Analysis of Bridge Structures Comprising Two Continuous Curved Main Box Girders, Whose Supports are Staggered or not, and That are Connected by Cross Beams having Flexural but not Torsional Rigidity

Calcul des structures comprenant deux poutres caisson maîtresses continues et courbes, à supports décalés ou non, et reliées par des traverses sans rigidité torsionelle

Berechnung von Brücken mit zwei durchlaufenden, gekrümmten, kastenförmigen Hauptträgern, deren Auflager beweglich oder fest sind, und die mit biegesteifen, jedoch drillweichen Querträgern verbunden sind

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Assumptions

1) The structure behaves elastically.
2) The supports of the box girders are unyielding and the intermediate supports provide vertical reactions only.
3) The loads act on the two box girders.
4) The formula's for uniform torsion are valid. It is well known that the errors resulting from this assumption are small in the case of box girders.

The two main girders may or may not have the same number of spans. Their flexural rigidity EI and torsional rigidity GC may be variable. The flexural rigidity of the cross beams may be infinite or finite. If the distance a between the box girders varies, the rate of variation must be small enough for the transverse beams to be practically perpendicular to the girders. The supports of the girders may or may not coincide with the locus of the shear center of their cross sections, which we shall henceforth call the center line. The end supports may or may not allow flexural or torsional rotation of the ends of the box girders. The loads may act on or off the center line of the main girders.

Nodes

In figure 1 each box girder is represented by its center line.

We first consider the box girder on the outside of the curve, together with one half of each tie beam (fig. 2). Along its center line nodes are introduced : at each support , at each junction with a connecting beam, at the point of application of every concentrated external load or moment,

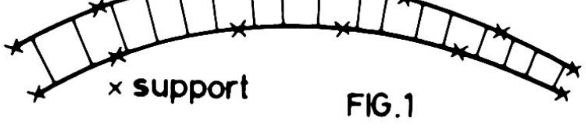

FIG.1

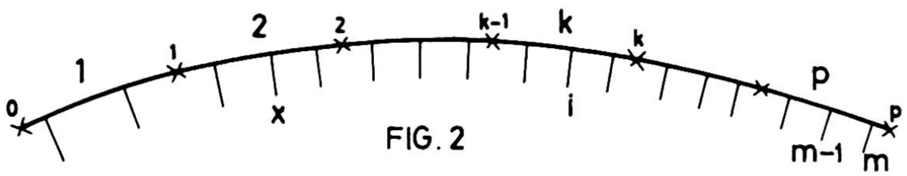

FIG. 2

at the boundaries between zones of constant curvature, or of constant flexural or torsional rigidity, or of constant distributed load,

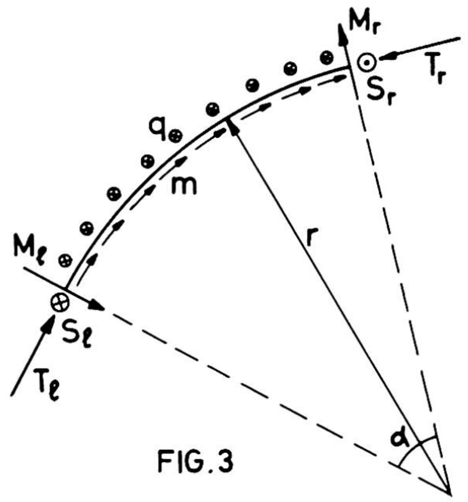

FIG.3

and at all points of the girder where it is desired to know the stress resultants or displacement components. Hence each girder element between two successive nodes has or is assumed to have a constant radius r (fig. 3) and constant rigidities EI and GC = $\frac{EI}{\rho}$, and it carries or is assumed to carry a uniform downward load q along its center line and a constant moment m (per unit length) about the center line. m is taken positive when it acts in the direction of the rotation of a corkscrew that moves forward in the direction of the arrow in figure 3.

<u>Element transmission matrix</u>

The forces at the left end of the element considered as a free body are the shear force S_ℓ , the bending moment M_ℓ and the torque T_ℓ . Similar forces act at the right end of the element. S_ℓ is considered positive when it acts downwards on the element, S_r when it acts upwards. The positive direction of the bending moments and torques is defined by the corkscrew rule, as is that of the rotations φ in the vertical plane tangent to the center line and ψ in the plane perpendicular to that line, both rotations being represented in figure 4 by arrows perpendicular to the plane of rotation. The vertical (downward) displacements at the ends of the element are denoted by w_ℓ and w_r .

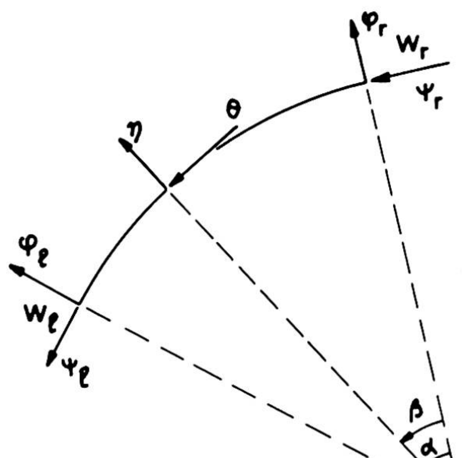

FIG.4

Statics provides the following relationships between the internal forces at the right end and at the left end of the element :

$$S_r = S_\ell + \alpha q r \qquad (1)$$

$$M_r = M_\ell \cos\alpha - T_\ell \sin\alpha + S_\ell r \sin\alpha + r(qr-m)(1-\cos\alpha) \qquad (2)$$

$$T_r = M_\ell \sin\alpha + T_\ell \cos\alpha + S_\ell r(1-\cos\alpha) - r(qr-m)\sin\alpha + \alpha q r^2 \qquad (3)$$

