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Dynamics of Continuous Structures with Repeated Elements

Vibrations des ouvrages continus formés d'éléments identiques

Dynamische Lösung der durchlaufenden Systeme mit sich wiederholenden Elementen

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Introduction

Structural systems with repeated elements are to be found in all historical periods. Bridges, continuous over several spans, may be regarded as a typical example of this kind (Fig. 1). The arches were first made of stone, later we find continuous structures of reinforced concrete, the structural systems being either continuous straight girders or continuous arches, and quite recently, elements of prestressed concrete have been used on an extensive scale for

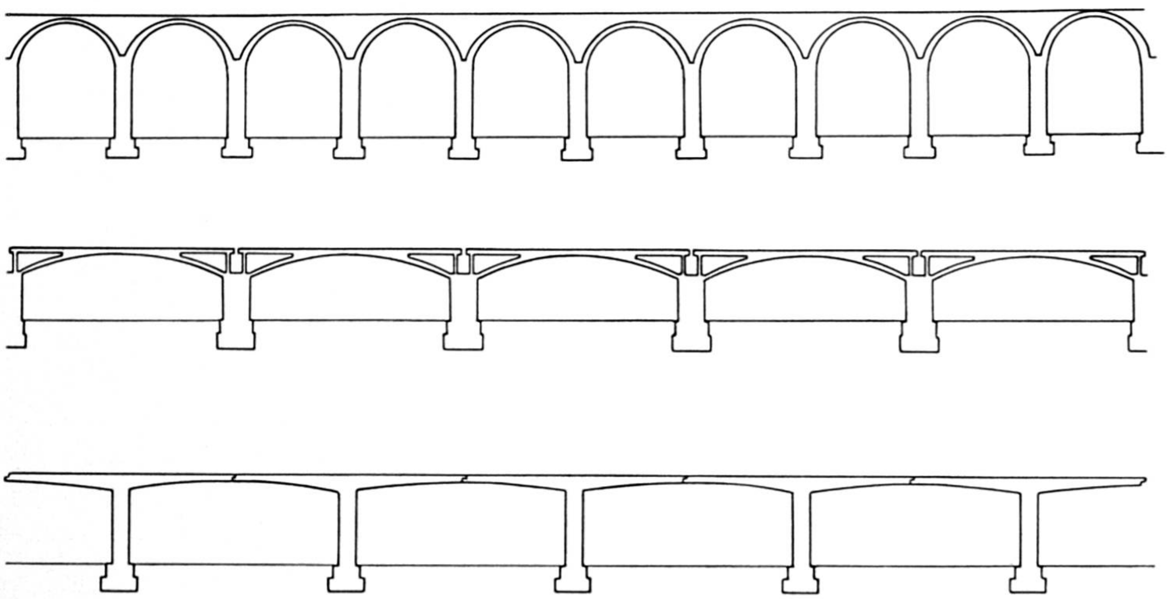

Fig. 1. Types of Continuous Bridge Structure with Repeated Elements.

continuous structures. In the structural systems used in building houses there are also many parts built up of repeated elements. Generally, it may be said that the increasing use of precast elements will naturally result in still wider application of systems built up of repeated elements, since these systems offer advantages from the point of view of economy.

In this paper some methods for investigating the vibrations of such systems are discussed, and it will be shown that both the theoretical and the numerical work involved may be considerably simplified, if we make use of all the advantages which the application of systems with repeated elements presents.

1. Continuous Beam of Uniform Section

A beam of uniform section, continuous over several equal spans, rigidly fixed at the end-supports, may be regarded as the simplest possible example of a system with repeated elements. In this case, the method of dynamical solution is well known, and we shall give here only a brief review of the analysis.

