AII 1

An approximate method for treatment of some plate bending problems

Méthode approchée pour l'etude de quelques problèmes concernant la flexion des dalles

Eine Näherungsmethode zur Behandlung einiger Probleme der Plattenbiegung

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Consider a rectangular plate, fig. 1, simply supported along the edges $x=0$ and $x=a$, whereas the other edges are either simply supported or rigidly clamped. Suppose, furthermore, that this plate is submitted to a load which can be expanded into a Fourier series in x. Then any quantities relating to the plate can be calculated using the well-known method involving simple corrections of the corresponding quantities for a simply supported beam. When, however, the boundary conditions at $x=0$ and $x=a$ are changed, the calculation is considerably complicated by time-wasting numerical computations, which can seldom be managed when a design problem calls for a rapid solution. Some cases have been treated in publications. Reference is made to S. Timoshenko,* and D. Young.† Special mention is also made to S. Levy,‡ the immediate source of inspiration for the present paper.

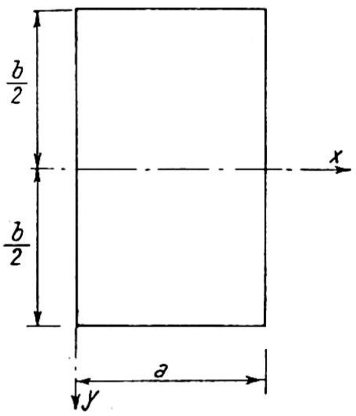

Fig. 1.

In the following paragraphs a very simple but somewhat rough-and-ready method, which is applicable under any arbitrary boundary conditions, is given. The easiest way to demonstrate this method is to adduce two examples which permit comparison with previously known "exact" solutions.

* "Bending of Rectangular Plates with Clamped Edges," *Proc. Fifth Int. Congr. Appl. Mech.*, 1939.

† "Deflection and Moments for Rectangular Plates with Hydrostatic Loading," *J. Appl. Mech.*, 1943.

‡ "Square Plate with Clamped Edges under Normal Pressure producing Large Deflections," *N.A.C.A. Report*, No. 740.

EXAMPLE NO. 1

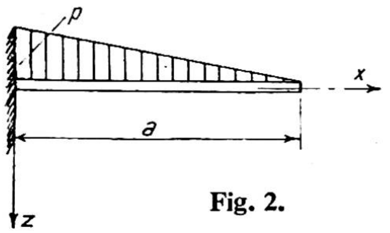

Fig. 2.

A triangular load, two opposite edges clamped, the third edge clamped, and the fourth edge free.

Consider the corresponding beam, fig. 2.

$$\frac{\partial^4 w}{\partial x^4}=\frac{p}{D}\left(1-\frac{x}{a}\right)=\frac{p}{D}\left[A\left(1-\frac{x}{A\,.\,a}\right)-(A-1)\right]$$

$$=\frac{2Ap}{\pi D}\sum_{n=1.2.3\ldots}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{A\,.\,a}-\frac{4(A-1)p}{\pi D}\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{\mathrm{B}\,.\,a}$$

$$\left.\begin{array}{l}x=0\\ \dfrac{\partial^3 w}{\partial x^3}=-\dfrac{pa}{2D}\end{array}\right\}\text{ and }\left.\begin{array}{l}x=a\\ \dfrac{\partial^3 w}{\partial x^3}=0\end{array}\right\}\text{ yields}$$

$$\left.\begin{array}{l}-2A^2+3(A-1)B=-3\\ -\dfrac{2A^2pa}{\pi^2 D}\displaystyle\sum_{n=1.2.3\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{A}+\frac{4(A-1)Bpa}{\pi^2 D}\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{B}=0\end{array}\right\}$$

and hence $A=\frac{3}{2}$; $B=1$.

