Initially deflected thin plate with initial deflection affine to additional deflection

Initially Deflected Thin Plate with Initial Deflection Affine to Additional Deflection

Die anfänglich gekrümmte Platte mit zur zusätzlichen Einsenkung affiner anfänglicher Verformung

La plaque initialement incurvée avec déformation initiale affine du fléchissement ultérieur

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Introduction

It is generally known that there is no linear relation between the load and the deformations, or between the load and the stresses, in the case of thin plates. The stress distribution is dependent on the type and the magnitude of the deformations. Therefore, it is not possible to disregard the effect of an initial deflection if it is of the same order of magnitude as the additional deflection caused by the load. Since thin plates are often subjected to initial deflections, it is desirable to acquire knowledge of the general effect of initial deflections, as this knowledge is useful in drawing up design rules, and frequently also in the interpretation of test results and in the design of measuring instruments.

The most probable form of the initial deflection is difficult to determine in advance. As will be shown below, the problem can be considerably simplified if the initial deflection is assumed to be affine with the additional deflection If the treatment of the problem is confined to this special case, it will also afford information on the general effect of other types of initial deflections. The fact that the form of the initial deflection is supposed to vary with the thickness of the plate at a given load and with the load at a given thickness — which is a consequence of the assumption that the initial deflection and the additional deflection are affine — is of minor importance in studying the general effect of initial deflections.

Fundamental Relations

We use as a point of departure the fundamental equations deduced by MARGUERRE for an initially deformed thin plate. These equations represent a further development of VON KÁRMÁN'S equations for an initially plane thin plate [1]).

If use is made of the notations:

Φ = Airy's stress function for a plane state of stress (in the x-y-plane),
w = the deflection of the plate due to the load (additional deflection) (in the z-direction),
W = the initial deflection,
h = the thickness of the plate,
N = $\frac{E h^3}{12(1-\nu^2)}$ = the stiffness of the plate, then MARGUERRE'S equations are as follows [2])

$$\Delta\Delta\Phi = E\left[(w_{xy})^2 - w_{xx}w_{yy} + 2W_{xy}w_{xy} - W_{xx}w_{yy} - W_{yy}w_{xx}\right] \tag{1}$$

$$\Delta\Delta w = \frac{h}{N}\left[\Phi_{yy}(W_{xx}+w_{xx}) + \Phi_{xx}(W_{yy}+w_{yy}) - 2\Phi_{xy}(W_{xy}+w_{xy})\right] - \frac{q(xy)}{N} = 0 \tag{2}$$

In these equations, the partial derivatives are denoted by subindices, viz.,

$$w_{xy} = \frac{\partial^2 w}{\partial x\,\partial y}; \qquad \Phi_{xy} = \frac{\partial^2 \Phi}{\partial x\,\partial y}$$

$$w_{xx} = \frac{\partial^2 w}{\partial x^2}; \qquad \Phi_{xx} = \frac{\partial^2 \Phi}{\partial x^2} \qquad \text{etc.}$$

As the initial deflection is assumed to be affine with the additional deflection we can write

$$W = k\cdot w \tag{3}$$

where k is a constant.

Eqs. (1) and (2) can then be written

$$\Delta\Delta\Phi = E(1+2k)\left[(w_{xy})^2 - w_{xx}w_{yy}\right] \tag{1'}$$

$$\Delta\Delta w - \frac{h}{N}(1+k)\left[\Phi_{yy}w_{xx} + \Phi_{xx}w_{yy} - 2\Phi_{xy}w_{xy}\right] - \frac{q(xy)}{N} = 0 \tag{2'}$$

This manner of writing Eqs. (1) and (2) indicates that a comparison between the equations of the initially deflected plate and those of the initially plane plate can contribute to the solution of the problem.

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[1]) MARGUERRE, K.: Zur Theorie der gekrümmten Platte mit großer Formänderung. Proc. of 5th Int. Congr. for Appl. Mech., Vol. V, p. 93, Cambr., Mass., 1939. — v. KÁRMÁN, Encyklopädie der Math. Wissenschaften. Vol. IV, p. 349, 1910.

[2]) Cf. also BERGMAN, STEN G. A., Behaviour of Buckled Rectangular Plates under the Action of Shearing Forces. Doctor's Thesis, Stockholm 1948, p. 49.

For comparison, we write the equations of an initially plane plate (they are obtained from Eqs. (1) and (2) by putting W equal to 0).

$$\varDelta\varDelta\,\varPhi_0 = E\left[(w_{0_{xy}})^2 - w_{0_{xx}}\,w_{0_{yy}}\right] \tag{1''}$$

$$\varDelta\varDelta\,w_0 - \frac{h_0}{N_0}\left[\varPhi_{0_{yy}}\,w_{0_{xx}} + \varPhi_{0_{xx}}\,w_{0_{yy}} - 2\,\varPhi_{0_{xy}}\,w_{0_{xy}}\right] - \frac{q_0\,(x\,y)}{N_0} = 0 \tag{2''}$$

This initially plane plate has the same extent in the plane, and the co-ordinate system $x\,y\,z$ coincides with the co-ordinate system used for the initially deflected plate. In comparison with the initially deflected plate, the initially plane plate has a thickness h_0, a flexural rigidity N_0, and a distributed load $q_0\,(x\,y)$, which are so far unknown. The deflection is denoted by w_0 and Airy's stress function is designated by $\varPhi_0$.

A comparison between Eqs. (1′) and (2′) on the one hand and Eqs. (1″) and (2″) on the other hand shows that the functions $\varPhi$ and $\varPhi_0$, as well as w and w_0 are affine under certain conditions. Therefore, we shall determine the conditions which must be fulfilled in order that the relations

$$\varPhi_0 = c_1\,\varPhi \tag{4}$$

$$w_0 = c_2\,w \tag{5}$$

where c_1 and c_2 are constants, shall hold good.