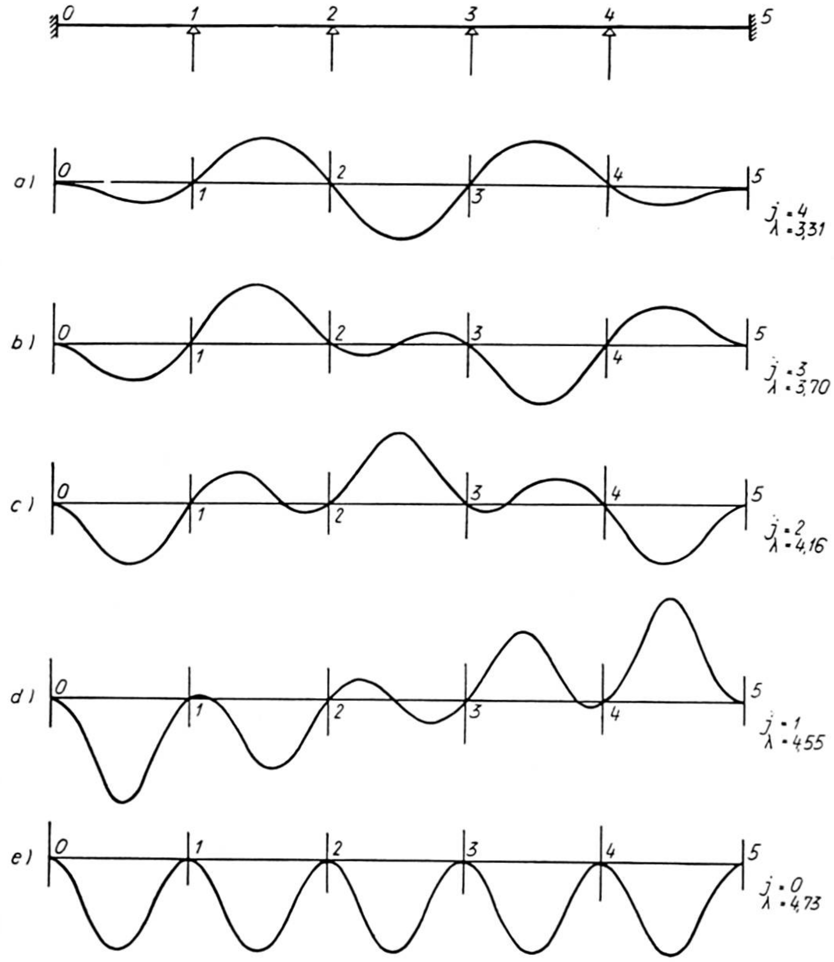

Fig. 2. Continuous Five-Span Beam, Rigidly Fixed at End-Supports. First Five Natural Modes of Vibration.

From the equilibrium of the moments at any isolated joint K we have

$$M_{K,K-1} + M_{K,K+1} = \mathfrak{M}_K, \tag{1}$$

where $M_{K,K-1}$, $M_{K,K+1}$ are the end-moments of the bars $K, K-1$ and $K, K+1$ respectively, and $\mathfrak{M}_K$ is the external moment loading at the joint K. If we express the end-moments by means of the end-rotations γ_K, we obtain from Eq. (1)

$$b\,\gamma_{K-1} + a\,\gamma_K + b\,\gamma_{K+1} = \mathfrak{M}_K. \tag{2}$$

Eq. (2) holds true for all intermediate supports, and we thus obtain a set of $n-1$ algebraic equations for determining the rotations γ_K. For free oscillations the external moment loading $\mathfrak{M}_K$ equals zero, and the set of algebraic equations in question is homogeneous.

As an illustration the analysis of a five-span beam of uniform section, shown in Fig. 2, will now be given.

The set of four equations, written down according to Eq. (2) is shown, in general form, in Table I.

Table I

γ_1	γ_2	γ_3	γ_4	
a	b			$=0$
b	a	b		$=0$
	b	a	b	$=0$
		b	a	$=0$

The coefficients a and b are functions of the natural frequency $f = \frac{\omega}{2\pi}$, and are defined as follows:

$$a = \frac{2EJ}{l} F_2(\lambda), \qquad b = \frac{EJ}{l} F_1(\lambda), \tag{2a}$$

where $$\lambda = l \sqrt[4]{\frac{\mu\,\omega^2}{EJ}}.$$

The functions $F_2(\lambda)$ and $F_1(\lambda)$ are tabulated in [1]) and [2]).

The equations as given in Table I are cyclically symmetrical, and may be solved by expanding the unknowns γ_K into finite trigonometrical series, viz.

$$\gamma_K = \sum_{j=1}^{n-1} \zeta_{[j]} \sin \frac{\pi}{n} j\,K, \tag{3}$$

where n denotes the number of spans.

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[1]) V. Koloušek: «Baudynamik der Durchlaufträger und Rahmen», Leipzig 1953.

[2]) V. Koloušek: «Calcul des efforts dynamiques dans les ossatures rigides», Dunod, Paris 1960.

If the values of γ_K according to Eq. (3) are introduced into Table I, the set of simultaneous equations reduces to four independent equations, which appear in the general form,