The summation of $\displaystyle\sum_{n=1.2.3\ldots}^{\infty}\frac{1}{n^2}\frac{2n\pi x}{3a}$ is carried out by substituting $S_0+S_1x+S_2x^2$ and by inserting $x=0$, $x=3a/2$, and $x=3a$. For these values, the sum is known.

$$\frac{\partial^2 w}{\partial x^2}=-\frac{27pa^2}{4\pi^3 D}\sum_{n=1.2.3\ldots}^{\infty}\frac{1}{n^3}\sin\frac{2n\pi x}{3a}+\frac{2pa}{\pi^3 D}\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{a}+C$$

$$\left.\begin{array}{l}x=0\\ \dfrac{\partial^2 w}{\partial x^2}=\dfrac{pa^2}{6D}\end{array}\right\}\text{ and }\left.\begin{array}{l}x=a\\ \dfrac{\partial^2 w}{\partial x^2}=0\end{array}\right\}\text{ yields }C=\frac{pa^2}{6D}=\frac{2pa^2}{3\pi D}\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{F\,.\,a}$$

$$\left.\begin{array}{l}x=0\\ \dfrac{\partial w}{\partial x}=0\end{array}\right\}\text{ gives }F=\frac{11}{10}\text{ whence, finally,}$$

$$w=\frac{243pa^4}{16\pi^5 D}\sum_{n=1.2.3\ldots}^{\infty}\frac{1}{n^5}\sin\frac{2n\pi x}{3a}-\frac{2pa^4}{\pi^5 D}\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{a}$$

$$-\frac{121pa^4}{150\pi^3 D}\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n^3}\sin\frac{10n\pi x}{11a}\quad .\;.\;.\quad(1)$$

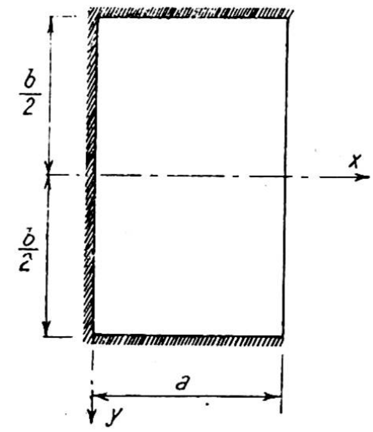

Fig. 3.

This is the equation of the elastic curve, which is generally assumed to be known. The third term represents the influence of the restraining moment.

For the plate shown in fig. 3 with the loading as indicated in fig. 2, the elastic surface is chosen:

$$w=\frac{243pa^4}{16\pi^5 D}\sum_{n=1.2.3\ldots}^{\infty}\frac{1}{n^5}(1+Y_{1_n})\sin\frac{2n\pi x}{3a}-\frac{2pa^4}{\pi^5 D}\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n^5}(1+Y_{2_n})\sin\frac{n\pi x}{a}$$

$$+M\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n^3}(1+Y_{3_n})\sin\frac{10n\pi x}{11a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where Y_n are functions of y.

Y_{1_n} is determined by:

$$\Delta\Delta Y_{1_n}\sin\frac{2n\pi x}{3a}=0 \text{ and}$$

$$\left.\begin{array}{l} y=\pm\frac{b}{2} \\ 1+Y_{1_n}=0, \text{ and } \frac{\partial Y_{1_n}}{\partial y}=0 \end{array}\right\}$$

From

$$\Delta\Delta Y_n \sin G\frac{n\pi x}{a}=0 \text{ when } \frac{n\pi b}{2a}=\alpha_n$$

is thus generally obtained:

$$1+Y_n=1-\frac{G\alpha_n\cosh G\alpha_n+\sinh G\alpha_n}{\sinh G\alpha_n\,.\cosh G\alpha_n+G\alpha_n}\,.\cosh G\frac{n\pi y}{a}$$

$$+\frac{\sinh G\alpha_n}{\sinh G\alpha_n\,.\cosh G\alpha_n+G\alpha_n}\,.\,G\frac{n\pi y}{a}\,.\sinh G\frac{n\pi y}{a} \quad . \quad . \quad . \quad (3)$$

