphi}{4}\,(A_o - A_a)$$

For the energy of the internal forces per unit area we can write:

$$v' = E'h\left[\alpha^2 + \beta^2 + 2\,\sigma\,\alpha\,\beta + \frac{1-\sigma}{2}c^2\right] + \frac{E'J}{2}\left[c_1^2 + c_2^2 + 2\sigma c_1 c_2 + 2(1-\sigma)(T-T_o)^2\right]$$

where $E' = \dfrac{E}{1-\sigma^2}$ $\quad\sigma$ Poisson's coefficient.

$$\alpha = \frac{\partial u}{\partial x} \qquad\qquad I = \frac{2h^3}{3}$$

$$\beta = \frac{w}{r} + \frac{\partial n}{\partial \vartheta}$$

$$c = r\frac{\partial n}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \vartheta}$$

$$c_1 = \frac{\partial^2 w}{\partial x^2}$$

$$c_2 = \frac{1}{r^2}\left(w + \frac{\partial^2 w}{\partial \vartheta^2}\right)$$

$$T = \frac{1}{r}\,\frac{\partial^2 w}{\partial x\,\partial \vartheta} - \frac{\partial n}{\partial x}$$

$$T_o = o$$

The preceding expressions become for the chosen displacements:

$$v' = E'\,h\left\{\left(\frac{\partial F}{\partial x}\right)^2 + \frac{c^2}{r^2}\sin^2\frac{a\,\pi\,\vartheta}{\varphi}\sin^2\frac{\varphi\,\pi\,x}{l} + 2\,\sigma\frac{c}{r}\,\frac{\partial F}{\partial x}\sin\frac{\alpha\,\pi\,\vartheta}{\varphi}\sin\frac{q\,\pi\,x}{l}\right.$$

$$\left. + \frac{1-\sigma}{2\,r^2}\left(\frac{\partial F}{\partial \vartheta}\right)^2\right\}$$

$$+\frac{E'h^3}{3}\left\{c^2\left(\frac{q\pi}{l}\right)^4 \sin^2\frac{a\,\pi\,\vartheta}{\varphi}\sin^2\frac{q\,\pi\,x}{l} + \frac{c^2}{r^4}\left[1-\left(\frac{a\,\pi}{\varphi}\right)^2\right]\sin^2\frac{a\,\pi\,\vartheta}{\varphi}\sin^2\frac{q\,\pi\,x}{l}\right.$$

$$-\,2\sigma\frac{c^2}{r^2}\left(\frac{q\pi}{l}\right)^2\left[1-\left(\frac{a\pi}{\varphi}\right)^2\right]\sin^2\frac{a\,\pi\,\vartheta}{\varphi}\sin^2\frac{q\,\pi\,x}{l}$$

$$\left. +\,2\left(\frac{1-\sigma}{r^2}\right)c^2\left(\frac{a\pi}{\varphi}\right)^2\left(\frac{q\pi}{l}\right)^2\cos^2\frac{a\,\pi\,\vartheta}{\varphi}\cos^2\frac{q\,\pi\,x}{l}\right\}$$

For the total energy for the internal and external forces can be written

$$v = r\int_o^{\varphi}\int_o^{l} v'\,d\,x\,d\,\vartheta - w$$

and according to what preceded it assumes the following form:

$$v = r E' h \left\{ \int_0^{\varphi} \int_0^{1} \left[\left(\frac{\partial F}{\partial x}\right)^2 + 2\sigma \frac{c}{r} \frac{\partial F}{\partial x} \sin \frac{q \pi x}{l} \sin \frac{a \pi \vartheta}{\varphi} + \frac{1-\sigma}{2 r^2} \left(\frac{\partial F}{\partial \vartheta}\right)^2 \right] dx\, d\vartheta \right\}$$