$$a + 2\,b\cos\frac{\pi}{n}\,j = 0 \qquad j = 1\,,\, n-1 \tag{4}$$

and introducing for a and b the values from Eq. (2a) we obtain

$$F_2(\lambda) + F_1(\lambda)\cos\frac{\pi}{5}\,j = 0\,. \tag{4b}$$

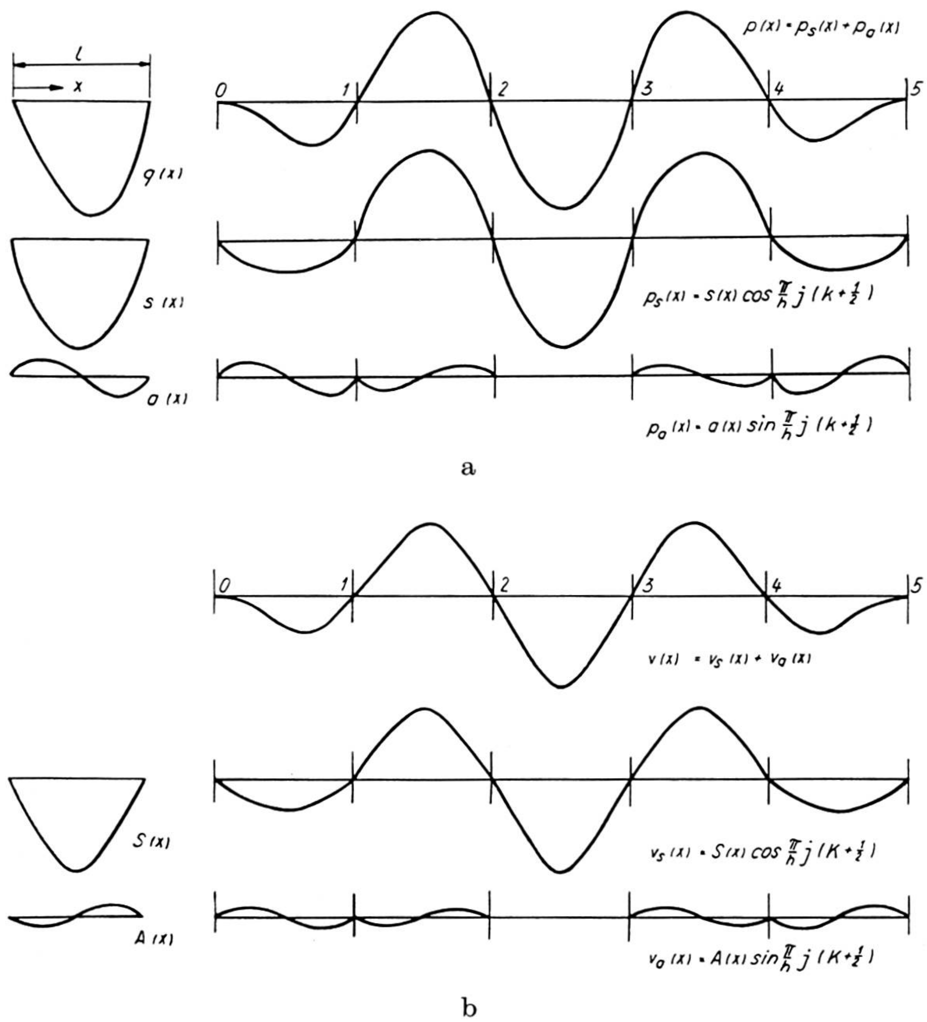

Fig. 3. Load and Deflections of a Five-Span Continuous Beam, for $j = 4$.

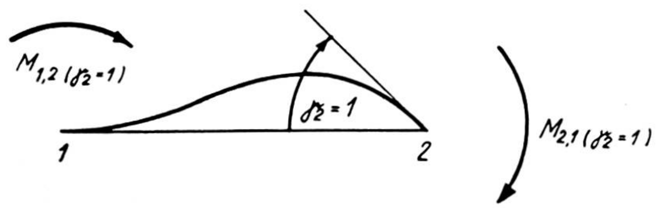

Fig. 4. End Moments.

The values of λ for which Eq. (4b) holds true determine, according to Eq. (2b), the first four natural frequencies of the system. The numerical values calculated for the system as shown in Fig. 2 are

$$j = 4 \quad \frac{\omega_{(1)}}{2\pi} = f_{(1)} = 1{,}74\,\alpha,$$

$$j = 3 \quad f_{(2)} = 2{,}18\,\alpha,$$

$$j = 2 \quad f_{(3)} = 2{,}75\,\alpha, \qquad \alpha = \frac{1}{l^2}\sqrt{\frac{E\,J}{\mu}}.$$

$$j = 1 \quad f_{(4)} = 3{,}30\,\alpha.$$

The fifth natural frequency of our system is identical with the first natural frequency of the single-span beams, of which the system is composed, if the individual beams were rigidly fixed at both ends, so that we have

$$f_{(5)} = 3{,}56\,\alpha.$$

The shapes of the first five natural modes of vibration are shown in Fig. 2. If the end-supports 0 and 5 of the system are hinged, the analysis remains, in principle, the same, but consideration must be given to the different end-conditions.

2. Continuous Beam of Non-Uniform Section

The dynamical analysis becomes complicated, if the section of the beam varies within the individual spans. Eq. (2) still holds true, but the coefficients a and b, although they are again functions of the natural frequency f, are no longer defined by Eq. (2a), the value of J, in this case, not being a constant, so that the functions cannot be tabulated. Direct solution would be thus very tedious, as the amount of numerical work might increase considerably. A convenient method of solution, in this case, is a combination of the direct method, described in the preceding paragraph, with the method of stepwise approximation. This method of analysis will now be illustrated for the case of a five-span beam, as shown in Fig. 5.

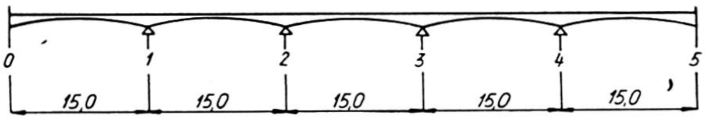

Fig. 5. Continuous Beam of Non-Uniform Section.

