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Contribution to Discussion on Theory on Plasticity.

Diskussion über die Plastizität.

Discussion relative à la plasticité.

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I. *On the general theory of plasticity, the significance of yield lines, and the boundary of the elastic and plastic regions.*

1) Definitions of plasticity in general.

To avoid any risk of misinterpretation it may be well to recall that plastic strain is said to exist in parts of a body, or the material in question is said to be plastic, when the strain which there exists is not entirely of an elastic nature, and at the same time the cohesion of the material is not entirely destroyed, even though some change in the structural lattice may have taken place.

This definition is a general one, and the expressions "plastic strain" and "permanent deformation without destruction of cohesion" are, therefore, synonymous, the latter being in contradistinction to cases where permanent deformation is attended by cracking and an implication that the cohesion has been partially destroyed.

2) Definition of the phenomenon of plastic flow and of creep lines in mild steel.

One reason for the special interest which attaches to the study of plasticity is that it covers the existence of a very important property in mild steel: this metal, when tested in simple tension or compression, exhibits a very special kind of plastic creep. The phenomenon in question is such that when a particular intensity of stress is reached the longitudinal extension suddenly increases in an unmistakable manner. In an idealised form the phenomenon is represented by the horizontal interlude (in French, *palier,* "stair landing") in the stress-strain curve for tension or compression. It does not extend to a very large range of deformation, but its technical consequence is to bring about an appreciable amount of plastic strain, which follows upon a phase in which the elastic strains are very small.

This phenomenon and its results have, indeed, been the starting point of modern researches into plasticity, from which it is sought to profit by economising in the design of mild steel structures, the plastic effect serving as a valuable buffer against local increases of stress. It is also partly this phenomenon, idealised by the break in the stress-strain curve, which has given rise to the notion of perfect plasticity — a condition under which the deformation is assumed to increase while the stress remains constant. This conception implies the existence of single, duo-axial or tri-axial state of stress, and of the condition of perfect plasticity. Hence the justification, as well as the practical need, to develop a suitable hypothesis to account for this condition.

3) Significance of the creep lines which appear on the surface of mild steel pieces when plastically strained.

The author is inclined to accept the opinion of Messrs. Ititaro Takaba and Katumi Okuda as quoted in Paper N° 1 of this group: "The appearance of creep lines, and the sudden break in the stress-strain curve, are the result of one and the same phenomenon, that is to say, of the deformation in groups of a large number of crystalline grains".

There would appear, then, to exist a true discontinuity of the state of strain, which occurs in zones, and which appears to affect a whole region of the material instead of being localised. It appears plausible to suppose that this abrupt occurrence is due to a condition of molecular instability, comparable with the phenomenon of delayed deformation.

This would imply that at the moment when the change occurs there is also an alteration in the lattice structure, an alteration which is manifested in an appreciable amount of irreversible slip — in other words of plastic slip — which is attended by an increase in hardness through blocking of the slipped surfaces (see *Moser*).

It would appear manifest from the foregoing that the true boundary line between that portion of a body which has remained in the elastic condition and that portion which has changed to the plastic condition need not necessarily coincide with the lines of creep, for the latter appear to be related to the phenomenon of delayed action and to affect the whole of a zone. Where the portion of the body in question is subjected to a condition of simple stress the creep lines may be very wide apart, but where this condition does not obtain they are, on the contrary, often very close together.

4) Some characteristics of the creep lines.

Dr. *Freudenthal* states that the most important property of the creep lines appearing on the surface is that they coincide with the direction of maximum shear. The present writer would observe, however, that this is incorrect except in so far as the creep lines constitute a network made up of two sets of lines at right angles, and that other cases occur where these lines are merely one set of "slip crazings". Such cases are common enough but appear to have been overlooked, even though they are clearly indicated in the early descriptive paper by *Hartmann,* as well as in those by *Frémont,* and are easy to reproduce. In such cases the craze lines are evidently not associated with the direction of maximum shear, but on the contrary with that of one of the two principal stresses (isostatic lines). It may be observed, moreover, that such a creep line may disappear in the middle of its course, or if the stress increases it may spread while at the same time others are originated. Clearly, then, a creep line is not necessarily something which is originated once and for all, but it frequently happens that such a line may develop by successive steps as the load is increased.