$$+ r E' h c^2 \frac{\varphi}{2} \cdot \frac{l}{2} \left\{ \frac{1}{r^2} + \frac{h^2}{3} \left[\left(\frac{q\pi}{l}\right)^4 + \frac{1}{r^4} \left[1 - \left(\frac{a\pi}{\varphi}\right)^2 \right]^2 \right. \right.$$

$$\left.\left. - \frac{2\sigma}{r^2} \left(\frac{q\pi}{l}\right)^2 \left[1 - \left(\frac{a\pi}{\varphi}\right)^2 \right] + 2 \left(\frac{1-\sigma}{r^2}\right) \left(\frac{a\pi}{\varphi}\right)^2 \left(\frac{q\pi}{l}\right)^2 \right] \right\} - w$$

The dissolution of this expression leads to a new form for the total energy which can be written

$$\frac{4}{\varphi l r E' h} V = \left(\frac{\pi}{l}\right)^2 \cdot \frac{1}{2} \sum_1^{\infty} (2s-1)^2 (a^o_{2s-1})^2 + \left(\frac{\pi}{l}\right)^2 \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} (2s-1)^2 \left[(a^n_{2s-1})^2 + \right.$$

$$\left. + (b^n_{2s-1})^2 \right] + \frac{1-\sigma}{2 r^2} \left(\frac{2\pi}{\varphi}\right)^2 \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} n^2 \left[(a^n_{2s-1})^2 + (b^n_{2s-1})^2 \right] +$$

$$+ \frac{2\sigma}{r} c \begin{cases} \text{a odd} & \begin{cases} -\dfrac{2q}{l} \left[\dfrac{a^o_q}{a} + a \displaystyle\sum_{n=1}^{\infty} \dfrac{2 a^n_q}{4n^2 - a^2} \right] & \text{q odd} \\ o \;\; \ldots\ldots\ldots\ldots & \text{q even} \end{cases} \\ \text{a even} & \begin{cases} -\dfrac{q\pi}{l} 6^{1/2}_q \;\; \ldots\ldots & \text{q odd} \\ o \;\; \ldots\ldots\ldots\ldots & \text{q even} \end{cases} \end{cases}$$

$$+ \frac{8p}{lE'} \left\{ \frac{1}{2} A_o \sum_{n=1}^{\infty} a^o_{2s-1} + \sum_{n=1}^{\infty} A_n \sum_{s=1}^{\infty} a^n_{2s-1} + \sum_{n=1}^{r} B_n \sum_{s=1}^{\infty} b^n_{2s-1} \right\}$$

$$+ c^2 \left(\frac{q\pi}{l}\right)^2 \left\{ \frac{l^2}{r^2 q^2 \pi^2} + \frac{h^2}{3} \left[\frac{q^2\pi^2}{l^2} + \frac{l^2}{q^2 \pi^2 r^4} \left[1 - \left(\frac{a\pi}{\varphi}\right)^2 \right]^2 - \frac{2\sigma}{r^2} \right. \right.$$

$$\left. \left. + \frac{2}{r^2} \left(\frac{a\pi}{\varphi}\right)^2 \right] - \frac{1}{2} (A_o - A_a) \frac{p}{E'} \right\}$$

This is a function for the constants

$$a^o_{2s-1}; \quad a^n_{2s-1}; \quad b^n_{2s-1}; \quad c.$$

In order that the energy becomes a maximum it is only necessary to form the partial derivatives of the first order for these constants and let the terms (equations) so obtained, be zero. The solutions of these equations are the values wanted.

The fact, that we have an unlimited number of constants, in other words equations, is of no hindrance, as it permits only to consider a definite number, for which the boundary conditions have to be determined. Apart from this the variables are separated in such a way that each equation contains one variable only.