2.1. The Deformations of a Single-Span Beam

a) We consider first a single-span beam, rigidly fixed at both ends (Fig. 6), loaded by distributed statical weight $q(x)$. The load may be resolved into a symmetrical component $s(x)$ and antimetrical component $a(x)$, as shown in Fig. 3a (left). Thus we have

$$q(x) = s(x) + a(x),$$
$$s(x) = \tfrac{1}{2}[q(x) + q(l-x)] = s(l-x),$$
$$a(x) = \tfrac{1}{2}[q(x) - q(l-x)] = -a(l-x).$$

The load component $s(x)$ causes symmetrical deflections $V_s(x)$ of the beam, while the moments produced at the fixed supports are $\pm M_s$. The antimetrical component $a(x)$ produces antimetrical deflections $V_a(x)$, and the fixed-end moments in this case are M_a. The total deflection at the point x is thus

$$V(x) = V_s(x) + V_a(x).$$

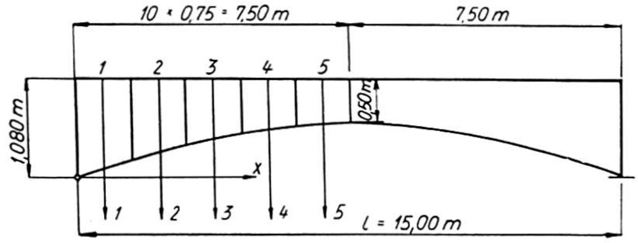

Fig. 6. Dimensions of Any Single Span for the Beam Shown in Fig. 5.

b) We now assume the beam to be simply supported, and consider the case where both ends are rotated simultaneously through a unit angle, in opposite directions. These end-rotations produce a symmetrical curve of deflections δ_{M_s}, which is at the same time the influence line for the moment M_s produced by symmetrical loading (Fig. 7a). The ordinate of the line δ_{M_s} at the point x gives the value of the moment M_s which is produced (with both ends fixed) at the left-hand support of the beam if two single loads $P = 1$ are applied to the points x and $l-x$.

c) The end sections of the simple beam are now rotated simultaneously through a unit angle in the same direction. These rotations produce an antimetrical curve of deflection δ_{M_a} (Fig. 7b), the curve being again the influence line for the fixed-end moment M_a at the left-hand support, produced in this case by two single unit loads, applied antimetrically at the abscissae x and $l-x$.

2.2. The Deformations of the Continuous Beam

We shall now consider a continuous beam of n equal spans, as shown in Fig. 5, where $n = 5$. We assume the beam to be loaded by distributed statical weight $p(x)$, the variation of which at the span $K, K+1$ may be expressed as follows:

symmetrical component $$p_s(x) = s(x)\cos\frac{\pi}{n} j\,(K + \tfrac{1}{2}) \tag{5}$$

antimetrical component $$p_a(x) = a(x)\sin\frac{\pi}{n} j\,(K + \tfrac{1}{2}). \tag{6}$$

The letter j here denotes an arbitrary whole number between 0 and $n-1$.

For the five-span beam we have $n=5$, and the case for $j=4$ is shown in Fig. 3.

If all rotations at the supports were prevented, the deflection produced by the load components as given by Eqs. (5) and (6) would be

$$V_s(x)\cos\frac{\pi}{n}j\left(K+\tfrac{1}{2}\right)+V_a(x)\sin\frac{\pi}{n}j\left(K+\tfrac{1}{2}\right), \tag{7}$$

while the loading moment at the support K would have the value

$$\begin{aligned}\mathfrak{M}_K &= M_s\left[\cos\frac{\pi}{n}j\left(K+\tfrac{1}{2}\right)-\cos\frac{\pi}{n}j\left(K-\tfrac{1}{2}\right)\right]\\ &\quad+M_a\left[\sin\frac{\pi}{n}j\left(K+\tfrac{1}{2}\right)+\sin\frac{\pi}{n}j\left(K-\tfrac{1}{2}\right)\right]=\mathfrak{M}_{[j]}\sin\frac{\pi}{n}j\,K,\end{aligned} \tag{8}$$

where

$$\mathfrak{M}_{[j]}=-2\left[M_s\sin\frac{\pi}{2n}j-M_a\cos\frac{\pi}{2n}j\right]. \tag{9}$$

If the temporarily "locked" supports are released, the loading moments $\mathfrak{M}_K$ produce the rotations

$$\gamma_K=\zeta_{[j]}\sin\frac{\pi}{n}j\,K, \tag{10}$$

where

$$\begin{aligned}\zeta_{[j]}&=\frac{\mathfrak{M}_{[j]}}{a_{[j]}},\\ a_{[j]}&=a+2b\cos\frac{\pi}{n}j=2M_{2,1(\gamma_2=1)}+2M_{1,2(\gamma_2=1)}\cos\frac{\pi}{n}j\end{aligned} \tag{11}$$

and the moments M are shown in Fig. 4.

The rotations γ_K may be again resolved into symmetrical and antimetrical components, according to the following formulæ:

symmetrical components

$$\begin{aligned}\tfrac{1}{2}(\gamma_K-\gamma_{K+1})&=\tfrac{1}{2}\zeta_{[j]}\left[\sin\frac{\pi}{n}j\,K-\sin\frac{\pi}{n}j\,(K+1)\right]=\\ &\quad-\zeta_{[j]}\sin\frac{\pi}{2n}j\cos\frac{\pi}{n}j\left(K+\tfrac{1}{2}\right)\end{aligned} \tag{12}$$

antimetrical components

$$\tfrac{1}{2}(\gamma_K+\gamma_{K+1})=\zeta_{[j]}\cos\frac{\pi}{2n}j\sin\frac{\pi}{n}j\left(K+\tfrac{1}{2}\right). \tag{13}$$