On the other hand, if the boundary conditions are:

$$\left.\begin{array}{l} y=\pm\frac{b}{2} \\ 1+Y_n=0 \text{ and } \frac{\partial^2 Y_n}{\partial y^2}=0 \end{array}\right\}$$

then

$$1+Y_n=1-\frac{G\alpha_n\,.\sinh G\alpha_n+2\cosh G\alpha_n}{2\cosh^2 G\alpha_n}\,.\cosh G\frac{n\pi y}{a}$$

$$+\frac{1}{2\cosh G\alpha_n}G\frac{n\pi y}{a}\sinh G\frac{n\pi y}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In this example, M is determined by the condition:

$$\left.\begin{array}{l} x=0 \\ y=0 \end{array}\right\}\frac{\partial w}{\partial x}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The approximation consists in assuming that the function in y represented by the first two terms in $\partial w/\partial x$ is affined to the function represented by the third term. This is not the case, and the angular deviation at the boundary becomes zero at one point only. In the remaining region, the angular deviation becomes negative.

M being determined, all requisite quantities can be calculated from eqn. (2). Suffice it to say that, for $x=0, y=0$, $Y_n=0$ can be put in the calculation of $\partial^2 w/\partial x^2$. When x is small, contributions to Y_n are furnished by the terms where n is large only, and for these terms $Y_n=0$. The calculation can be made rapidly by using the functions shown in figs. 4 and 5, and the summations given below:

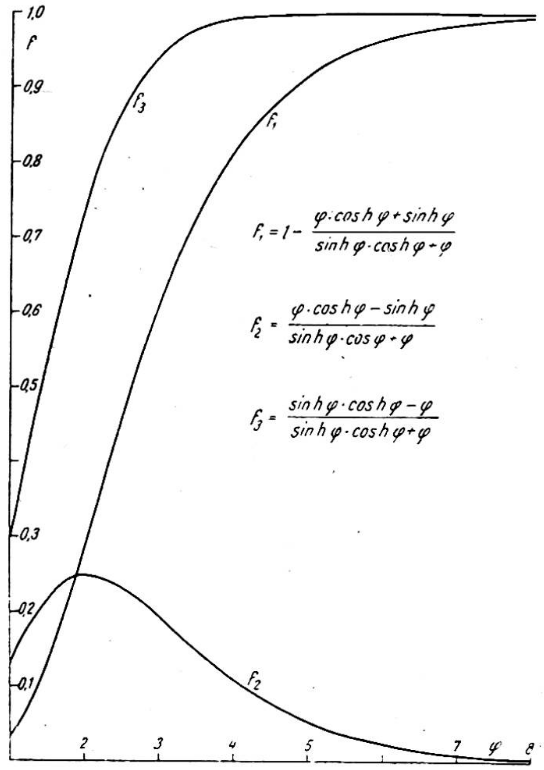

Fig. 4.

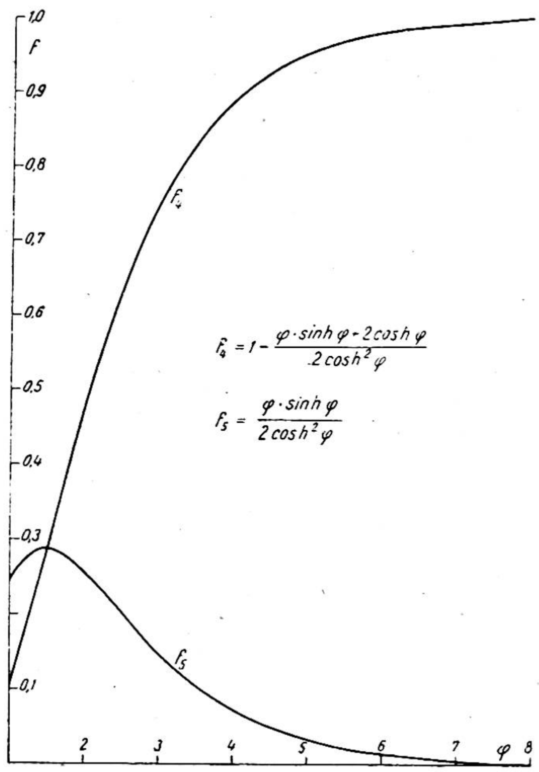

Fig. 5.