All the evidence tends, then, to suggest that creep lines do not as a rule constitute boundary lines between the elastic and plastic regions. This is a conclusion which seems to follow from the fact that a line which does not make its appearance at one definite juncture, but which propagates itself as the stress increases, cannot be a boundary line: for if it were the latter it would, presumably, be a closed curve conforming in part to the shape of the material.

The conclusion reached by Dr. *Freudenthal* would seem, therefore, to be untenable: but to say this is very far from asserting that creep lines (particularly) where they very well defined) are devoid of mathematical interest.

5. The plastic condition.

In the case of metals which are capable of plastic deformation with or without a definite creep limit, the main hypotheses which have been put forward to account for the conditions that obtain at the boundary separating the regions of plastic and elastic strain are as follows[1]:

Hypothesis of Saint-Venant, Maurice Lévy, and Guest:

$$\tau_{max} \quad \text{or} \quad \frac{\sigma_I - \sigma_{III}}{2} = k = \frac{R_e}{2}$$

(R_e being the limit of elasticity for pure tension).

Hypothesis of Beltrami and Haigh: Here the criterion is the amount of specific energy involved in the elastic strain, and the condition to be satisfied may be written as follows:

$$(\sigma^2_I + \sigma^2_{II} + \sigma^2_{III}) - \frac{2}{m}(\sigma_{II} \cdot \sigma_{III} + \sigma_{III} \cdot \sigma_I + \sigma_I \cdot \sigma_{II}) = R_e^2$$

This may be represented in space as an ellipsoid, or in the case of duo-axial stress as an ellipse.

Hypothesis of von Mises and Hencky[2]: Here the criterion adopted is the value of the specific energy of change of shape by slip, and may be expressed as follows:

$$(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 = 2\,R_e^2 = 8\,k^2$$

or, in terms of the maximum tangential strains:

$$\tau^2_{I.\,II} + \tau^2_{II.\,III} + \tau^2_{III.\,I} = \frac{1}{2} R_e^2 = 2\,k^2$$

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[1] See *L. Baes:* Résistance des matériaux et éléments de la théorie de l'élasticité et de la plasticité des corps solides. Vol. 1, Chapter XI: «Le problème des critères de la résistance des matériaux». Brussels 1930—34.

[2] The French translation of *Dr. Freudenthal's* paper refers to: «3° Hypothèse de travail constant de déformation suivant la relation . . .». This is a dangerous way of putting the matter, for it should be made clear that only part of the strain energy is involved — that part implied by the expression "energy of change of shape due to slip". The qualification is all the more necessary because the expression is not well known in French.

This is represented in space by a cylinder of revolution, or in the case of duo-axial stress by an ellipse.

Hypothesis of von Mises and Hencky as modified by Huber: Here the criterion is the value of the specific energy of change of shape due to slip in so far as the cubical expansion, or the average stress

$$\frac{\sigma_{I} + \sigma_{II} + \sigma_{III}}{3}$$

are negative. If, on the contrary these quantities are positive, the criterion of Beltrami should be used instead. (It may be as well to draw attention here to the criterion of *Huber,* which is not identical with that of *von Mises* and *Hencky,* being a great deal more than the latter. This is often overlooked.)

Beltrami's criterion may be represented graphically as in Fig. 1 or in the case duo-axial stress it corresponds to a figure made up of two ellipses and differs little from that of *von Mises* and *Hencky.*

It is to be observed that the experiments now on record, notably those carried out by *Roš* and *Eichinger,* have shown that *Huber's* hypothesis is very satisfactory for mild steel or similar materials.