If use is made of the notations given by Eqs. (4) and (5), then Eqs. (1″) and (2″) become

$$\varDelta\varDelta\,\varPhi = \frac{(c_2)^2}{c_1}\,E\left[(w_{xy})^2 - w_{xx}\,w_{yy}\right] \tag{1'''}$$

$$\varDelta\varDelta\,w - c_1\,\frac{h_0}{N_0}\left[\varPhi_{yy}\,w_{xx} + \varPhi_{xx}\,w_{yy} - 2\,\varPhi_{xy}\,w_{xy}\right] - \frac{1}{c_2}\,\frac{q_0\,(x\,y)}{N_0} = 0 \tag{2'''}$$

A comparison between Eqs. (1′) and (2′) on the one hand and Eqs. (1‴) and (2‴) on the other hand yields the following necessary conditions which must be satisfied in order that Eqs. (4) and (5) shall be fulfilled

$$\frac{(c_2)^2}{c_1} = (1 + 2\,k) \tag{6}$$

$$c_1\,\frac{h_0}{N_0} = (1 + k)\,\frac{h}{N} \tag{7}$$

$$\frac{1}{c_2}\,\frac{q_0\,(x\,y)}{N_0} = \frac{q\,(x\,y)}{N} \tag{8}$$

Since

$$N_0 = \frac{E\,h_0^{\,3}}{12\,(1-\nu^2)}$$

$$N = \frac{E\,h^3}{12\,(1-\nu^2)}$$

it follows that the three equations (6) to (8) can be satisfied by an appropriate choice of the relations between the quantities

$$c_1\,,\quad c_2\,,\quad \frac{h_0}{h}\,,\quad \text{and}\quad \frac{q_0\,(x\,y)}{q\,(x\,y)}\,.$$

If Eqs. (6), (7) and (8) are satisfied, then Eq. (1‴) is identically equal to Eq. (1′) and Eq. (2‴) is identically equal to Eq. (2′).

In that case, a solution of the fundamental equations of the initially plane plate also involves a solution of the fundamental equations of the initially deflected plate.*)

The fact that the fundamental equations are satisfied implies that the conditions for equilibrium and continuity are fulfilled.

On the other hand, it is not certain beforehand that the boundary conditions for the initially deflected plate are fulfilled if they are satisfied for the initially plane plate. This question should therefore be examined separately in each special case.

One or several of the boundary conditions are usually expressed as conditions for the deformation components u and v in the plane of the plate. Before passing to the examples, we shall therefore study the relations between u and v and between the deflection w of the plate and the stress components.

If the plate is initially plane, these relations are [3])

$$E\left[u_{0\,x} + \frac{w_{0\,x}^{\,2}}{2}\right] = \sigma_{x\,0} - \nu\,\sigma_{y\,0} \tag{9}$$

$$E\left[v_{0\,y} + \frac{w_{0\,y}^{\,2}}{2}\right] = \sigma_{y\,0} - \nu\,\sigma_{x\,0} \tag{10}$$

$$G\,[u_{0\,y} + v_{0\,x} + w_{0\,x}\,w_{0\,y}] = \tau_0 \tag{11}$$

For an initially deformed plate, the corresponding relations are [3])

$$E\left[u_x + \frac{w_x^{\,2}}{2}\,(1+2\,k)\right] = \sigma_x - \nu\,\sigma_y \tag{9'}$$

$$E\left[v_y + \frac{w_y^{\,2}}{2}\,(1+2\,k)\right] = \sigma_y - \nu\,\sigma_x \tag{10'}$$

$$G\,[u_y + v_x + w_x\,w_y\,(1+2\,k)] = \tau \tag{11'}$$

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*) The fundamental equations (1′), (2′), (1″), (2″), (1‴) and (2‴) can also be written as relations between $\frac{\Phi}{h^2}$, $\frac{w}{h}$ and $\frac{q\,(xy)}{h\cdot N}$ and between $\frac{\Phi_0}{h_0^{\,2}}$, $\frac{w_0}{h_0}$, $\frac{q_0\,(xy)}{h_0 N_0}$. Then the eqs. (4) and (5) may be written $\frac{\Phi_0}{h_0^{\,2}} = \alpha\,\frac{\Phi}{h^2}$ resp $\frac{w_0}{h_0} = \beta\,\frac{w}{h}$. In this way the results in the examples, which are given in dimensionless form, may be obtained more directly.

[3]) Cf. MARGUERRE, footnote [1]).

By integrating the relations (9) to (10) and (9′) to (10′), and in virtue of Eqs. (4) to (6), (11) and (11′) it can be shown that*)

$$u = \frac{1}{c_1} u_0 + A \cdot y + B = \frac{1}{c_1} u_0 \tag{12}$$

$$v = \frac{1}{c_1} v_0 - A \cdot x + C = \frac{1}{c_2} v_0 \tag{13}$$

Consequently, the deformations in the plane of the initially deformed plate are affine to the deformations in the plane of the initially plane plate, if the conditions in Eqs. (6)–(8) are fulfilled.

Example No. 1

Rectangular plate submitted to a uniformly distributed load. The plate is clamped at the edges. The deformations in the plane of the plate are equal to zero along the edges (Fig. 1).

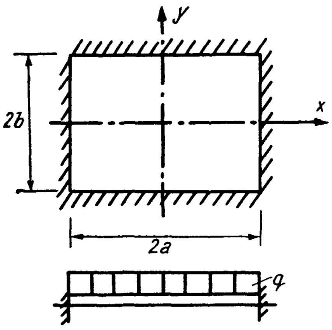

Fig. 1. Notations and co-ordinate system used in Example No. 1

The boundary conditions are

$$\left.\begin{aligned} x &= -a \\ x &= +a \end{aligned}\right\} \quad \begin{aligned} u &= 0 \\ v &= 0 \end{aligned} \qquad w = 0 \qquad \frac{\partial w}{\partial x} = 0$$

$$\left.\begin{aligned} y &= -b \\ y &= +b \end{aligned}\right\} \quad \begin{aligned} u &= 0 \\ v &= 0 \end{aligned} \qquad w = 0 \qquad \frac{\partial w}{\partial y} = 0$$

In view of the relations (5), (12) and (13), it is seen that if these boundary conditions are fulfilled for the initially plane plate, they are also fulfilled for the initially deflected plate.