One obtains the bending moment M and torque T in the section defined by the angle β by replacing α by $\alpha-\beta$ in the expressions 2 and 3. The curvature η and the twist per unit length θ at the same point are given by $\eta = \frac{M}{EI}$ and $\theta = \frac{T}{GC} = \frac{\rho T}{EI}$

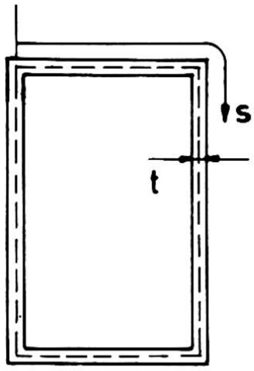

FIG. 5

with C given by Bredt's formula $C = \dfrac{4A^2}{\oint \frac{ds}{t}}$ when the box girder is monocellular (fig. 5 - A : cross-sectional area bordered by the center line of the girder walls).

Geometry, as applied to small angles and deflections, allows the following relationships between the vertical and rotational displacements at both ends of the girder element to be written :

$$\begin{aligned}
\varphi_r &= \varphi_\ell \cos\alpha - \psi_\ell \sin\alpha + \int_0^\alpha \cos\beta . \eta r d\beta - \int_0^\alpha \sin\beta . \theta r d\beta \\
&= \varphi_\ell \cos\alpha - \psi_\ell \sin\alpha + \frac{r}{EI}\int_0^\alpha M\cos\beta d\beta - \frac{\rho r}{EI}\int_0^\alpha T\sin\beta d\beta \qquad (4)
\end{aligned}$$

$$\begin{aligned}
\psi_r &= \varphi_\ell \sin\alpha + \psi_\ell \cos\alpha + \int_0^\alpha \sin\beta . \eta r d\beta + \int_0^\alpha \cos\beta . \theta r d\beta \\
&= \varphi_\ell \sin\alpha + \psi_\ell \cos\alpha + \frac{r}{EI}\int_0^\alpha M\sin\beta d\beta + \frac{\rho r}{EI}\int_0^\alpha T\cos\beta d\beta \qquad (5)
\end{aligned}$$

$$\begin{aligned}
w_r &= w_\ell + \varphi_\ell r\sin\alpha - \psi_\ell r(1-\cos\alpha) + \int_0^\alpha r\sin\beta . \eta r d\beta - \int_0^\alpha r(1-\cos\beta) . \theta r d\beta \\
&= w_\ell + \varphi_\ell r\sin\alpha - \psi_\ell r(1-\cos\alpha) + \frac{r^2}{EI}\int_0^\alpha M\sin\beta d\beta - \frac{\rho r^2}{EI}\int_0^\alpha T(1-\cos\beta) d\beta \qquad (6)
\end{aligned}$$

Substituting the expressions of M and T into 4, 5 and 6, and performing the integrations, one finds three equations, which may be assembled with 1, 2 and 3 into the matrix equation

$$\begin{bmatrix} w_r \\ \varphi_r \\ \psi_r \\ M_r \\ T_r \\ S_r \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & r\sin\alpha & -r(1-\cos\alpha) & b_{14} & b_{15} & b_{16} & b_{17} \\ 0 & \cos\alpha & -\sin\alpha & b_{24} & b_{25} & b_{26} & b_{27} \\ 0 & \sin\alpha & \cos\alpha & b_{34} & b_{35} & b_{36} & b_{37} \\ 0 & 0 & 0 & \cos\alpha & -\sin\alpha & r\sin\alpha & r(qr-m)(1-\cos\alpha) \\ 0 & 0 & 0 & \sin\alpha & \cos\alpha & r(1-\cos\alpha) & -r(qr-m)\sin\alpha+\alpha q r^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha q r \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} w_\ell \\ \varphi_\ell \\ \psi_\ell \\ M_\ell \\ T_\ell \\ S_\ell \\ 1 \end{bmatrix}$$

or $\quad V_r = B V_\ell \quad$, with $\qquad (7)$

$b_{14} = \frac{r^2}{2EI}\left[(1+\rho)\alpha\sin\alpha - 2\rho(1-\cos\alpha)\right]$ $\quad b_{15} = \frac{r^2}{2EI}(1+\rho)(\alpha\cos\alpha-\sin\alpha)$

$b_{24} = \frac{r}{2EI}\left[(1-\rho)\sin\alpha + (1+\rho)\alpha\cos\alpha\right]$ $\quad b_{25} = -\frac{r}{2EI}(1+\rho)\alpha\sin\alpha$

$b_{34} = \frac{r}{2EI}(1+\rho)\alpha\sin\alpha$ $\quad b_{35} = \frac{r}{2EI}\left[(1+\rho)\alpha\cos\alpha - (1-\rho)\sin\alpha\right]$

$b_{16} = -\frac{r^3}{2EI}\left[(1+\rho)(\alpha\cos\alpha - \sin\alpha) + 2\rho(\alpha - \sin\alpha)\right]$