Now follow the equations which we received by putting the partial derivatives equal zero

$$a^o_{2s-1}; \quad a^o_q; \quad a^n_{2s-1}; \quad a^n_q; \quad b^n_{2s-1}; \quad b^n_q \text{ and c}$$

$$\left(\frac{\pi}{l}\right)^2 (2s-1)^2 a^o_{2s-1} + \frac{A_o}{2} \frac{8\pi}{lE'} = o$$

q odd, a odd

$$\frac{q^2\pi^2}{l^2}\, a_q^o + \frac{A_o}{2}\frac{8\,p}{l\,E'} - \frac{2\,\sigma}{r}\cdot\frac{2\,q}{a\cdot l}\,c = o$$

q odd, a odd

$$\frac{q^2\pi^2}{l^2}\, a_{2s-1}^o + \frac{A_o}{2}\frac{8\,p}{l\,E'} = o$$

$$\left[\left(\frac{\pi}{l}\right)^2(2\,s-1)^2 + \frac{1-\sigma}{2\,r^2}\left(\frac{2\,\pi}{\varphi}\right)^2 n^2\right] 2\,a_{2s-1}^n + A_n\frac{8\,p}{l\,E'} = o$$

q odd, a even

$$\left[\frac{q^2\pi^2}{l^2} + \frac{1-\sigma}{2\,r^2}\left(\frac{2\,\pi}{\varphi}\right)^2 n^2\right] 2\,a_q^n + A_n\frac{8\,p}{l\,E'} + \frac{2\,\sigma}{r}\frac{4\,a\,q}{l}\frac{1}{4\,n^2-a^2}\,c = 0$$

q odd, a even

$$\left[\left(\frac{\pi}{2}\right)^2 q^2 + \frac{1-\sigma}{2\,r^2}\left(\frac{2\,\pi}{\varphi}\right)^2 n^2\right] 2\,a_q^n + A_n\frac{8\,p}{l\,E'} = 0$$

$$\left[\frac{\pi^2}{l^2}(2\,s-1)^2 + \frac{1-\sigma}{2\,r^2}\left(\frac{2\,\pi}{\varphi}\right)^2 n^2\right] 2\,b_{2s-1}^n + B_n\frac{8\,p}{l\,E'} = 0$$

q odd, a even

$$\left[\frac{\pi^2}{l^2}\, q^2 + \frac{1-\sigma}{2\,r^2}\left(\frac{2\,\pi}{\varphi}\right)^2\frac{a^2}{4}\right] 2\,b_q^{\frac{a}{2}} + B_{\frac{a}{2}}\cdot\frac{8\,p}{l\,E'} - \frac{2\sigma}{r}\,c\cdot\frac{q\,\pi}{l} = 0$$

etc.

In these equations care has to be taken not to loose sight of the function $p \cdot f(\delta)$.

We shall be satisfied to solve this system for the case of the function $f(\delta)$ being even. The general case does not offer more difficulties, only that in this case we receive simpler results.

If q is even we receive at once:

$$a_{2s-1}^o = -\frac{A_o}{2}\frac{8\,p}{l\,E'}\left(\frac{l}{\pi}\right)^2\frac{1}{(2\,s-1)^2}$$

$$2\,a_{2s-1}^n = -A_n\frac{8\,p}{l\,E'}\,\frac{1}{\left(\frac{\pi}{l}\right)^2(2\,s-1)^2 + \frac{1-\sigma}{2\,r^2}\left(\frac{2\,\pi}{\varphi}\right)^2 n^2}$$

$$b_{2s-1}^n = 0.$$

Since C cannot be nil, we receive from the last equation of the system a term for the compressive force, which can be written as under:

$$p = \frac{E}{1-\sigma^2}\cdot\frac{2}{A_o - A_a}\left\{\frac{l^2}{r^2\,q^2\,\pi^2} + \frac{h^2}{3}\left[\frac{q^2\pi^2}{l^2} + \frac{l^2}{\varphi^2\,\pi^2\,r^4}\left(1-\left(\frac{a\,\pi}{\varphi}\right)^2\right)^2 - \frac{2\sigma}{r^2} + \frac{2}{r^2}\left(\frac{a\,\pi}{\varphi}\right)^2\right]\right\}$$

This expression is dependent on q and a. The determination is carried out in such a way that p is a minimum, hence equal to the critical load searched for.