We shall determine the deflection as a function of the load.

It is convenient to represent the solutions in a dimensionless form. For an initially plane plate, TIMOSHENKO (Theory of Plates and Shells, p. 348) gives $\frac{w_0}{h_0}$ as a function of

*) The terms with the constants of integration A, B and C represent a small relative rotation and translation of the plates regarded as rigid bodies. The relative position in the x-y-plane is fixed by putting $A = B = C = 0$.

$$\frac{q_0 b^4}{N_0 h_0}.$$

For the initially deflected plate, we obtain from the condition expressed by Eq. (8)

$$\frac{q b^4}{N h} = \frac{1}{c_2} \frac{q_0 b^4}{N_0 h_0} \frac{h_0}{h} \tag{a}$$

From Eq. (7) we determine

$$\frac{h_0}{h} = \sqrt{\frac{c_1}{1+k}} \tag{b}$$

Inserting (b) in (a) gives in view of Eq. (6)

$$\frac{q b^4}{N h} = \frac{q_0 b^4}{N_0 h_0} \frac{1}{\sqrt{(1+2k)(1+k)}} \tag{c}$$

Furthermore, Eqs. (5) and (7) yield the relation

$$\frac{w}{h} = \frac{w_0}{h_0} \cdot \frac{1}{c_2} \frac{h_0}{h}$$

or

$$\frac{w}{h} = \frac{w_0}{h_0} \frac{1}{\sqrt{(1+2k)(1+k)}} \tag{d}$$

The constant k is determined as follows.

The maximum initial deflection W_{max} is expressed in terms of the thickness of the plate by the relation

$$W_{max} = \alpha \cdot h \tag{e}$$

Then we obtain from Eq. (3)

$$W_{max} = k\, w_{max} = \alpha \cdot h$$

and from (d)

$$k = \frac{\alpha}{\frac{w_{max}}{h}} = \frac{\alpha}{\frac{w_{0max}}{h_0}} \sqrt{(1+2k)(1+k)} \tag{f}$$

When the values of α and $\frac{w_{0max}}{h_0}$ are known, we can determine k from this equation. After that, $\frac{q b^4}{N h}$ and $\frac{w}{h}$ can be determined in relation to $\frac{q_0 b^4}{N_0 h_0}$ and $\frac{w_0}{h_0}$ respectively from Eqs. (c) and (d).

Fig. 2 shows curves representing $\frac{w_{0max}}{h_0}$ as a function of $\frac{q_0 b^4}{N_0 h_0}$ (these curves are reprinted from the above-mentioned book by TIMOSHENKO).

Fig. 2 shows the construction of $\frac{w_{max}}{h}$ for $W_{max} = \pm \frac{1}{4} h$ i.e. for $\alpha = \pm \frac{1}{4}$ and for $\frac{w_{0max}}{h_0} = 1$.

With the values of k calculated from Eq. (f), we obtain from Eqs. (c) and (d) two points B and C for the initially deformed plate. According to Eqs. (c) and (d), these points are situated on the secant $O-A$ to the point $\frac{w_{0max}}{h_0} = 1$ on the curve for the initially plane plate, and correspond to the positive and negative values of α.

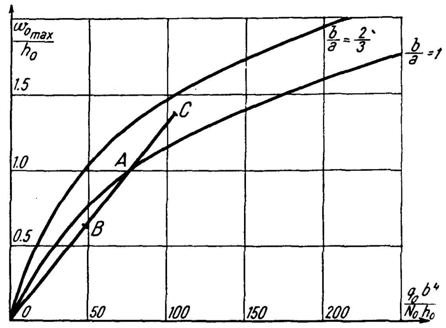

Fig. 2. Variation in $\frac{w_{0max}}{h_0}$ with $\frac{q_0 b^4}{N_0 h_0}$ for an initially plane plate in Example No. 1 determined according to TIMOSHENKO. The points B and C for the initially deformed plate $\left(\frac{a}{b} = 1\right)$ lie on the secant 0—A. The point B corresponds to a positive initial deflection, whereas the point C corresponds to a negative initial deflection

For any other value of $\frac{w_{0max}}{h_0}$, the calculation (or construction) is carried out in an analogous manner.

The tangent at the origin can be determined for the different curves.

From Eqs. (c) and (d) we obtain

$$\left(\frac{\dfrac{w_{max}}{h}}{\dfrac{q\,b^4}{E\,h^4}}\right)_{\frac{q\,b^4}{E\,h^4} = 0} = \left(\frac{\dfrac{w_{0max}}{h_0}}{\dfrac{q_0\,b^4}{E\,h_0^4}}\right)_{\sqrt{(1+2k)(1+k)} = \infty} \tag{g}$$

For $\sqrt{(1+2k)(1+k)} = \infty$, we have $k = \pm\infty$. Eq. (f) gives,

$$\text{for } k = +\infty, \quad \frac{w_{0max}}{h_0} = \alpha\sqrt{2},$$

$$\text{for } k = -\infty, \quad \frac{w_{0max}}{h_0} = -\alpha\sqrt{2}.$$

Then Eq. (g) gives, for the tangent at the origin

$$\left(\frac{\dfrac{w_{max}}{h}}{\dfrac{q\,b^4}{E\,h^4}}\right)_{\frac{q\,b^4}{E\,h^4} = 0} = \left(\frac{\dfrac{w_{0max}}{h_0}}{\dfrac{q_0\,b^4}{E\,h_0^4}}\right)_{\frac{w_{0max}}{h_0} = \pm\alpha\sqrt{2}} \tag{g'}$$

that is to say, for an initially deflected plate, the tangent at the origin consists of the secant to the curve for the initially plane plate at the point $\frac{w_{0max}}{h_0} = \alpha\sqrt{2}$ (positive initial deflection) and $\frac{w_{0max}}{h_0} = -\alpha\sqrt{2}$ (negative initial deflection).