$$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\;\ldots\;=\frac{\pi}{4}$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\;\ldots\;=\frac{\pi^2}{6}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\;\ldots\;=\frac{\pi^2}{8}$$

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}\;\ldots\;=\frac{\pi^3}{32}$$

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}\;\ldots\;=\frac{\pi^4}{90}$$

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}\;\ldots\;=\frac{\pi^4}{96}$$

$$\frac{1}{1^5}-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}\;\ldots\;=\frac{5\pi^5}{1536}$$

For $b/a=1$, 2, and 3, some quantities have been computed on the assumption that Poisson's ratio is equal to zero. In fig. 6, they are compared with previously known "exact" values.

EXAMPLE NO. 2

A uniformly distributed load; all edges clamped. For the corresponding beam shown in fig. 7, the equation is:

$$w=\frac{4pa^4}{\pi^5 D}\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{a}-\frac{pa^4}{3\pi^3 D}\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{a}\quad\ldots\quad(6)$$

For the plate shown in fig. 8, the following is chosen:

$$w=\frac{4pa^4}{\pi^5 D}\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n^5}(1+Y_n)\sin\frac{n\pi x}{a}+M\sum_{n=1.3.5\ldots}^{\infty}\frac{1}{n^3}(1+Y_n)\sin\frac{n\pi x}{a}\quad.\quad(7)$$

$1+Y_n$ is determined from eqn. (3) and M from eqn. (5).

Just as in Example No. 1, some quantities have been calculated for Poisson's ratio=0, and are compared in fig. 9 with previously known "exact" values.

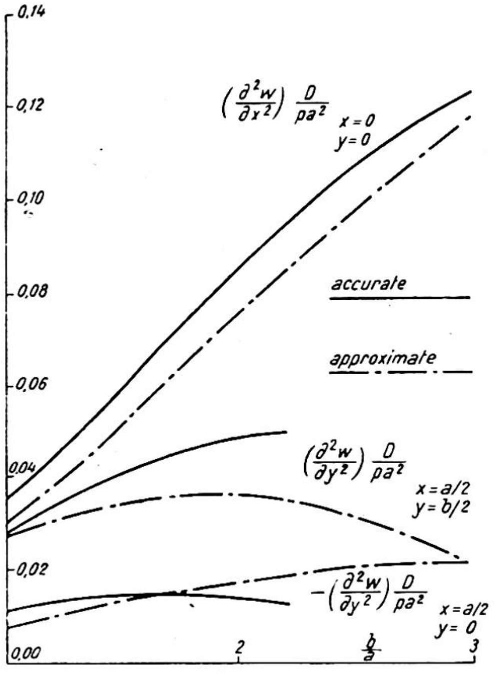

Fig. 6.

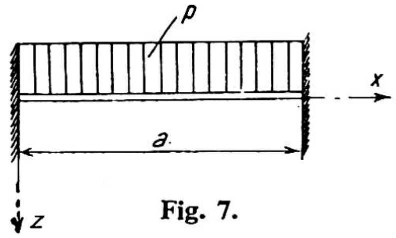

Fig. 7.

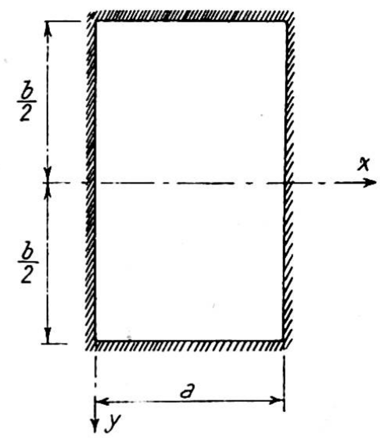

Fig. 8.