It is therefore only necessary to let the partial derivatives of p for q and a be nil. The equations received this way are equal to those obtained by derivation of the expression for the energy for q and a. The first of these expressions is as follows:

$$\frac{h^2 \pi^2 q}{2 \quad l^2} - \left[\frac{l^2}{r^2 \pi^2} + \frac{h^2}{3} \cdot \frac{l^2}{r^4 \pi^2}\left(1 - \left(\frac{a\pi}{\varphi}\right)^2\right)^2\right] \cdot \frac{1}{q^3} = 0.$$

The second expression cannot be formed as long as the value of A is not known. A is dependent on a and can be determined for every case by the function f (δ).

From the equation above we receive:

$$q_o = \frac{l\sqrt[4]{3}}{r\sqrt{rh}} \cdot \sqrt[4]{1 + \frac{h^2}{3r^2}\left[1 - \left(\frac{a\pi}{\varphi}\right)^2\right]^2}$$

For q an even round figure has to be chosen which is nearest to the value of q_o.

With these values the critical loading can be written:

$$p = \frac{E}{1-\sigma^2} \cdot \frac{2}{A_o - A_a}\left\{\left[1 + \frac{h^2}{3r^2}\left(1 - \frac{a^2\pi^2}{\varphi}\right)^2\right] \frac{l^2}{r^2\pi^2} \cdot \frac{1}{q^2} + \frac{h^3}{3} \frac{\pi^2}{l^2} q^2 + \frac{2h^2}{3r^2}\left(-\sigma + \frac{a^2\pi^2}{\varphi^2}\right)\right\}.$$

For the particular case of uniform pressure we receive: f (δ) = 1, in which case $A_o = 2$ and $A_u = 0$, the quantity u is independent of δ, and we receive:

$$u(\vartheta, o) = -u(\vartheta, l) = -\frac{4pl}{\pi^2 E'} \sum_{s=1}^{\infty} \frac{1}{(2s-1)^2}$$

This signifies that the pressure is transmitted through the means of two stiff panels, and this was applied for most tests carried out at St. Chamond. A few numerical examples will be shown below.

It will be seen that p varies the same way as a and a minimum is reached if a = 1.

In this case the loading assumes the form

$$p = \frac{E}{1-\sigma^2}\left\{\left[1 + \frac{h^2}{3r^2}\left(1 - \frac{\pi^2}{\varphi^2}\right)^2\right] \frac{l^2}{r^2\pi^2} \cdot \frac{1}{q^2} + \frac{h^2\pi^2}{3l^2} q^2 + \frac{2h^2}{3r^2}\left(-\sigma + \frac{\pi^2}{\varphi^2}\right)\right\}$$

If q and a have odd values we find:

$$p = \frac{E}{1-\sigma^2} \cdot \frac{2}{A_o - A_a}\left\{\left[1 + \frac{h^2}{3r^2}\left(1 - \frac{a^2\pi^2}{\varphi^2}\right)^2\right] \frac{l^2}{r^2\pi^2} \cdot \frac{1}{q^2} + \frac{h^2\pi^2}{3l^2} q^2 + \frac{2h^2}{3r^2}\left(-\sigma + \frac{a^2\pi^2}{\varphi^2}\right) - \frac{\sigma^2}{r^2} \cdot \frac{1}{\frac{q^2\pi^2}{l^2} + \frac{1-\sigma}{2r^2}\left(\frac{2\pi}{\varphi}\right)^2 \cdot \frac{a^2}{4}}\right\}$$

At present we have examined together with Mr. *Vasilesco* the same problems relating to sections with variable curvature. The calculations for these problems will be a separate study; what we have shown in the present paper is an abstract from "Comptes Rendues de l'Académie des Sciences de Paris 201", 1935, p. 537 and 201, 1935 p. 642.