The curves corresponding to the negative initial deflection (α negative) for small $\frac{q\,b^4}{E\,h^4}$ have been omitted in this example because they would impair the legibility of the diagram. (Cf. Example No. 4.)

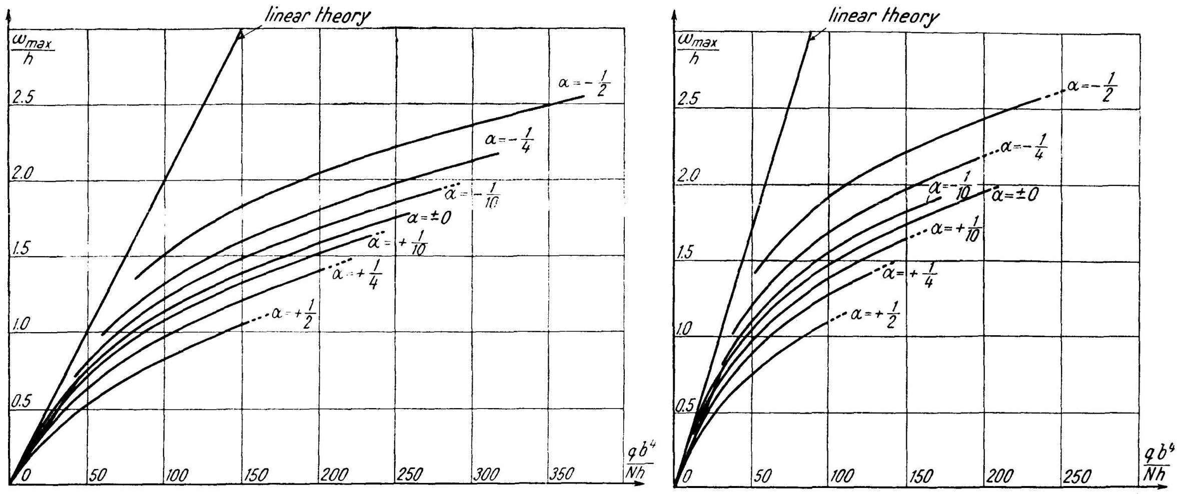

Fig. 3. Deflection at the centre of an initially deflected plate for various values of the initial deflection at the centre of the plate in Example No. 1. The initial deflection at the centre of the plate W_{max} is expressed in terms of the thickness h of the plate by the relation $W_{max} = \alpha \cdot h$. When the value of α is negative, this implies that the direction of the initial deflection is opposed to that of the additional deflection. The curves for negative values of α and for small values of $\frac{qb^4}{Nh}$ are omitted in these diagrams.

a) $\frac{b}{a} = 1$ b) $\frac{a}{b} = \frac{2}{3}$

The results are reproduced in Fig. 3 for the plates having the side ratios $\frac{b}{a} = 1{,}0$ and $\frac{b}{a} = \frac{2}{3}$.

The stresses can be determined by a correspondingly simple method. A distinction must however be made between membrane and bending stresses. The procedure to be followed in the determination of the stresses is illustrated by Examples Nos. 3 and 5.

Example No. 2

Rectangular plate subjected to shearing forces applied along the periphery. The plate is simply supported at the edges, which remain straight and do not change in length during the deformation (cf. Fig. 4).

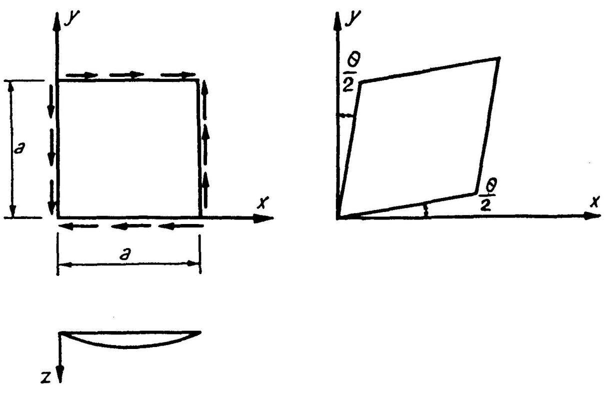

Fig. 4. Notations and coordinate system used in Example No. 2

The boundary conditions are

$$\left.\begin{aligned} x &= 0 \\ x &= a \end{aligned}\right\} \qquad w = 0 \qquad \frac{\partial^2 w}{\partial x^2} = 0$$

$$\left.\begin{aligned} y &= 0 \\ y &= a \end{aligned}\right\} \qquad w = 0 \qquad \frac{\partial^2 w}{\partial y^2} = 0$$

$$x = 0 \qquad u = y\frac{\theta}{2} \qquad v = 0$$

$$x = a \qquad u = y\frac{\theta}{2} \qquad v = a\frac{\theta}{2}$$

$$y = 0 \qquad u = 0 \qquad v = x\frac{\theta}{2}$$

$$y = a \qquad u = a\frac{\theta}{2} \qquad v = x\frac{\theta}{2}$$

In this example, too, it follows from the relations (5), (12) and (13) that if the above boundary conditions are fulfilled for the initially plane plate, they are also fulfilled for the initially deflected plate.