$b_{26} = \frac{r^2}{2EI}\left[(1+\rho)\alpha\sin\alpha - 2\rho(1-\cos\alpha)\right]$ $\quad b_{36} = -\frac{r^2}{2EI}(1+\rho)(\alpha\cos\alpha-\sin\alpha)$

$$b_{17} = \frac{r^3}{EI}\left[(1+\rho)(1-\cos\alpha-\frac{\alpha}{2}\sin\alpha)(qr-m)+\rho(1-\cos\alpha-\frac{\alpha^2}{2})qr\right]$$

$$b_{27} = -\frac{r^2}{EI}\left[\tfrac{1}{2}(1+\rho)(\alpha\cos\alpha-\sin\alpha)(qr-m)+\rho(\alpha-\sin\alpha)qr\right]$$

$$b_{37} = \frac{r^2}{EI}\left[(1+\rho)(1-\cos\alpha-\frac{\alpha}{2}\sin\alpha)(qr-m)+\rho(1-\cos\alpha)m\right]$$

One finds the column vector V_r, made up of the displacement components and stress resultants pertaining to the right end of the girder element by premultiplication of the column vector V_ℓ pertaining to the left end by the element transmission matrix B.

<u>Boundary vector and boundary matrix</u>

A boundary vector L and a boundary matrix D are associated with each node at which an external load acts on the box girder. This applies in particular to the junctions of the girder with the tie beams, but not to its supports.

Discontinuities Δw, $\Delta\varphi$ or $\Delta\Psi$ in the displacement components w, φ and Ψ at the node may also be included in the vector

$$L = \begin{bmatrix}\Delta w\\ \Delta\varphi\\ \Delta\Psi\\ \Delta M\\ \Delta T\\ \Delta S\\ 0\end{bmatrix} \text{ and in the matrix } D = \begin{bmatrix}0&0&0&0&0&0&\Delta w\\ 0&0&0&0&0&0&\Delta\varphi\\ 0&0&0&0&0&0&\Delta\Psi\\ 0&0&0&0&0&0&\Delta M\\ 0&0&0&0&0&0&\Delta T\\ 0&0&0&0&0&0&\Delta S\\ 0&0&0&0&0&0&0\end{bmatrix} \quad (8)$$

The positive directions of the Δ-quantities are shown in figure 6.

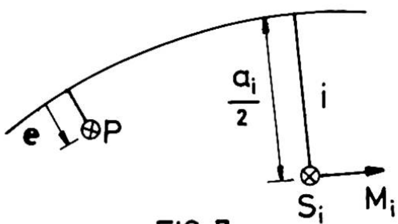

FIG.7

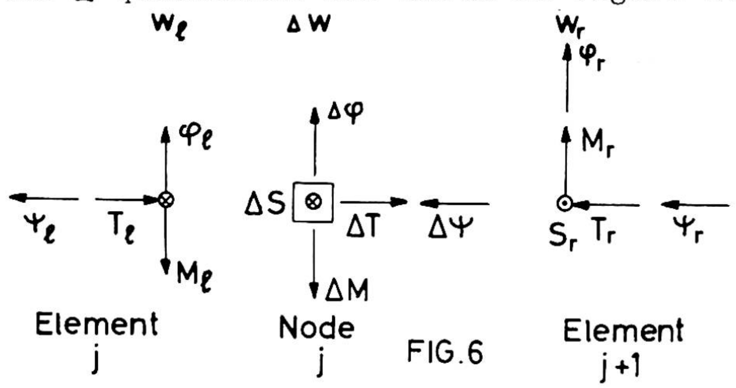

FIG.6

If the load consists of a downward force P acting with the eccentricity e (positive toward the center of curvature) with respect to the center line (fig. 7) : $\Delta S = P$ and $\Delta T = Pe$. At the junction of the girder with the cross beam i : $\Delta S = S_i$ and

$\Delta T = M_i + \frac{a_i}{2} S_i$ (S_i and M_i : shearing force and bending moment at mid span of the transverse beam i).

Denoting by L_j the boundary vector and by D_j the boundary matrix associated with the node j, that separates the girder elements j and j+1, and observing that the quantities on the right hand, resp. the left hand side of figure 6 are the components of the vector $V_{j+1,\ell}$ relating to the left end of the element j+1, resp. of the vector V_{jr} relating to the right end of the element j, one sees from considerations of geometry and equilibrium that

$V_{j+1,\ell} = L_j + V_{jr}$.

Using equation 7, we obtain $V_{j+1,\ell} = L_j + B_j V_{j\ell}$ (9)

Span transmission matrix

Numbering 1 to n the elements in any given span k of the box girder (fig. 8), we may write $V_{nr} = B_n V_{n\ell}$ and then repeatedly apply equation 9:

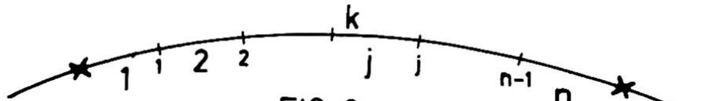

FIG. 8

$$V_{nr} = B_n V_{n\ell} = B_n\left(L_{n-1}+B_{n-1}V_{n-1,\ell}\right) = B_n\left(L_{n-1}+B_{n-1}\left(L_{n-2}+B_{n-2}V_{N-2,\ell}\right)\right)$$
$$= \ldots = B_n\left(L_{n-1}+B_{n-1}\left(L_{n-2}+B_{n-2}\left(\cdots\cdots\left(L_2+B_2\left(L_1+B_1V_{1\ell}\right)\right)\right)\right)\right)$$