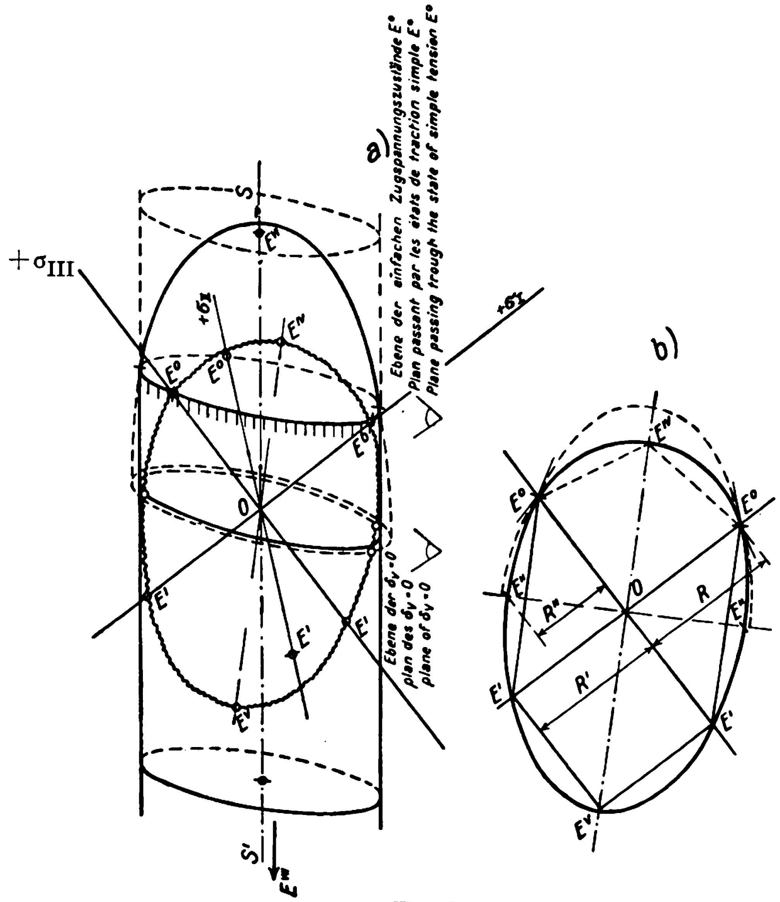

Fig. 1.
Huber's theory of limiting elastic equilibrium. (Graphical form).
a) Tri-axial stress conditions.
b) Duo-axial stress conditions.

It will now be expedient to consider two special cases which are of frequent occurrence:

Special case of deformation in a single plane, with perfect plasticity.

Here the plastic strain occurs in parallel planes, and if it be assumed that these are the planes of the principal stresses σ_{I} and σ_{III}, having perpendicular to them the plane of σ_{II}, then for perfect plasticity we shall have at all points

$$\sigma_{II} = \frac{\sigma_{I} + \sigma_{III}}{2}$$

whence

$$\sigma_{I} < \sigma_{II} < \sigma_{III}.$$

The conditions of plasticity between stresses, according to *St. Venant* and *Maurice Levy,* will then be expressible as follows:

$$\sigma_{I} - \sigma_{III} = 2\,k = R_e$$

The condition of plasticity according to *von Mises* and *Hencky* will be

$$(\sigma_{I} - \sigma_{III}) = \frac{4}{\sqrt{3}}\,k = \frac{2}{\sqrt{3}}\,R_e.$$

It will be noticed that these two conditions agree in a coefficient $\frac{2}{\sqrt{3}}$ and correspond to a particular value of τ_{max}. It follows from the fact that σ_{II} is intermediate between σ_{I} and σ_{III} that the facets at which plastic slip occurs will be those perpendicular to the plane I, III.

The slip surfaces are cylinders of which the axes are normal to this plane; the slip occurs parallel to this plane and is marked in the latter by two conjugate families of slip lines forming an orthogonal network, bisecting that of the isostatic network. Along these lines, as and when they occur, the tangential stress reaches its critical value.

Particular case of stress in one plane, or of duo-axial stress with perfect plasticity.

This case is of very frequent occurrence on the surface of pieces. One of the principal stresses is zero, say $\sigma_{II} = 0$, and if σ_{I} and σ_{III} are of contrary sign the condition of plasticity according to *St. Venant* and *Maurice Levy* is

$$\tau_{max} = \frac{\sigma_{I} - \sigma_{III}}{2} = \pm\,k = \pm\frac{R_e}{2}.$$

If, however σ_{I} and σ_{III} are of the same sign, the condition becomes σ_{I} or $\sigma_{III} = R_e$.

The condition of plasticity according to *von Mises* and *Hencky* is then

$$\sigma_{I}^2 - \sigma_{I} \cdot \sigma_{III} + \sigma_{III}^2 = 4\,k^2 = R_e^2$$

and in Cartesian co-ordinates σ_{I}, σ_{III} is represented by an ellipse.

It will be seen that where the principal stresses σ_{I} and σ_{II} are of different signs there is scarcely any numerical difference between the conditions of *St. Venant* and *von Mises.* The two conditions are not, however, proportional, as was the case for plane deformation. In the case where the two principal stresses σ_{I} and σ_{II} are of contrary sign, as in Fig. 2, there is formed a network of slip lines. According to the hypothesis of *St. Venant* this network bisects the network of isostatic lines, and at any point in such a line the corresponding τ_{max} reaches its critical value at the moment when slip occurs.