We shall determine the deflection as a function of the change in angle θ. We take as a starting-point the solutions given for the initially plane plate by Sten G. A. Bergman [4]). He has represented

$\frac{w_{0\,max}}{h_0}$ as a function of $\theta_0\frac{a^2}{h_0^2}$, where θ_0 is determined by the relation

$$\theta_0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \cdot \frac{\partial w_0}{\partial y}$$

For the initially deformed plate, we have

$$\theta = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}(1+2k)$$

or, in view of Eqs. (5), (6), (12) and (13),

$$\theta = \frac{1}{c_1}\theta_0$$

Then, in virtue of the relations in Eq. (7), we obtain

$$\theta\frac{a^2}{h^2} = \theta_0\frac{a^2}{h_0^2}\,\frac{1}{c_1}\,\frac{c_1}{1+k} = \theta_0\frac{a^2}{h_0^2}\,\frac{1}{1+k} \tag{a}$$

and, as in Example No. 1,

$$\frac{w}{h} = \frac{w_0}{h_0}\,\frac{1}{\sqrt{(1+2k)(1+k)}} \tag{b}$$

Eq. (f) and the definition in Eq. (e) in Example No. 1 hold in this case too.

4) See footnote 2).

For known values of $\frac{w_{0\,max}}{h_0}$ in Fig. 5 and of α, we calculate k by means of Eq. (f) in Example No. 1. After that, $\theta\frac{a^2}{h^2}$ and $\frac{w_{max}}{h}$ are determined from the above equations (a) and (b).

It is of interest to determine the tangent at the origin. We get

$$\left(\frac{\frac{w_{max}}{h}}{\theta\frac{a^2}{h^2}}\right)_{\substack{\theta\frac{a^2}{h^2}=0\\ \frac{w_{max}}{h}=0}} = \left(\frac{\frac{w_{0\,max}}{h_0}}{\theta_0\frac{a^2}{h_0^2}}\cdot\frac{1+k}{\sqrt{(1+2k)(1+k)}}\right)_{k=\infty} \tag{c}$$

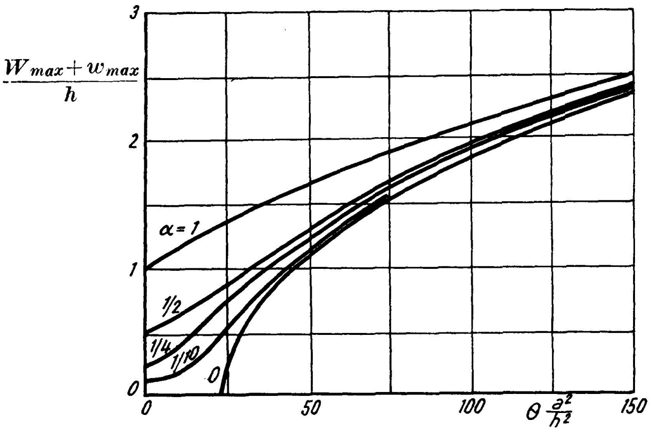

Fig. 5. Deflection at the centre of the plate in Example No. 2 as a function of $\theta\frac{a^2}{h^2}$ for various values of the initial deflection (for the angle θ, see Fig. 4). The curve for $\alpha = 0$, initially plane plate, has been calculated by BERGMAN. The initial deflections are included.

The circumstance that k shall be put equal to ∞ in the right-hand member of Eq. (c) follows from Eqs. (a) and (b) which give $\theta\frac{a^2}{h^2}$ and $\frac{w_{max}}{h} = 0$ respectively for $k = \infty$.

Eq. (f) in Example No. 1, which is also applicable in this case, yields the corresponding value of $\frac{w_{0\,max}}{h_0}$

$$\frac{w_{0\,max}}{h_0} = \alpha\sqrt{2} \tag{d}$$

Then we obtain from Eq. (c)

$$\left(\frac{\frac{w_{max}}{h}}{\theta\frac{a^2}{h^2}}\right)_{\substack{\theta\frac{a^2}{h^2}=0\\ \frac{w_{max}}{h}=0}} = \left(\frac{\frac{w_{0\,max}}{h_0}}{\theta_0\frac{a^2}{h_0^2}}\,\frac{1}{\sqrt{2}}\right)_{\frac{w_{0\,max}}{h_0}=\alpha\sqrt{2}} \tag{e}$$

For several initial deflections, the results are given in Fig. 5.

BERGMAN has determined corresponding curves by means of detailed calculations made on the assumption that the surface of the initially deflected plate is sine-shaped. The curves shown in Fig. 5, which are based on the assumption that the initial deformation and the additional deformation are affine, are in close agreement with BERGMAN's curves.

The conclusions which can be drawn from the above results from the standpoint of building statics agree, on the whole, with BERGMAN's inferences.

The procedure in the determination of the stresses is the same, in principle, as in Examples Nos. 3 and 5.

Example No. 3

Rectangular plate compressed in one direction. The plate is simply supported at all four edges. Along those edges at which the compressive forces are applied, the deformation in the direction of compression is the same along the whole edge (Fig. 6).

Using the notations given in Fig. 6, we have the boundary conditions

$$y = \pm \frac{b}{2}; \qquad w = 0; \qquad \frac{\partial^2 w}{\partial y^2} = 0;$$

$$x = \pm \frac{a}{2}; \qquad w = 0; \qquad \frac{\partial^2 w}{\partial x^2} = 0;$$

$$y = \pm \frac{b}{2}; \qquad \tau_{xy} = 0; \qquad \sigma_y = 0;$$

$$u_{x=\frac{a}{2}} - u_{x=-\frac{a}{2}} = \int_{-\frac{a}{2}}^{+\frac{a}{2}} \frac{1}{E} \sigma_{x y=\frac{b}{2}} \, dx;$$

$$x = \pm \frac{a}{2} \qquad \tau_{xy} = 0 \quad \text{or} \quad v = 0 \quad \text{(extreme cases)}$$

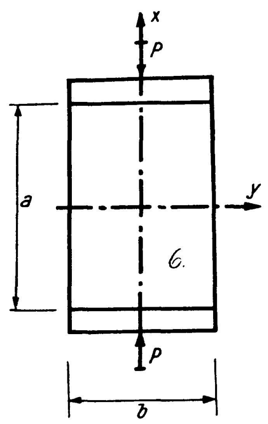

Fig. 6. Notations and co-ordinate system used in Example No. 3

These boundary conditions shall be satisfied both for the initially deflected plate and for the initially plane plate used for comparison.