Since $D_jV_{1\ell} = L_j$ $(j=1,2,\ldots,n-1)$, the above equation may be transformed into $V_{nr} = B_n\left(D_{n-1}+B_{n-1}\left(D_{n-2}+B_{n-2}\left(\cdots\left(D_2+B_2\left(D_1+B_1\right)\right)\right)\right)\right) V_{1\ell}$

or $$V_{nr} = U_kV_{1\ell} \qquad (10)$$

with the span transmission matrix U_k defined by
$$U_k = B_n\left(D_{n-1}+B_{n-1}\left(D_{n-2}+B_{n-2}\left(\cdots\left(D_2+B_2\left(D_1+B_1\right)\right)\right)\right)\right) \qquad (11)$$

Now denoting by V_{kr} and $V_{k\ell}$ the vectors V pertaining to the right end and to the left end of the span k, equation 10 is identical with $V_{kr} = U_kV_{k\ell}$ (12)

The product of any two matrices B has zero elements and unit elements in the same places as the matrices B themselves. Hence, the span transmission matrix is of the type

$$U_k = \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} & u_{26} & u_{27} \\ 0 & u_{32} & u_{33} & u_{34} & u_{35} & u_{36} & u_{37} \\ 0 & 0 & 0 & u_{44} & u_{45} & u_{46} & u_{47} \\ 0 & 0 & 0 & u_{54} & u_{55} & u_{56} & u_{57} \\ 0 & 0 & 0 & 0 & 0 & 1 & u_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If all the loads on the box girder, including the shearing forces S_i and bending moments M_i in the tie beams, and the displacement discontinuities Δw, $\Delta\varphi$ and $\Delta\Psi$, if any, are known, equation 11 yields the numerical value of the 24 elements u of every one of the p matrices U_k (p is the number of spans of the box girder - fig.2).

Expanding equation 12, we obtain

$$w_{kr} = w_{k\ell} + u_{12}\varphi_{k\ell} + u_{13}\Psi_{k\ell} + u_{14}M_{k\ell} + u_{15}T_{k\ell} + u_{16}S_{k\ell} + u_{17} \qquad (14)$$
$$\varphi_{kr} = u_{22}\varphi_{k\ell} + u_{23}\Psi_{k\ell} + u_{24}M_{k\ell} + u_{25}T_{k\ell} + u_{26}S_{k\ell} + u_{27} \qquad (15)$$
$$\Psi_{kr} = u_{32}\varphi_{k\ell} + u_{33}\Psi_{k\ell} + u_{34}M_{k\ell} + u_{35}T_{k\ell} + u_{36}S_{k\ell} + u_{37} \qquad (16)$$
$$M_{kr} = u_{44}M_{k\ell} + u_{45}T_{k\ell} + u_{46}S_{k\ell} + u_{47} \qquad (17)$$
$$T_{kr} = u_{54}M_{k\ell} + u_{55}T_{k\ell} + u_{56}S_{k\ell} + u_{57} \qquad (18)$$
$$S_{kr} = S_{k\ell} + u_{67} \qquad (19)$$

Girder transmission matrix

1) All supports coincide with the center line of the girder

Then $w_{kr} = w_{k\ell} = 0$ for all values of k. We obtain φ_{kr} , ψ_{kr} , M_{kr} and T_{kr} as functions of $\varphi_{k\ell}$, $\psi_{k\ell}$, $M_{k\ell}$ and $T_{k\ell}$ only by adding 14 successively to 15, 16, 17 and 18, after multiplying 14 with respectively $-\frac{u_{26}}{u_{16}}$, $-\frac{u_{36}}{u_{16}}$, $-\frac{u_{46}}{u_{16}}$ and $-\frac{u_{56}}{u_{16}}$. The resulting expressions may be written as the matrix equation

$$\begin{bmatrix} \varphi_{kr} \\ \psi_{kr} \\ M_{kr} \\ T_{kr} \\ 1 \end{bmatrix} = \begin{bmatrix} u_{22}-\frac{u_{26}}{u_{16}}u_{12} & u_{23}-\frac{u_{26}}{u_{16}}u_{13} & u_{24}-\frac{u_{26}}{u_{16}}u_{14} & u_{25}-\frac{u_{26}}{u_{16}}u_{15} & u_{27}-\frac{u_{26}}{u_{16}}u_{17} \\ u_{32}-\frac{u_{36}}{u_{16}}u_{12} & u_{33}-\frac{u_{36}}{u_{16}}u_{13} & u_{34}-\frac{u_{36}}{u_{16}}u_{14} & u_{35}-\frac{u_{36}}{u_{16}}u_{15} & u_{37}-\frac{u_{36}}{u_{16}}u_{17} \\ -\frac{u_{46}}{u_{16}}u_{12} & -\frac{u_{46}}{u_{16}}u_{13} & u_{44}-\frac{u_{46}}{u_{16}}u_{14} & u_{45}-\frac{u_{46}}{u_{16}}u_{15} & u_{47}-\frac{u_{46}}{u_{16}}u_{17} \\ -\frac{u_{56}}{u_{16}}u_{12} & -\frac{u_{56}}{u_{16}}u_{13} & u_{54}-\frac{u_{56}}{u_{16}}u_{14} & u_{55}-\frac{u_{56}}{u_{16}}u_{15} & u_{57}-\frac{u_{56}}{u_{16}}u_{17} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \varphi_{k\ell} \\ \psi_{k\ell} \\ M_{k\ell} \\ T_{k\ell} \\ 1 \end{bmatrix} \qquad (20)$$

or, denoting the two column matrices in this equation by W_{kr} and $W_{k\ell}$, and the square matrix by F_k :