According to the hypothesis of von Mises, also, there is a network formed by two families of lines, but along the direction of slips in this network τ_{max} reaches no definite value, for it is the critical value that is obtained, and its amount is not based on a definite value of τ_{max}.

Where the two principal stresses σ_I and σ_{III} are of the same sign, as in Fig. 4, there no longer occurs any formation of a network of two families of lines at right angles, the slip faces not being normal to the free surface according to either of the hypothesis, and there is merely formed a single family of slip crazings common to the two groups of slip faces (Fig. 5 and 6).

According to the hypothesis of *St. Venant*, every element in these crazings coincides, at the moment of its formation, with the element of the isostatic line

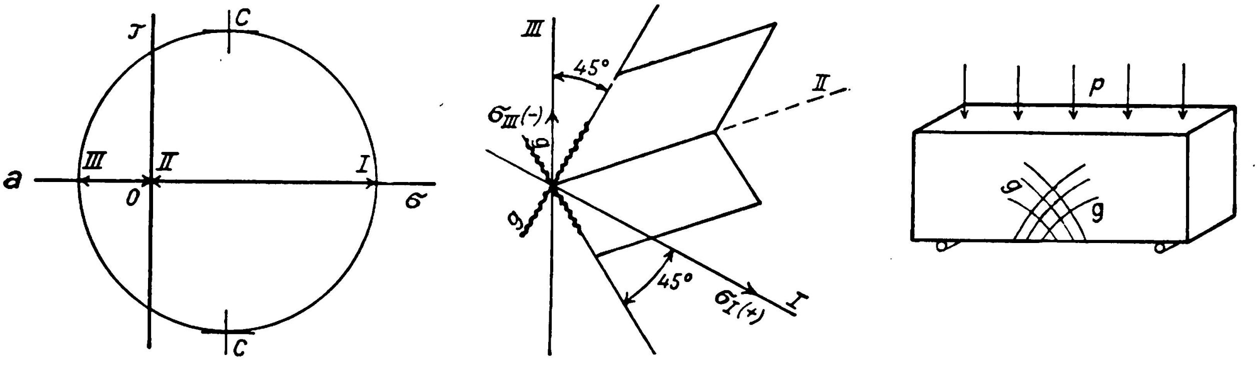

Fig. 2.

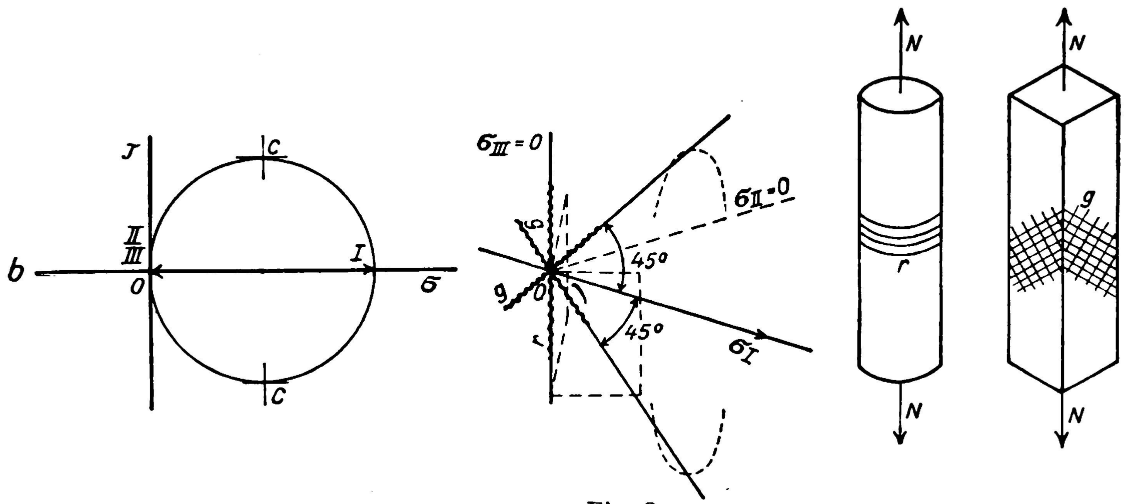

Fig. 3.