The deflections produced at the span $K, K+1$ by the symmetrical components of the end-rotations are

$$\tfrac{1}{2}(\gamma_K-\gamma_{K+1})\,\delta_{M_s},$$

while the antimetrical components of the end-rotations produce at the span $K, K+1$ the deflections

$$\tfrac{1}{2}(\gamma_K+\gamma_{K+1})\,\delta_M\,.$$

Table II

Symmetric vibrations $j=4$

[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]
		Influence co-ordinate						1st Approximation				
Point i	Mass m_i in $tm^{-1}\,s^2$	for the Deflection $\delta^s_{i,k}\cdot 10^5$ in mt^{-1} — Point K					for the Moment at Support δ_{M_s} in m	S_i	[2]×[9] $m_i S_i$ in $tm^{-1}\,s^2$	[8]×[10] $m_i S_i \delta_{M_s}$ in t s^2	$\sum m_k S_k \delta^s_{ik}\cdot 10^5$ in s^2	$-\xi_{[4]}\sin\frac{2\pi}{5}\cdot 10^5\,\delta_{M_s}$ in s^2
		1	2	3	4	5						
1	0,373	0,058	0,112	0,262	0,335	0,378	0,745	1	0,373	0,278	1,0	90,6
2	0,301	0,112	1,530	3,280	4,525	5,222	2,168	1,8	0,542	1,175	12,6	263,8
3	0,247	0,262	3,280	9,395	14,290	17,035	3,427	3,2	0,791	2,711	39,5	416,9
4	0,212	0,335	4,525	14,290	26,275	33,370	4,417	4,4	0,933	4,121	69,6	537,4
5	0,194	0,378	5,222	17,035	33,370	47,540	4,972	4,8	0,933	4,629	92,0	604,9

$a=0{,}454\cdot 10^5$ tm, $b=0{,}162\cdot 10^5$ tm

$$\sin\frac{2\pi}{5}=0{,}951,\quad \cos\frac{2\pi}{5}=0{,}309,\quad \cos\frac{4\pi}{5}=-0{,}809$$

$$a_{[4]}=a+2b\cos\frac{4\pi}{5}=[0{,}454+2\cdot 0{,}162\,(-0{,}809)]\,10^5=$$
$$=0{,}192\cdot 10^5 \text{ tm}$$

$$\sum=12{,}914=M_s$$

$$-\mathfrak{M}_{[4]}=2\,M_s\sin\frac{2\pi}{5}=2\cdot 12{,}914\cdot 0{,}951=24{,}56\text{ t s}^2$$

$$\xi_{[4]}=\frac{\mathfrak{M}_{[4]}}{a_{[4]}}=-\frac{24{,}56}{0{,}192\cdot 10^3}=-127{,}9\cdot 10^{-5}\text{ m}^{-1}\text{ s}^2$$

$$-10^5\,\xi_{[4]}\sin\frac{2\pi}{5}=127{,}9\cdot 0{,}951=121{,}7\text{ m}^{-1}\text{ s}^2$$

$$-10^5\,\xi_{[4]}\cos\frac{2\pi}{5}=127{,}9\cdot 0{,}309=39{,}5\text{ m}^{-1}\text{ s}^2$$

Antisymmetric vibrations

[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]
		Influence co-ordinate						1st Approximation				
Point i	Mass m_i in $tm^{-1}\,s^2$	for the Deflection $\delta^a_{i,k}\cdot 10^5$ in mt^{-1} — Point K					for the Moment at Support δ_{M_a} in m	A_i	[2]×[9] $m_i A_i$ in $tm^{-1}\,s^2$	[8]×[10] $m_i A_i \delta_{M_a}$ in t s^2	$\sum m_k A_k \delta^a_{ik}\cdot 10^5$ in s^2	$\xi_{[4]}\cos\frac{2\pi}{5}\cdot 10^5\cdot\delta_{M_a}$ in s^2
		1	2	3	4	5						
1	0,373	0	0,061	0,062	0,054	0,023	0,735	0	0	0	0	−29,1
2	0,301	0,061	1,324	1,994	1,791	0,726	1,795	0	0	0	0	−71,0
3	0,247	0,062	1,994	4,828	4,922	2,080	2,240	0	0	0	0	−88,5
4	0,212	0,054	1,791	4,922	7,091	3,320	1,865	0	0	0	0	−73,7
5	0,194	0,023	0,726	2,080	3,320	2,484	0,725	0	0	0	0	−28,7