$$W_{kr} = F_k W_{k\ell} \qquad (21)$$

Neither of the quantities φ, ψ , M and T varies suddenly at any intermediate support. Therefore $W_{k\ell} = W_{k-1,r}$ $(k = 2,3,\ldots,p)$ and

$$W_{pr} = F_p W_{p\ell} = F_p F_{p-1} W_{p-1,\ell} = F_p F_{p-1} F_{p-2} W_{p-2,\ell} = \cdots\cdots$$
$$= F_p F_{p-1} F_{p-2} \cdots\cdots F_2 F_1 W_{1\ell}$$

or

$$W_{pr} = Z W_{1\ell} \qquad (22)$$

with the girder transmission matrix defined by

$$Z = F_p F_{p-1} F_{p-2} \cdots\cdots F_2 F_1 \qquad (23)$$

Z is a 5x5 matrix ; its fifth row consists of four zero elements and one unit element.

2) Some or all of the supports are located off the center line of the box girder

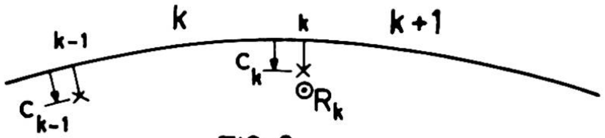

FIG.9

The eccentricity of support k, between the spans k and k+1, towards the center of curvature of the girder is denoted by c_k (fig. 9). The downward movement of point k on the center line is $w_{kr} = c_k\psi_{kr} = c_k\psi_{k+1,\ell}$, and that of point k-1 is $w_{k\ell} = c_{k-1}\psi_{k\ell}$. Equation 14 thus becomes

$$c_k\psi_{kr} = u_{12}\varphi_{k\ell} + \left(c_{k-1}+u_{13}\right)\psi_{k\ell} + u_{14}M_{k\ell} + u_{15}T_{k\ell} + u_{16}S_{k\ell} + u_{17} \qquad (24)$$

We multiply both members of 16 with c_k and subtract from 24 to obtain an equation that we solve for $S_{k\ell}$:

$$S_{k\ell} = -\frac{u_{12} - c_k u_{32}}{u_{16} - c_k u_{36}}\varphi_{k\ell} - \frac{c_{k-1} + u_{13} - c_k u_{33}}{u_{16} - c_k u_{36}}\psi_{k\ell} - \frac{u_{14} - c_k u_{34}}{u_{16} - c_k u_{36}}M_{k\ell}$$
$$- \frac{u_{15} - c_k u_{35}}{u_{16} - c_k u_{36}}T_{k\ell} - \frac{u_{17} - c_k u_{37}}{u_{16} - c_k u_{36}} \qquad (25)$$

Substitution of this expression for $S_{k\ell}$ in 15, 16, 17 and 18 yields a set of four equations that may be written in matrix form :

$$\begin{bmatrix}\varphi_{kr}\\ \psi_{kr}\\ M_{kr}\\ T_{kr}\\ 1\end{bmatrix} = \begin{bmatrix} h_{22} & h_{23} & h_{24} & h_{25} & h_{27}\\ h_{32} & h_{33} & h_{34} & h_{35} & h_{37}\\ h_{42} & h_{43} & h_{44} & h_{45} & h_{47}\\ h_{52} & h_{53} & h_{54} & h_{55} & h_{57}\\ 0 & 0 & 0 & 0 & 1\end{bmatrix} \times \begin{bmatrix}\varphi_{k\ell}\\ \psi_{k\ell}\\ M_{k\ell}\\ T_{k\ell}\\ 1\end{bmatrix} \qquad (26)$$

or
$$W_{kr} = H_k W_{k\ell} \qquad (27)$$

with $$h_{ij} = u_{ij} - \frac{u_{1j} - c_k u_{3j}}{u_{16} - c_k u_{36}}u_{i6} \quad (i=2,3,4,5\ ;\ j=2,3,4,5,7) \qquad (28)$$

on the understanding that $u_{42} = u_{43} = u_{52} = u_{53} = 0$ and that for $j=3$ $c_{k-1}+u_{13}$ must be substituted for u_{13} in the numerator.
Obviously all the elements u appearing in the matrices U_k and F_k and in the equations 25 and 28 relate to span k of the girder, although this is not specified explicitly in the notation.
The matrices H_k and F_k are identical when $c_{k-1} = c_k = 0$.

Equilibrium of the infinitely short portion of the girder resting on support k (fig. 10) requires $S_{k+1,\ell} = S_{kr} - R_k$ and $T_{k+1,\ell} = T_{kr} - c_k R_k$. Elimination of R_k leads to $T_{kr} - T_{k+1,\ell} = c_k\left(S_{kr} - S_{k+1,\ell}\right)$. For T_{kr} we substitute in this equation the expression included in equation 26. For S_{kr} we substitute, in accordance with equation 19, $u_{67} + S_{k\ell}$, with expression 25 substituted for $S_{k\ell}$. For $S_{k+1,\ell}$ we substitute expression 25, written for span k+1 instead of span k ; we differentiate the elements u pertaining to span k+1 from those pertaining to span k by denoting the former by u' ; the quantities $\varphi_{k+1,\ell}$, $\psi_{k+1,\ell}$ and $M_{k+1,\ell}$ appearing in the expression for $S_{k+1,\ell}$ are equal to φ_{kr} , ψ_{kr} and M_{kr} , and consequently they can be written as functions of $\varphi_{k\ell}$, $\psi_{k\ell}$, $M_{k\ell}$ and $T_{k\ell}$ by means of equation 26.