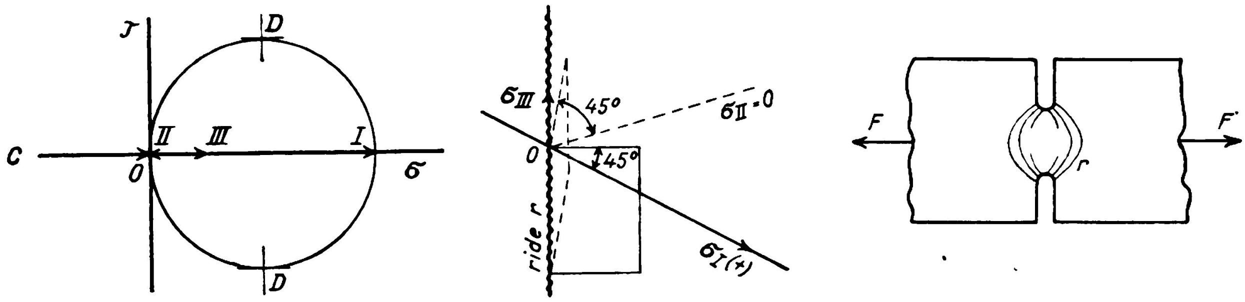

Fig. 4.

Figs. 2—4. Appearance of slip networks g or lines r on the outer faces of parts subjected to uni-planar stress.

Case a: $\sigma_{II} = 0$; σ_I and σ_{III} of opposite sign,

Case b: $\sigma_{II} = \sigma_{III} = 0$,

Case c: $\sigma_{II} = 0$; σ_I and σ_{III} of same sign.

which corresponds to the smallest absolute value of principal stress at that moment and in that place, and the maximum principal stress attains double the value of the critical tangential stress.

In the old papers by *Hartmann*, the distinction between the crazings and the slip surfaces is clearly apparent — see Fig. 7 — though *Hartmann* has not put forward an explanation of this difference.

According to the hypothesis of *von Mises* the stress attained at a given point in a craze line at the moment of its formation corresponds to the critical condition, but this is not altogether simple to understand; indeed the circumstances in which the crazing occurs would appear to have been overlooked, though the phenomenon is one of fairly frequent occurrence and the problem to which it gives rise is then altogether different from that of the formation of the network. It is a condition which often arises in flat pieces with lateral notches. Fig. 6.

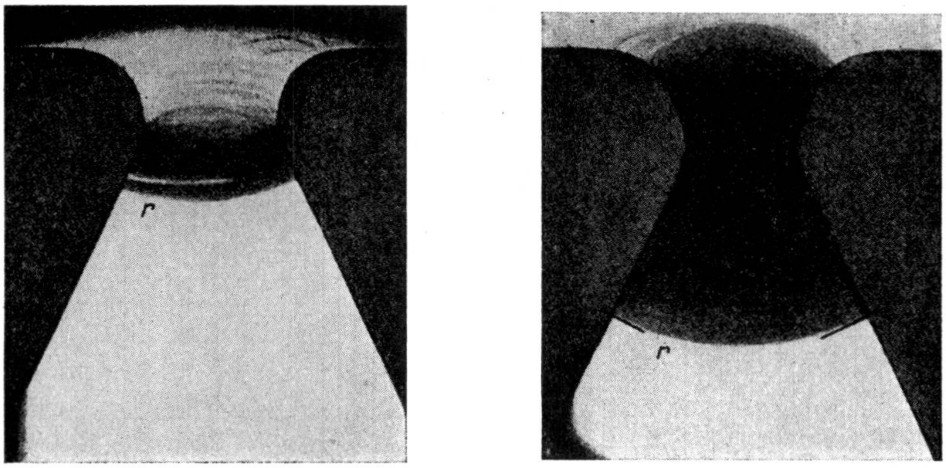

Fig. 5.

Sample of plate subjected to tension, showing the gradual development of slip lines r. (From "Mesure de la limite élastique des métaux", Ch. Frémont, 1903.)

Where only one of the principal stresses differs from zero, there is a theoretical possibility of the formation of the network of slip lines g or of the family of craze lines r (Fig. 3). In practice it is probably the network of slip lines g which will be formed, and so far as this is concerned the two stress hypotheses of *St. Venant* and *von Mises* are evidently identical.