[1]	[14]	[15]	[16]	[17]	[18]	[19]	[20]
	2nd Approximation					3rd App.	
Point i	[12]+[13]	[2]×[14]	[8]×[15]	$\sum m_k S_k \delta^s_{ik}\cdot 10^8$ in s^4	$-\xi_{[4]}\sin\frac{2\pi}{5}\cdot 10^8\,\delta_{M_s}$ in s^4	$S_i\cdot 10^8$ in s^4	[14]:[19]
	$S_i\cdot 10^5$ in s^2	$m_i S_i\cdot 10^3$ in $tm^{-1}\,s^4$	$m_i S_i\,\delta_{M_s}\cdot 10^3$ in $t\,s^4$				$\omega^2_{(1)}$
1	91,6	0,342	0,255	1,4	131,6	133,0	689
2	276,4	0,832	1,803	17,9	382,9	400,8	690
3	456,4	1,128	3,865	54,9	605,2	660,1	690
4	607,0	1,287	5,684	99,0	780,1	879,1	690
5	696,9	1,352	6,722	131,2	878,1	1009,3	690

$$\sum = 18{,}329 = M_s\cdot 10^3$$

$$-\mathfrak{M}_{[4]} = 2\left(M_s\sin\frac{2\pi}{5} - M_a\cos\frac{2\pi}{5}\right) = 2\,(18{,}329\cdot 0{,}951 + 1{,}287\cdot 0{,}309)\cdot 10^{-3} =$$

$$= 35{,}66\cdot 10^{-3}\,t\,s^4$$

$$\xi_{[4]} = \frac{\mathfrak{M}_{[4]}}{a_{[4]}} = -\frac{35{,}66\cdot 10^{-3}}{0{,}192\cdot 10^5} = -185{,}7\cdot 10^{-8}\,m^{-1}\,s^4$$

$$-10^8\,\xi_{[4]}\sin\frac{2\pi}{5} = 185{,}7\cdot 0{,}951 = 176{,}6\,m^{-1}\,s^4$$

$$-10^8\,\xi_{[4]}\cos\frac{2\pi}{5} = 185{,}7\cdot 0{,}309 = 57{,}4\,m^{-1}\,s^4$$

[1]	[14]	[15]	[16]	[17]	[18]	[19]	[20]
	2nd Approximation					3rd App.	
Point i	[12]+[13]	[2]×[14]	[8]×[15]	$\sum m_k A_k \delta^a_{i,k}\cdot 10^8$ in s^4	$\xi_{[4]}\cos\frac{2\pi}{5}\cdot 10^8\,\delta_{M_a}$ in s^4	$A_i\cdot 10^8$ in s^4	[14]:[19]
	$A_i\cdot 10^5$ in s^2	$m_1 A_i\cdot 10^3$ in $tm^{-1}\,s^4$	$m_i A_i\,\delta_{M_a}\cdot 10^3$ in $t\,s^4$				$\omega^2_{(1)}$
1	−29,1	−0,109	−0,079	0,0	− 42,2	− 42,2	689
2	−71,0	−0,214	−0,384	−1,0	−103,0	−104,0	682
3	−88,5	−0,219	−0,492	−2,4	−128,5	−130,9	675
4	−73,7	−0,156	−0,292	−2,8	−107,0	−109,8	670
5	−28,7	−0,056	−0,040	−1,3	− 41,6	− 42,9	668

$$\sum = -1{,}287$$

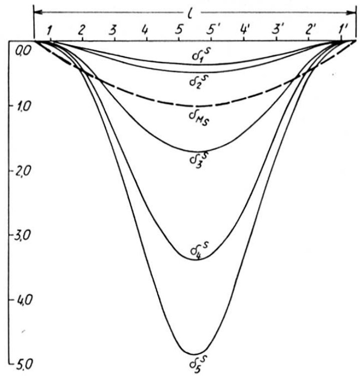

Fig. 7a. Influence Lines for Symmetrical Load. Scale: Line δ^s_i 1 Grad. = 10^{-4} m/t, Line δ_{M_s} 1 Grad. = 5 m.

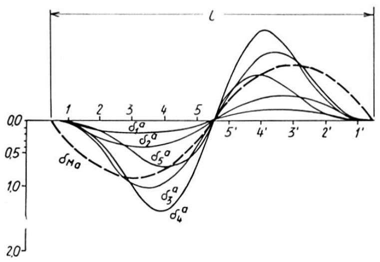

Fig. 7b. Influence Lines for Non-Symmetrical Load. Scale: Line δ^a_i 1 Grad. = $0{,}5\cdot 10^{-4}$ m/t, Line δ_{M_a} 1 Grad. = 2,5 m.

The total deflections (see Fig. 3b) produced by the load $p(x)$ at any particular span are then given as the sum of the symmetrical and antimetrical components, and the variations of the symmetrical components $v_s(x)$ and antimetrical components $v_a(x)$ are represented as follows

$$v_s(x) = S(x) \cos \frac{\pi}{n} j \left(K + \tfrac{1}{2}\right), \tag{14}$$

$$v_a(x) = A(x) \sin \frac{\pi}{n} j \left(K + \tfrac{1}{2}\right), \tag{15}$$

where

$$S(x) = V_s(x) - \delta_{M_s} \zeta_{[j]} \sin \frac{\pi}{2n} j, \tag{16}$$

$$A(x) = V_a(x) + \delta_{M_a} \zeta_{[j]} \cos \frac{\pi}{2n} j. \tag{17}$$

The values of $S(x)$ are thus obtained by superposition of

1. the deflection $V_s(x)$, which is produced by the load-component $s(x)$ acting on a rigidly fixed single-span beam, and
2. the deflection of the single-span simple beam, the supports of which are rotated symmetrically through the angle $\mp \zeta_{[j]} \sin \frac{\pi}{2n} j$.

(The signs minus and plus are to be taken for the left-hand and the right-hand supports respectively.)