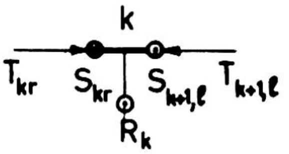

FIG.10

The operations described finally yield an equation that we solve for $T_{k+1,\ell}$: $T_{k+1,\ell} = \bar{h}_{52}\varphi_{k\ell} + \bar{h}_{53}\psi_{k\ell} + \bar{h}_{54}M_{k\ell} + \bar{h}_{55}T_{k\ell} + \bar{h}_{57}$ (29)

the quantities $\bar{h}_{5j}$ being defined by

$$\left(1 + c_k \frac{u'_{15} - c_{k+1}u'_{35}}{u'_{16} - c_{k+1}u'_{36}}\right)\bar{h}_{5j} = h_{5j} + c_k \frac{u_{1j} - c_k u_{3j}}{u_{16} - c_k u_{36}} - \frac{c_k}{u'_{16} - c_{k+1}u'_{36}} \cdot$$
$$\left[h_{2j}\left(u'_{12} - c_{k+1}u'_{32}\right) - h_{3j}\left(c_k + u'_{13} - c_{k+1}u'_{33}\right) - h_{4j}\left(u'_{14} - c_{k+1}u'_{34}\right)\right] \quad (30)$$

on the understanding that $u_{13} + c_{k-1}$ must be substituted for u_{13} and that, for $j = 7$, $-c_k\left(u_{67} + \frac{u'_{17} - c_{k+1}u'_{37}}{u'_{16} - c_{k+1}u'_{36}}\right)$ must be added to the second member of equation 30.

The first three relationships contained in 26 may be assembled with equation 29 into the matrix equation

$$\begin{bmatrix} \varphi_{k+1,\ell} \\ \psi_{k+1,\ell} \\ M_{k+1,\ell} \\ T_{k+1,\ell} \\ 1 \end{bmatrix} = \begin{bmatrix} h_{22} & h_{23} & h_{24} & h_{25} & h_{27} \\ h_{32} & h_{33} & h_{34} & h_{35} & h_{37} \\ h_{42} & h_{43} & h_{44} & h_{45} & h_{47} \\ \bar{h}_{52} & \bar{h}_{53} & \bar{h}_{54} & \bar{h}_{55} & \bar{h}_{57} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \varphi_{k\ell} \\ \psi_{k\ell} \\ M_{k\ell} \\ T_{k\ell} \\ 1 \end{bmatrix} \quad (31)$$

or

$$W_{k+1,\ell} = \bar{H}_k W_{k\ell} \quad (32)$$

The matrices $\bar{H}_k$ and F_k are identical when $c_{k-1} = c_k = 0$.

Now we successively apply equation 27 for $k = p$ and equation 32 for $k = p-1, p-2, \ldots, 2, 1$:

$$W_{pr} = H_p W_{p\ell} = H_p \bar{H}_{p-1} W_{p-1,\ell} = H_p \bar{H}_{p-1} \bar{H}_{p-2} W_{p-2,\ell} = \ldots\ldots$$
$$= H_p \bar{H}_{p-1} \bar{H}_{p-2} \ldots\ldots \bar{H}_2 \bar{H}_1 W_{1\ell}$$

or

$$W_{pr} = Z W_{1\ell} \quad (22)$$

with the girder transmission matrix defined by

$$Z = H_p \bar{H}_{p-1} \bar{H}_{p-2} \ldots\ldots \bar{H}_2 \bar{H}_1 \quad (33)$$

End support conditions

We expand matrix equation 22 :

$$\begin{aligned} \varphi_{pr} &= z_{11}\varphi_{1\ell} + z_{12}\psi_{1\ell} + z_{13}M_{1\ell} + z_{14}T_{1\ell} + z_{15} \\ \psi_{pr} &= z_{21}\varphi_{1\ell} + z_{22}\psi_{1\ell} + z_{23}M_{1\ell} + z_{24}T_{1\ell} + z_{25} \\ M_{pr} &= z_{31}\varphi_{1\ell} + z_{32}\psi_{1\ell} + z_{33}M_{1\ell} + z_{34}T_{1\ell} + z_{35} \\ T_{pr} &= z_{41}\varphi_{1\ell} + z_{42}\psi_{1\ell} + z_{43}M_{1\ell} + z_{44}T_{1\ell} + z_{45} \end{aligned} \quad (34)$$

1) Both end supports coincide with the center line of the girder

Various boundary conditions at the left end support may occur :

a) The end of the girder rotates freely about its tangent and also in the vertical plane containing the tangent : $T_{1\ell} = M_{1\ell} = 0$

b) The end of the girder cannot rotate in either direction : $\psi_{1\ell} = \varphi_{1\ell} = 0$

c) The end of the girder rotates freely in one of those directions, but is fixed in the other direction :
either $\psi_{1\ell} = 0$ and $M_{1\ell} = 0$
or $T_{1\ell} = 0$ and $\varphi_{1\ell} = 0$ (unlikely combination).

One of these pairs of conditions obtains at the right end support.

Whatever the combination of end support conditions may be, two terms are zero in the right hand member of each equation 34. Moreover, the left hand member of two equations is also zero. These may be solved for the two unknown components of the vector $W_{1\ell}$.

2) The end supports are located off the center line of the girder

In this case practically the only boundary conditions imaginable are : free rotation of the girder ends about their tangent and in the vertical plane containing the tangent. The following relationships then obtain :
$M_{1\ell} = 0$ and $T_{1\ell} = -c_o R_o = +c_o S_{1\ell}$ at the left end support, and
$M_{pr} = 0$ and $T_{pr} = +c_p R_p = +c_p S_{pr}$ at the right end support.