We shall now examine shells made from rolled steel plates with different data, as used for our experimental trials. The tests were carried out under factory conditions. The results obtained are somewhat smaller than the theoretical values based on the formula given in this article.

If we consider the small deviations and discrepancies inherent to the shell before testing, and the imperfect mode of testing carried out with big presses, we can assume that there is good agreement between tests and theoretical calculation.

For a cylindrical circular shell with the following properties we receive:

$$2h = 0.003\ \text{m}; \qquad r = 2.1; \qquad \varphi = 160^0 = 2.795_G; \qquad l = 0.8$$

we find:

$$\sin\frac{\varphi_1}{4} = \sin 40^0 = 0{,}643$$

$$\sin\frac{\varphi}{4} = \frac{1}{\sqrt{2}} \cdot 0{,}643 = 0{,}455.$$

These constants indicate the zones of stress distribution of variable pressure along the section for the case of ordinary bending.

$$\frac{A_0 - A_1}{2} = 1 - 2{,}415 + 2{,}265$$
$$= 0{,}85$$
$$\frac{2}{A_0 - A_1} = 1{,}176.$$

For uniform pressure we obtain the following expression

$$q_0 = \frac{l\sqrt[4]{3}}{\pi\sqrt{rh}} = \frac{0{,}8\sqrt[4]{3}}{\pi\sqrt{2{,}1 \cdot \dfrac{3}{2 \cdot 10^3}}}$$
$$q_0 = 5{,}8$$

$$p = 2{,}2 \cdot 10^{10}\left\{\left[1 + \frac{3}{4}\cdot\frac{1}{10^6}\cdot\frac{1}{2{,}1^2}\left(1 - \left(\frac{\pi}{2{,}795}\right)^2\right)^2\right]\frac{8^2}{10^2 - 2{,}1^2\pi^2}\cdot\frac{1}{36} + \frac{3}{4\cdot 10^6}\cdot\frac{\pi^2\cdot 10^2}{8^2}\cdot 36 + \frac{3}{4\cdot 10^6} + \frac{2}{2{,}1^2}\left(-0{,}3 + \frac{\pi^2}{2{,}795^2}\right)\right\}$$

$$p = 2{,}2 \cdot 10^{10} \cdot 0{,}000825$$
$$= 18{,}5\ \text{kg/mm}^2.$$

The tests showed average values of 15 kg/mm².

It is of interest to know, that for the case of ordinary bending the maximum value of the vertical stress would be:

$$p_f = p \cdot \frac{2}{A_0 - A_1}$$
$$= 18{,}15 \cdot 1{,}176$$
$$= 21{,}35\ \text{kg/mm}^2.$$

For bending failure actually occurs on account of buckling due to compressive stresses which are slightly higher than those produced by uniformly distributed pressure.

Conclusions.

The present report is an addition to the application of plates in structural engineering. The import of this study lies in the application of steel plating which leads to more economy and new structural forms.

The material "plate" is very much different from the material "rolled sections".

The rolled sections, irrespective if U, I or angle section, represent only the skeleton in structures, prepared to carry the paneling.

The danger is, as is often seen, that a whole field of structural engineering is conquered by steel skeleton construction, this way not allow a development in other directions.

The engineering technique often fails to recognise the aiming at new proportion, new effects and new contrasts, which would help in improving the aesthetic side of structural engineering. The composition of angles etc. based merely on the strength of materials can hardly establish a perfect spatial effect. Our efforts were this to give the plate, a comparatively new structural element, the possibility of application in structural engineering.

Summary.

Tests carried out by the author throw hight on the behaviour of steel shells under different kind of loading.

Based on the test a mathematical investigation is carried through, which shows that, apart from small deviations, there is good agreement between theory and test.

By example of structures already executed the author shows the great possibilities offered by steel shell structures. The structures are both economical and satisfactory from the aesthetic point of view.