The values of $A(x)$ are obtained by superposition:

1. The deflection $V_a(x)$, which is produced by the load component $a(x)$, acting on a fixed single-span beam, with
2. the deflection of the single-span simple beam, the supports of which are rotated antimetrically through the angle $\zeta_{[j]} \cos \frac{\pi}{2n} j$.

If we compare Eqs. (14) and (15) with Eqs. (5) and (6) we see that the deflections and the load components have a similar mathematical representation.

2.3. The Vianello Method of Stepwise Approximation

The results of the analysis as given in the preceding paragraph may be used for calulating the frequencies and modes of vibration of continuous beams, by stepwise approximation. A suitable procedure will be briefly discussed in this paragraph.

As a first approximation we assume an arbitrary curve of deflection ${}_1v(x)$ which, however, must admit of being expanded into components according to Eq. (5) and (6), and we calculate the corresponding symmetrical and antimetrical components of the load intensity $\mu(x)\,{}_1v(x)$. (By $\mu(x)$ we denote the mass per unit length, as a function of the abscissa x of the beam.) These load components produce the deflections ${}_2v(x)$, as a second approximation to the true shape of the natural mode. The deflections ${}_2v(x)$ are resolved into symmetrical and antimetrical components according to Eqs. (14) and (15). It is evident that *only a single span of the structure has to be considered*, as the deflection curve at any span may be readily determined by means of the quantities $S(x)$ and $A(x)$, which are defined by Eqs. (16) and (17).

The third approximation to the true shape of natural mode may be then obtained by repeating the process, i. e. calculating the curve of deflection ${}_3v(x)$ produced by the load $\mu(x)\,{}_2v(x)$ etc. The process is repeated until concordance of the deflections ${}_{(k-1)}v(x)$ and ${}_kv(x)$ has been reached with the desired accuracy. In practical calculations we usually do not consider the continuously distributed mass, but divide the beam up into a finite number of strips, and then assume the mass to be concentrated at the centroids of the strips.

A numerical example will illustrate the practical procedure.

2.4. *Numerical example*

A five-span continuous beam, as shown in Figs. 5 and 6, will now be analysed, applying the method outlined in the preceding paragraph. The centre-line of the beam has been assumed to be straight, the modulus of elasticity has been taken to be $E=2{,}4\cdot 10^6\,\mathrm{t/m^2}$. For purposes of calculation the beam has been divided up into ten strips; the masses concentrated at the centres of the respective strips are tabulated in column 2 of Table II.

For symmetrical loading by two single unit loads the influence lines δ_i^s for the deflections of the single span fixed-end beam are shown in Fig. 7a, where the influence line δ_{M_s} for the fixed-end moment is also given. The lines δ_i^a and δ_{M_a} corresponding to antimetrical loading are given in Fig. 7b. The influence line ordinates at the centres of the individual strips are given in Table II. The coefficient a, which is also given in Table II, equals twice the value of the moment $M_{2,1}$, acting at the support 2 of the beam 1—2 when the end rotations are $\gamma_1=0$, $\gamma_2=1$. The coefficient b is the corresponding value of the moment $M_{1,2}$.

The numerical calculation by stepwise approximation has been carried out for $j=4$, $j=1$, and $j=0$. In Table II the calculation for $j=4$ is shown by way of illustration. For $j=4$ the natural frequency attains its lowest value. From the ratio of the last two approximations we obtain the square of the natural angular frequency.

$$\omega_{(1)}^2 = \frac{696{,}9\cdot 10^3}{1009{,}3} = 690\,\mathrm{sec}^{-2}$$

so that the first natural frequency is

$$f_{(1)} = 4{,}18\,\mathrm{sec}^{-1}.$$

The shape of the corresponding natural mode, at the span $K, K+1$, is given by the last approximations of S_i, and A_i, according to the formula.

$$v(x_i) = v_i = S_i \cos\frac{\pi}{n} j (K+\tfrac{1}{2}) + A_i \sin\frac{\pi}{n} j (K+\tfrac{1}{2}), \tag{18}$$

where $j=4$.

The fourth natural frequency $f_{(4)} = 10{,}73\,\mathrm{sec}^{-1}$ was calculated in a similar manner, but with $j=1$. The fifth natural frequency $f_{(5)} = 12{,}43\,\mathrm{sec}^{-1}$ is identical with the first natural frequency of a single span fixed-end beam. The shapes of the first, fourth and fifth natural modes of vibration are shown in Fig. 8.