Where a solid mass is in question, however, the hypothesis of *St. Venant* leads to the assumption of slip surfaces for all those elements which at the moment of slip have τ_{max} equal to the critical value; on the other hand by the hypothesis of *von Mises* no simple connection exists between the critical conditions and the maximum tangential stress.

6) Boundary between the region which has remained elastic and the region which has become plastic.

Dr. Freudenthal writes as follows:

"The acceptance of the slip lines as the boundaries of the plastic regions, and the development of solutions depending on properties of the slip lines themselves, will always lead to results which are not in agreement with reality."

The second part of this may be accepted with reserve, but the truth of the first part is evident in a general way. Generally speaking, inaccuracy must result from confusing the boundary of the plastic region with a slip surface, for it is evident that along the slip surfaces *within* the plastic zone the condition of plasticity is satisfied as it is on the actual boundary; actually at the boundary, however, it has to be reconciled with an elastic condition.

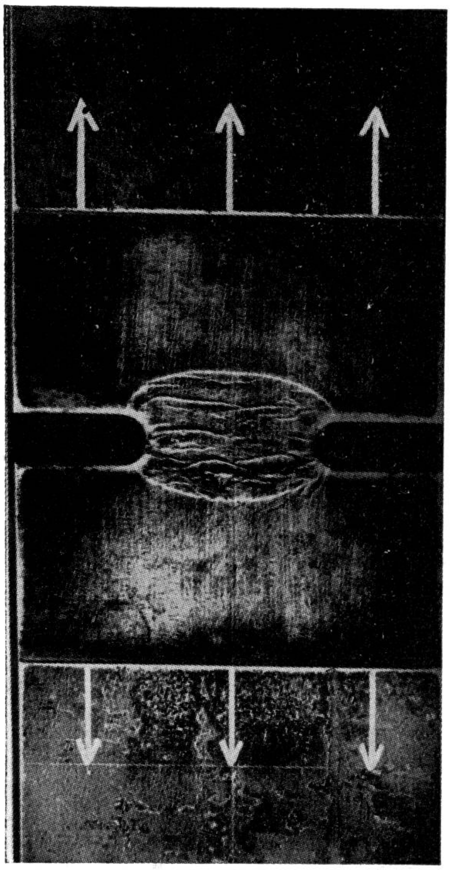

Fig. 6.
Slip bands in a notched piece of mild steel.

Generally speaking the boundary surface is not formed by any one slip surface, but by points on a number of slip surfaces. It is, therefore, inaccurate to assert generally "as is clearly indicated by all the observations, the shapes of these curves has nothing in common with the creep lines themselves, *but the curves correspond equally well with conditions of plastic or of elastic strain*".

The boundary surface must evidently be defined as a surface in the elastic state, wherein the function taken as a criterion is constant. In the case of a plane surface which is stressed in that plane, the bounding line corresponds with an isochromatic line in photo-elastic experiments, and this is true whether the criterion of *St. Venant* or that of *Von Mises* be applied. Where the same piece is stressed in another plane the boundary surface is not isochromatic except in accordance with the hypothesis of *St. Venant*, when the tensions σ_I and σ_{III} are of contrary sign, or when one of them is not equal to zero.

If the same piece, is subjected to two plane stresses of the same sign, then, in accordance with the hypothesis of *St. Venant*, the boundary line is a curve of equal value for the maximum principal stress, and is not an isochromatic curve as obtained in photo-elastic experiments.

According to the hypothesis of *Von Mises* in this condition of stress the boundary line is not an isochromatic curve. *The importance of distinguishing between the case of duo-axial stress and plane deformation will thus be apparent*, and this is the upshot of the author's present remarks.

In order to show that the boundary line is not, generally speaking, a slip line, it is merely necessary to mention two simple cases which are well known. The first of these is a thick cylindrical envelope subjected to a large difference of pressure, where on account of the axial symmetry the boundary surface between the plastic and the elastic regions is a cylinder concentric with the tube itself, whereas in each cross section the trace of the surfaces of slips are logarithmic spirals, there is nothing in common between these forms.

The second example is that of a plane slab which is stressed on its edge by the application of a load which is nearly concentrated. Here the lines of slip on the side of the slab are logarithmic spirals. The boundary line is an isochromatic line, that is to say a circumference which has its centre on the line of load and which is tangential to the boundary line of the piece. Yet another typical case is that of a circular disc loaded by two diametrically opposed loads.