It follows from Eqs. (4), (5), (12) and (13) that if the above boundary conditions are fulfilled for the initially plane comparison plate, they are also fulfilled for the initially deflected plate.

a) Maximum Deflection

A convenient method of dimensionless representation is to express the deflection w in relation to the thickness of the plate h as a function of P/P_k, where P denotes the compressive force applied to the plate, and P_k designates that compressive force at which the plane condition of equilibrium for the originally plane plate of the thickness h ceases to be stable.

If h is the thickness of the initially deformed plate and h_0 is the thickness of the initially plane plate, we have, for P_k[5]),

[5]) See BRYAN, Proc. London Math. Soc., Vol. 22, p. 54, 1891, and TIMOSHENKO, Theory of Elastic Stability, p. 327.

$$P_k = \beta \frac{\pi^2}{b} \frac{E\,h^3}{12\,(1-\nu^2)} \tag{a}$$

and for P_{k_0},

$$P_{k_0} = \beta \frac{\pi^2}{b} \frac{E\,h_0^{\,3}}{12\,(1-\nu^2)} \tag{b}$$

where β is a constant which varies with the side ratio $\frac{a}{b}$. The minimum value of β is 4.0, which holds good for $\frac{a}{b} = 1.0$, 2.0, 3.0, etc.

We obtain from Eq. (7)

$$\frac{h}{h_0} = \sqrt{\frac{1+k}{c_1}} \tag{c}$$

The relations (a) and (b) yield

$$\frac{P_k}{P_{k_0}} = \sqrt{\frac{(1+k)^3}{c_1^{\,3}}} \tag{d}$$

Furthermore, we have, for the initially deflected plate,

$$P = \int_{-\frac{b}{2}}^{+\frac{b}{2}} \sigma_x\,h\,dy \tag{e}$$

and for the initially plane plate,

$$P_0 = \int_{-\frac{b}{2}}^{+\frac{b}{2}} \sigma_{x_0}\,h_0\,dy \tag{f}$$

In virtue of Eqs. (4) and (7), we get

$$\frac{P}{P_0} = \sqrt{\frac{(1+k)}{c_1^{\,3}}} \tag{g}$$

Eqs. (d) and (g) give

$$\frac{P}{P_k} = \frac{1}{1+k}\,\frac{P_0}{P_{k_0}} \tag{h}$$

For the same reasons as in Example No. 1 and Example No. 2, the relation between $\frac{w}{h}$ and $\frac{w_0}{h_0}$ is

$$\frac{w_{max}}{h} = \frac{w_{0max}}{h_0}\,\frac{1}{\sqrt{(1+2\,k)\,(1+k)}} \tag{i}$$

To the author's knowledge, no strictly accurate solution of this problem has been found for the initially plane plate. Approximate solutions have been given by Cox, Yamamoto-Kondo [6]), and others. These solutions have been

[6]) Cox, H. L., Buckling of Thin Plates in Compression. Aer. Res. Com. Reports and Mem. No. 1554, London 1933. — Yamamoto, M. and Kondo, K., Buckling and Failure of Thin Rectangular Plates in Compression. Aer. Res. Inst., Tokyo 1934.

deduced on simplified assumptions, viz. in treating the membrane state of stress, the modulus of elasticity in shear G has been put equal to zero, and the shape of the deflection surface has been supposed to be of a certain definite character.

The curve for the initially plane plate shown in Fig. 7 has been calculated by the author on simplified assumptions which are similar to those made by Cox.