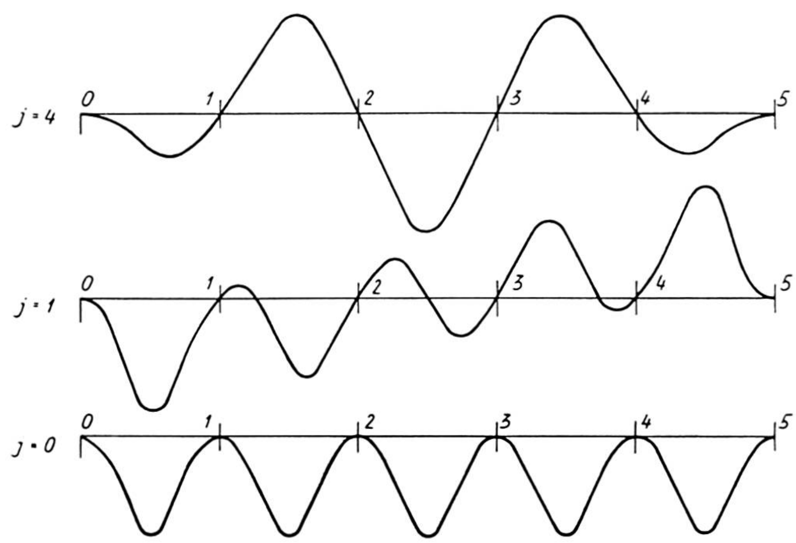

Fig. 8. Natural Modes of Vibration for the Beam of Fig. 5.

a) First Natural Mode ($j=4$),
b) Fourth Natural Mode ($j=1$),
c) Fifth Natural Mode ($j=0$).

The above described method of analysis may also be applied to the solution of continuous arch structures. In this case, however, the number of unknowns is larger, because the intermediate supports undergo not only rotations, but also vertical and horizontal translatory displacements. The vertical displacements of the supports may usually be neglected, and this simplifies the calculations, but in exceptional cases these displacements may also be taken into account. With continuous arch structures the basic equations are again cyclically symmetrical, and admit of a solution which, in principle, is the same as in the case of continuous beams. Continuous beams having elastic intermediate supports may also be solved in a similar manner.

The method, as described in this paper, can be applied not only to the dynamical but also to the statical analysis of the systems in question, and the numerical work involved may thus be considerably reduced.

Summary

In this paper some methods for investigating the vibrations of continuous structures with repeated elements are discussed. The structures in question are continuous beams with equal spans, and of either uniform or non-uniform section, continuous arch structures, continuous rigid frames, etc.

Finite trigonometrical series and a combination of the slope-deflection method with stepwise approximation enable the mathematical investigation to take advantage of all the specific simplifications which the repetition of equal elements presents. The analysis is first given for a continuous beam of uniform section, where a further simplification is possible if tabulated functions are used. In addition the frequencies and modes of vibration are investigated for a five-span beam of non-uniform section. The procedure is illustrated by a numerical example and it is shown that the numerical work involved is only slightly greater than that which a solution of a single-span beam requires.

The method may also be applied to continuous structures with elastic supports, and not only the dynamical but also the statical analysis can be thus considerably simplified.

Résumé

L'auteur présente des méthodes permettant d'étudier les vibrations des ouvrages continus formés d'éléments successifs identiques. Il s'agit de poutres continues de portées égales avec une section constante ou variable, de voûtes multiples, de cadres continus, etc.

Ce problème apparemment fastidieux peut être considérablement simplifié si l'on tire parti, dans la résolution mathématique même, de tous les avantages que présente la répétition d'éléments identiques; les équations des déformations étant cycliquement symétriques, l'introduction de séries trigonométriques finies permet de simplifier considérablement le problème. L'auteur traite tout d'abord les poutres continues de section constante, dont la résolution est grandement simplifiée par l'utilisation de fonctions disposées en tables. L'étude des poutres continues de section variable peut se faire en combinant la méthode des déformations avec celle des approximations successives. L'auteur donne une application numérique de son procédé en traitant une poutre continue comportant cinq travées identiques de section variable.

Cette méthode peut également être utilisée pour les systèmes continus sur appuis élastiques. De plus, elle s'applique au calcul statique des poutres et des arcs continus; elle y apporte une importante simplification des opérations numériques.

Zusammenfassung

In dieser Abhandlung sind Methoden der Berechnung der schwingenden Systeme, deren Elemente sich wiederholen, behandelt. Es handelt sich um Durchlaufträger mit gleichen Feldern mit konstantem oder veränderlichem Querschnitt, durchlaufende Bogenreihen, durchlaufende Rahmen usw.

Die scheinbar mühsame Aufgabe wird wesentlich vereinfacht, wenn man auch in der mathematischen Lösung alle Vorteile ausnützt, welche die Wiederholung von gleichen Elementen bietet. Die Formänderungsgleichungen sind zyklisch symmetrisch und die Einführung der endlichen trigonometrischen Reihen bringt deshalb eine äußerste Vereinfachung der Lösung. Es werden zuerst Durchlaufträger mit konstantem Querschnitt untersucht, bei denen die Benützung von tabellierten Funktionen eine weitere Vereinfachung ermöglicht. Bei den Durchlaufträgern mit variabler Steifigkeit kann die Aufgabe so gelöst werden, daß man die Deformationsmethode mit der Methode der schrittweisen Näherung kombiniert. Das Verfahren wird an einem numerischen Beispiel erläutert, in welchem ein Durchlaufträger mit fünf gleichen Elementen mit variablem Querschnitt bearbeitet wird.

Die Methode kann auch zur Berechnung der durchlaufenden Systeme auf elastischen Stützen benützt werden. Das Verfahren kann auch bei *statischer* Lösung der durchlaufenden Träger und Bogen angewendet werden, denn auch hier wird eine wesentliche Vereinfachung der numerischen Berechnungen erzielt.