We write equation 25 for $k = 1$, multiply by c_o , replace the left hand member $c_o S_{1\ell}$ by $T_{1\ell}$, and thus obtain a relationship containing only $\varphi_{1\ell}$, $\psi_{1\ell}$ and $T_{1\ell}$. A second such relationship is provided by the third equation 34, with $M_{1\ell} = M_{pr} = 0$. To obtain a third, we premultiply both members of equation 12, written for $k = p$, by U_p^{-1} : $V_{p\ell} = U_p^{-1} V_{pr}$, extract from this equation the expressions for $w_{p\ell}$ and $\psi_{p\ell}$ as functions of $w_{pr} = c_p \psi_{pr}$, φ_{pr} , ψ_{pr} , $M_{pr} = 0$, T_{pr} and S_{pr} , write that the former expression is equal to c_{p-1} times the latter expression, substitute $\frac{T_{pr}}{c_p}$ for S_{pr} in the resulting equation, and so arrive at a relationship between φ_{pr} , ψ_{pr} and T_{pr} , that we transform into a relationship $\varphi_{1\ell}$, $\psi_{1\ell}$ and $T_{1\ell}$ by using the first, second and fourth equations 34. Thus we finally have three equations that we may solve for $\varphi_{1\ell}$, $\psi_{1\ell}$ and $T_{1\ell}$.

Displacement components and stress resultants

Whether the end supports coincide with the center line of the girder or not, we know vector $W_{1\ell}$ completely after having performed the computations just described. We may now calculate $W_{k\ell}$ for all other values of k by repeatedly using equation 32. Equation 25 then yields the numerical value of $S_{k\ell}$ for $k = 1, 2, \ldots, p$. Since $w_{k\ell} = +c_{k-1} \psi_{k\ell}$, we know all the vectors $V_{k\ell}$ and are able to compute the vector V pertaining to any node in any span by means of equation 9.

Deformation induced by direct load - Deformation influence coefficients - Actual deflections and rotations

The above analysis is used to calculate the deflections w of the outside box girder and the rotations ψ about the tangent at the nodes coinciding with the m cross beams (fig. 2) for 2m + 1 different loading conditions. We denote

by w_{io} $(i=1,2,\ldots,m)$ the deflection and by ψ_{io} the rotation at the junction with cross beam i, produced by the given external loading on the box girder ;

by w_{ix} $(i=1,2,\ldots,m\ ;\ x=1,2,\ldots,m)$ the deflection and by ψ_{ix} the rotation produced at the same point by a downward force $\frac{2}{a_x}$ applied at the junction with cross beam x (fig. 11) ;

by $w_{ix'}$ the deflection and by $\psi_{ix'}$ the rotation produced at the same point by a unit moment acting at the junction with cross beam x, as shown in figure 12.

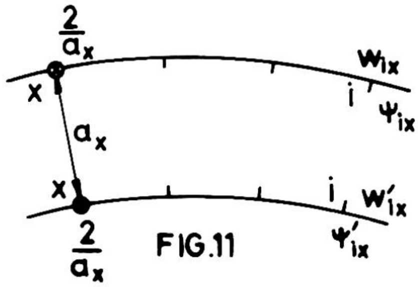

FIG.11

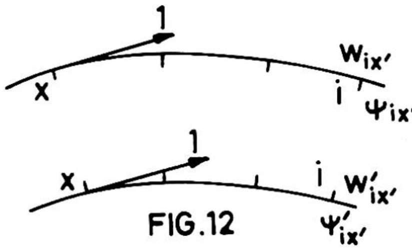

FIG.12

The corresponding quantities for the inside girder are calculated likewise. They are denoted by the same symbols, marked off by an ' (figures 11 and 12).

Maxwell's reciprocal theorem shows that the calculated deflection and rotation influence coefficients must satisfy the relations $a_x w_{ix} = a_i w_{xi}$; $a_x \psi_{ix} = -2 w_{xi'}$; $\psi_{ix'} = \psi_{xi'}$.

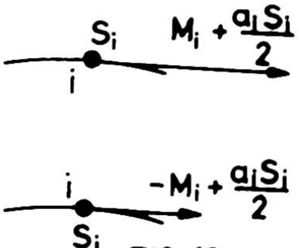

FIG. 13

The action of any cross beam i on the outside main girder consists of the downward force S_i and the moment $M_i + \frac{a_i S_i}{2}$, as shown in figure 13 ; its action on the inside girder consists of the upward force S_i and the moment $-M_i + \frac{a_i S_i}{2}$.