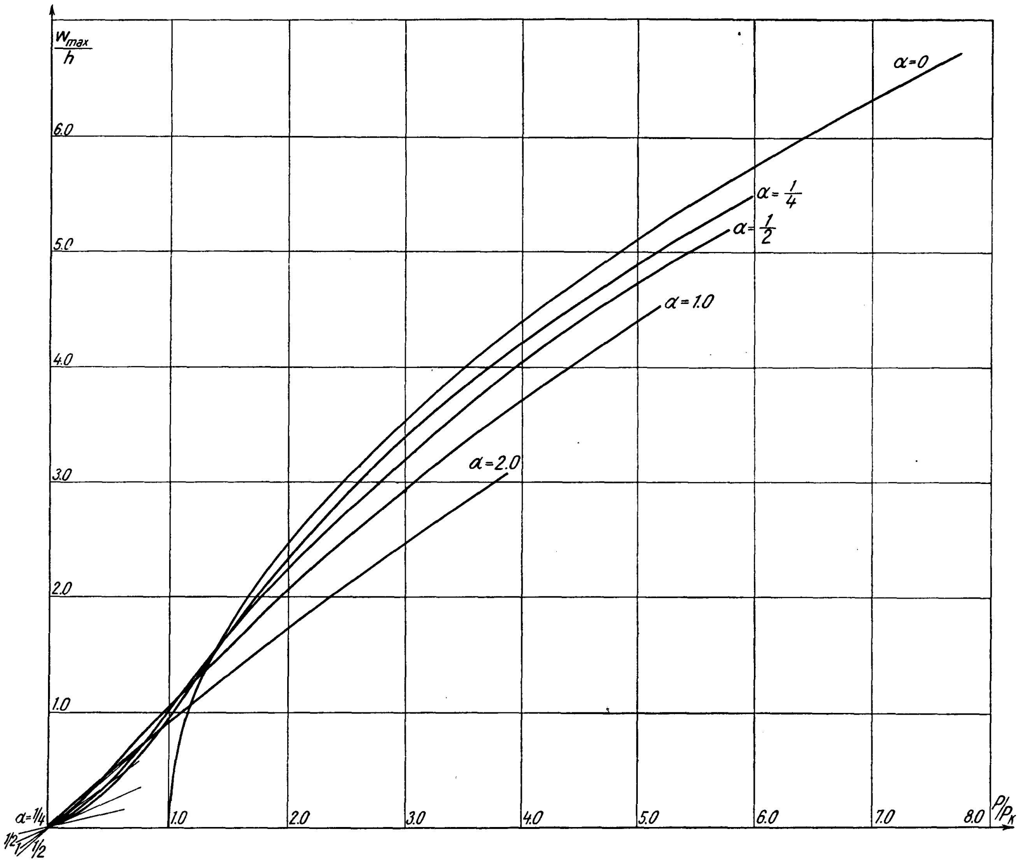

Fig. 7. Additional deflection at the centre of the plate in Example No. 3 as a function of $\frac{P}{P_K}$ for various values of the initial deflection at the centre of the plate

Just as in Example No. 2, it can be demonstrated that k is determined by the equation

$$k = \alpha \frac{h_0}{w_{0_{max}}} \sqrt{(1+2k)(1+k)} \tag{j}$$

where α is defined by the relation

$$W_{max} = \alpha \cdot h \tag{k}$$

As in Example No. 2, it can be shown that the tangent at the origin is determined by the relation

$$\left(\frac{\frac{w_{max}}{h}}{\frac{P}{P_k}}\right)_{\substack{\frac{P}{P_k}=0 \\ \frac{w_{max}}{h}=0}} = \left(\frac{\frac{w_{0\,max}}{h_0}}{\left(\frac{P}{P_k}\right)_0 \sqrt{2}}\right)_{\frac{w_{0\,max}}{h_0} = \alpha\sqrt{2}} \tag{l}$$

Fig. 7 shows curves for several initial deflections computed by means of the method outlined in the above.

b) *Design Stresses and Effective Width*

The system under consideration is statically indeterminate in a high degree. Therefore, if the yield point stress is reached in some portion of a plate made of steel, this involves at the beginning only a redistribution of stresses at an increasing load. The relation between the practical ultimate load and that load which causes the yield point to be reached in the most heavily stressed portion of the plate is dependent on the stress distribution.

Yamamoto and Kondo have found that the heaviest stress in an initially plane plate is obtained in their solution as a combination of the normal stress and the bending stress parallel to the direction of compression.

However, the load-carrying capacity is primarily determined by the ability of the plate to withstand the edge stresses which are parallel to the direction of compression, because the ability of the plate to bear a load above P_k is largely dependent on the redistribution of stresses resulting in a stress concentration at the edges.

In addition, as will be shown below, it is necessary to consider the effect of the initial deflection, which increases the highest compressive stress, but decreases, in relation to the mean compressive stress, the bending and torsional stresses due to the deflection of the plate in a large portion of the region used in design.

It is convenient to represent the results by expressing $\frac{\sigma_{edge}}{\sigma_{mean}}$ as a function of $\frac{P}{P_k}$, where σ_{edge} denotes the maximum compressive stress occurring at the edge, and $\sigma_{mean} = \frac{P}{b\,h}$ designates the mean compressive stress. Since the membrane state of stress in the initially deflected plate is a *uniform* enlargement of the membrane state of stress in the initially plane plate (the coefficient of enlargement being given by $\frac{1}{c_1}$ in accordance with Eq. (4)), we can write

$$\frac{\sigma_{edge}}{\sigma_{mean}} = \frac{\sigma_{edge_0}}{\sigma_{mean_0}} \tag{m}$$

Eqs. (h), (j), and (m) determine $\frac{\sigma_{edge}}{\sigma_{mean}}$ as a function of $\frac{P}{P_k}$ at varying initial deflections. The results are given in Fig. 8.

At comparatively high values of $\frac{P}{P_k}$, the importance of the bending and torsional stresses in the initially deflected plate becomes smaller. In what follows, we shall therefore deal only with the torsional stress for $x = \frac{1}{2}a$ and $y = \frac{1}{2}b$.

For the initially deflected plate, the torsional stress is given by the formula

$$(\tau_T) = \frac{m_{xy}}{\frac{1}{3}h^2} = \frac{E}{4(1+\nu)}\, h\, \frac{\partial^2 w}{\partial x\, \partial y} \tag{n}$$

We form the dimensionless expression

$$\frac{\tau_T}{\sigma_{mean}} = \frac{E}{4(1+\nu)}\, h\, \frac{\dfrac{\partial^2 w}{\partial x\, \partial y}}{\sigma_{mean}} \tag{o}$$

For the initially plane plate, the corresponding relation is

$$\left(\frac{\tau_T}{\sigma_{mean}}\right)_0 = \frac{E}{4(1+\nu)}\, h_0\, \frac{\dfrac{\partial^2 w_0}{\partial x\, \partial y}}{\sigma_{mean_0}} \tag{p}$$