It is manifest, then, that the boundary line between the plastic and elastic regions is not generally coincident with a slip line. The present writer is of an opinion that it may be useful to bring this point out more clearly and more simply than has been done by the author of the paper in question. But that is well established.

There are numerous and important questions which remain to be elucidated in the field of plasticity, for the present existing theories are no more than a simple outline needing to be filled in. It is apparent also from the point of view specially treated in Paper N° I 2 by *Dr. J. Fritsche,* that the condition of plastic creep is a function not of the local state of stress, but of the *stress prevailing over a whole region.* This new kind of creep is a very interesting one, and involves new features which are supported by undeniable experimental facts. Indeed the present writer has had an intuition of these ever since he undertook the tests to destruction on joists encased in concrete.

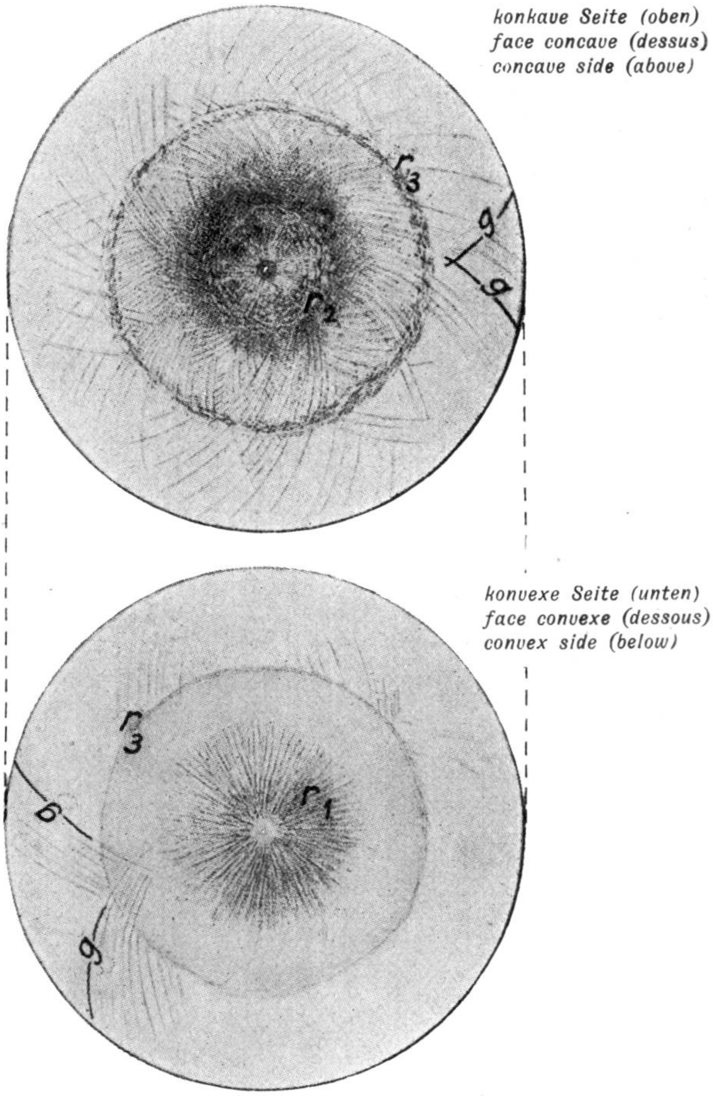

Fig. 7.
Steel plate deformed by an isolated point load.
Slip lines g and bands r.
(From „Distribution des déformations dans les métaux soumis à des efforts" L. Hartmann, 1896.)

II. Application to the design of steel structures.

The writer wishes to record his complete agreement with *Dr. F. Bleich,* who draws attention to the necessity for the exercise of great prudence in applying these principles to structures at the present time, and has rightly urged that the new method of calculation must not be applied to systems in which account has to be taken of the fatigue resistance of the material. For the present it should be applied only to simple systems which are hyperstatic to no more than a very limited degree, consisting of members subject to bending, where the part in compression cannot become uncased; it is applicable only to those girder constructions or building frameworks which are not subject to repeated loading or to vibration by machinery.

It would further be a wise precaution, when designing structures on the hypothesis of plastic equilibrium, to work with stresses such that the creep stress will not actually be reached, so that the adjustability due to plasticity will be held in reserve.