The actual deflection w_i and rotation ψ_i of the outside main girder at its junction with cross beam i under the influence of the given loading acting on the complete structural system are given by

$$w_i = w_{io} + \sum_{x=1}^{m}\left[\frac{a_x}{2} w_{ix} S_x + w_{ix'}\left(M_x + \frac{a_x S_x}{2}\right)\right]$$

$$= w_{io} + \sum_{x=1}^{m}\left[\left(w_{ix} + w_{ix'}\right)\frac{a_x S_x}{2} + w_{ix'} M_x\right] \qquad (35)$$

$$\psi_i = \psi_{io} + \sum_{x=1}^{m}\left[\frac{a_x}{2}\psi_{ix}S_x + \psi_{ix'}\left(M_x + \frac{a_xS_x}{2}\right)\right]$$

$$= \psi_{io} + \sum_{x=1}^{m}\left[\left(\psi_{ix} + \psi_{ix'}\right)\frac{a_xS_x}{2} + \psi_{ix'}M_x\right] \qquad (36)$$

The actual deflection w'_i and rotation ψ'_i of the inside main girder are likewise

$$w'_i = w'_{io} + \sum_{x=1}^{m}\left[\left(-w'_{ix} + w'_{ix'}\right)\frac{a_xS_x}{2} - w'_{ix'}M_x\right] \qquad (37)$$

$$\psi'_i = \psi'_{io} + \sum_{x=1}^{m}\left[\left(-\psi'_{ix} + \psi'_{ix'}\right)\frac{a_xS_x}{2} - \psi'_{ix'}M_x\right] \qquad (38)$$

Deformation of the cross beams

The loads acting on any tie beam i and emanating from the main girders are shown in figure 14. They bring about the end deflections w_i and w'_i and end rotations ψ_i and ψ'_i. Assuming that the braces between the box girders are full-webbed beams and not trusses, and that the shear deformation is therefore negligible with respect to the flexural deformation, one easily derives the following relations :

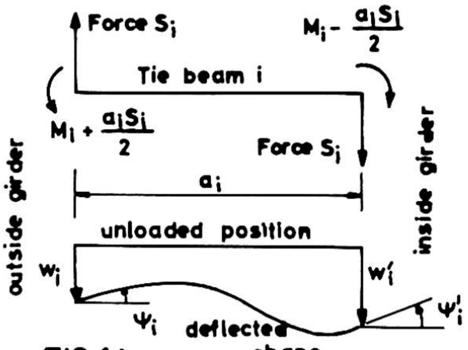

FIG.14

$$\psi_i - \psi'_i = \frac{a_iM_i}{EI_i} \qquad (39)$$

$$\psi_i + \psi'_i + 2\,\frac{w'_i - w_i}{a_i} = \frac{a_i^2S_i}{6EI_i} \qquad (40)$$

(I_i : moment of inertia of the tie beam i).

Similar relations between the shear force, the bending moment and the displacement components may be derived for trussed connections between the main girders.

Obtaining the values of the unknowns S_x and M_x

Substituting the expressions 35 to 38 for w_i , ψ_i , w'_i and ψ'_i into 39 and 40 we find

$$\sum_{x=1}^{m}\left[\left(\psi_{ix}+\psi_{ix'}+\psi'_{ix}-\psi'_{ix'}\right)\frac{a_xS_x}{2} + \left(\psi_{ix'}+\psi'_{ix'}\right)M_x\right] - \frac{a_iM_i}{EI_i} = \psi'_{io} - \psi_{io} \qquad (41)$$

$$(i=1,2,\ldots,m)$$

$$\sum_{x=1}^{m}\left\{\left[\psi_{ix} + \psi_{ix'} - \psi'_{ix} + \psi'_{ix'} - \frac{2}{a_i}\left(w_{ix}+w_{ix'}+w'_{ix}-w'_{ix'}\right)\right]\frac{a_xS_x}{2} + \left[\psi_{ix'}-\psi'_{ix'} - \frac{2}{a_i}\left(w_{ix'} + w'_{ix'}\right)\right]M_x\right\}$$

$$- \frac{a_i}{3EI_i}\cdot\frac{a_iS_i}{2} = -\psi_{io} - \psi'_{io} + \frac{2}{a_i}\left(w_{io} - w'_{io}\right) \quad (i=1,2,\ldots,m) \qquad (42)$$

This set of 2m simultaneous algebraic equations may be solved for the 2m unknowns $\frac{a_x S_x}{2}$ and M_x . The matrix to be inverted is of the order 2m, while the structure is 2m+p+p'-4 times statically indeterminate (p' : number of spans of the inside girder), if the ends of the main girders are free to rotate.

When the shearing forces and bending moments in the bracing are known, it is easy to determine the final internal stress resultants and displacement components for the box girders, either by superposing for each one 2m+1 cases of loading already analysed, or by analysing each box girder separately under its full loading, including the forces and moments emanating from the bracing.

The complexity of the structural system considered is such that the calculation is hardly feasible without a computer.

Remark about the location of the connecting beams

It was assumed implicitly in the above that the braces do not meet the box girders at the supports. Yet, it is almost natural for some tie beams to be connected with the main girders at supported cross sections. This situation can be handled in the analysis by assuming that the junction of any such cross beam is located beside, but close to the support, leaving an infinitely short girder element between itself and the support.

SUMMARY

The structural system described in the title is analysed by applying the transmission (or reduction) method to the curved main girders separately, thus obtaining deformation components and influence coefficients, and by using the force method to find the shear forces and bending moments in the connecting beams.

RÉSUMÉ

Les deux poutres maitresses courbes sont d'abord étudiées séparément par la méthode de transmission ou de réduction, ce qui fournit les composantes de la déformation de ces poutres, ainsi que les coefficients d'influence de ces composantes. Ensuite les efforts tranchants et les moments fléchissants dans les traverses sont déterminés au moyen de la méthode des forces.

ZUSAMMENFASSUNG

Die zwei gekrümmten Kastenträger werden zunächst gesondert mit dem Ubertragungsverfahren studiert. Diese Berechnung liefert die Verformungskomponenten, sowie Einfluszahlen für diese Komponenten. Nachher werden die Querkraft und das Biegemoment in den Querbalken mit dem Kraftverfahren ermittelt.