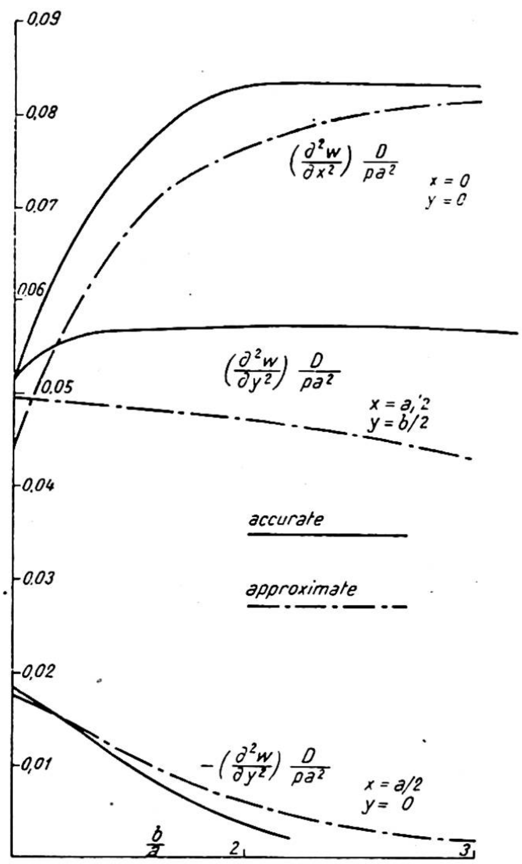

Fig. 9.

Summary

If a rectangular plate (fig. 1) is simply supported or clamped along the edges $y=\pm b/2$ and simply supported along the edges $x=0$ and $x=a$, and if this plate is submitted to a load which can be expanded into a Fourier series in x, then all quantities relating to the plate can be calculated in a simple manner by means of generally known methods. When, however, the boundary conditions at $x=0$ and $x=a$ are changed, the numerical computations are time-wasting. In this paper, the author demonstrates an approximate method which is characterised by the fact that the latter boundary conditions are satisfied on one point only. The calculations are very simple, and the results are sufficiently accurate for most design problems.

Résumé

L'auteur considère le cas d'une dalle rectangulaire suivant figure 1, portant librement ou totalement encastrée sur les bords $y=\pm b/2$, portant librement sur les bords $x=0$ et $x=a$ et soumise à une charge se prêtant à un développement en série de Fourier par rapport à x. Il montre que toutes les grandeurs qui caractérisent la dalle peuvent être calculées d'une manière simple, à l'aide de méthodes généralement connues. Les calculs numériques sont toutefois fastidieux lorsque l'on fait varier les conditions marginales sur les bords $x=0$ et $x=a$. L'auteur expose une méthode approchée caractérisée par ce fait que les conditions marginales latérales ne sont remplies qu'en un point. Les calculs sont très simples et la précision obtenue est généralement suffisante pour les besoins de la pratique.

Zusammenfassung

Für den Fall einer Rechteckplatte nach Abb. 1, die an den Rändern $y=\pm b/2$ frei aufliegt oder total eingespannt ist, an den Rändern $x=0$ und $x=a$ frei aufliegt und einer Belastung unterworfen ist, die nach einer Fourier-Reihe in x entwickelt werden kann, können alle die Platte betreffenden Grössen auf einfache Weise mittels allgemein bekannten Methoden berechnet werden. Die numerischen Berechnungen werden jedoch zeitraubend, wenn die Randbedingungen an den Rändern $x=0$ und $x=a$ geändert werden. Im vorliegenden Aufsatz wird eine Näherungsmethode beschrieben, die durch die Tatsache charakterisiert ist, dass die seitlichen Randbedingungen nur in einem Punkt erfüllt sind. Die Berechnungen werden sehr einfach und es wird eine für praktische Probleme meist genügende Genauigkeit erzielt.