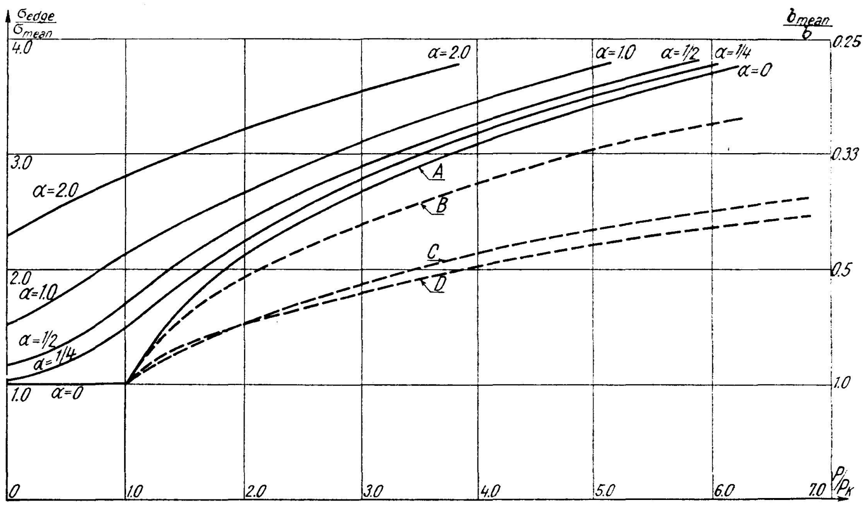

Fig. 8. $\frac{\sigma_{edge}}{\sigma_{mean}}$ and $\frac{b_{mean}}{b}$ as functions of $\frac{P}{P_k}$ for various values of the initial deflection in Example No. 3. Among the curves for the initially plane plate, it is Curve A that was used for the calculation of the curves for the initially deformed plate. This curve represents an approximate solution for the method of support in question. — Curve C expresses MARGUERRE's approximate solution $\frac{\sigma_{mean}}{\sigma_{edge}} = 0{,}81 \sqrt{\frac{P}{P_k}\frac{\sigma_{mean}}{\sigma_{edge}}} + 0{,}19$ in the case where those edges which are parallel to the direction of compression cannot be deformed in the plane of the plate. — Curve D represents MARGUERRE's strictly accurate solution, and refers to the same method of support as Curve C. Curve B expresses an estimated actual relation between $\frac{\sigma_{mean}}{\sigma_{edge}}$ and $\frac{P}{P_k}$ for the method of support in question. This curve was calculated on the assumption that the tangent at the point $\left(\frac{P}{P_k} = 1\ \frac{\sigma_{mean}}{\sigma_{edge}} = 1\right)$ is known (given by MARGUERRE), and was obtained by interpolation between Curves A and D

In virtue of the relation (c), Eq. (5), and Eq. (6), and in view of the fact that

$$\sigma_{mean} = \frac{1}{c_1} \sigma_{mean_0}$$

we obtain

$$\frac{\frac{\tau_T}{\sigma_{mean}}}{\left(\frac{\tau_T}{\sigma_{mean}}\right)_0} = \sqrt{\frac{1+k}{c_1}} \frac{1}{\sqrt{c_1 (1+2k)}} c_1 = \sqrt{\frac{1+k}{1+2k}} \tag{q}$$

The results are graphically represented in Fig. 9.

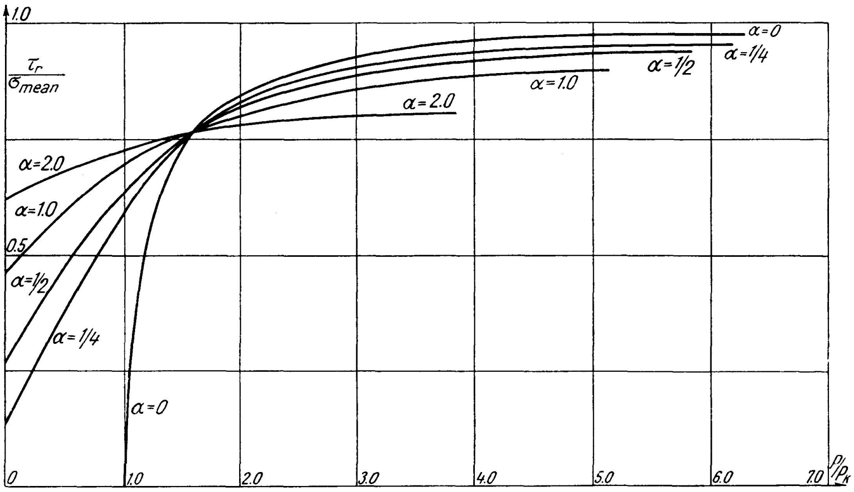

Fig. 9. Torsional stress due to the deflection of the plate at the point $x = \pm \frac{a}{2}\, y = \pm \frac{b}{2}$ as a function of the load for various values of the initial deflection

Eq. (q) and Fig. 9 indicate that torsional stresses due to the deflection of the plate are less sensitive to initial deformations than the greatest edge stress. An analogous conclusion can be drawn regarding the bending stresses because the expressions for these stresses are analogous to the expressions for the torsional stress.

Considering the above remark that the system is statically indeterminate in a high degree, and seeing that the edge stress in the direction of compression is predominant, particularly in the case of the initially deformed plate, this edge stress should be used as a basis for design.

In view of the great importance that is obviously to be attached to the initial deflection, a certain definite imaginable magnitude of this deflection should be fixed in drawing up design rules.

It is also to be observed that the diagram representing $\frac{\sigma_{edge}}{\sigma_{mean}}$ as a function of $\frac{P}{P_k}$ at the same time shows the inverse value of $\frac{b_{mean}}{b}$. This is a direct consequence of the definition of $\frac{b_{mean}}{b}$. The extremely great effect of the initial deflection is clearly seen from Fig. 8. For instance, when the initial deflection is twice the thickness of the plate, then $\frac{b_{mean}}{b} < 0{,}5$ even for $P = 0$.

In Fig. 10 the diagram representing $\frac{\sigma_{edge}}{\sigma_{mean}}$ as a function of $\frac{P}{P_k}$ for several values of the initial deflection contains test values obtained by SECHLER and WINTER. The extraordinary large dispersion of these test values is attributed by WINTER to the insufficient accuracy of the measurements and to the probable influence of initial deflections. The possible effect of initial deflections is strikingly illustrated by Fig. 10.

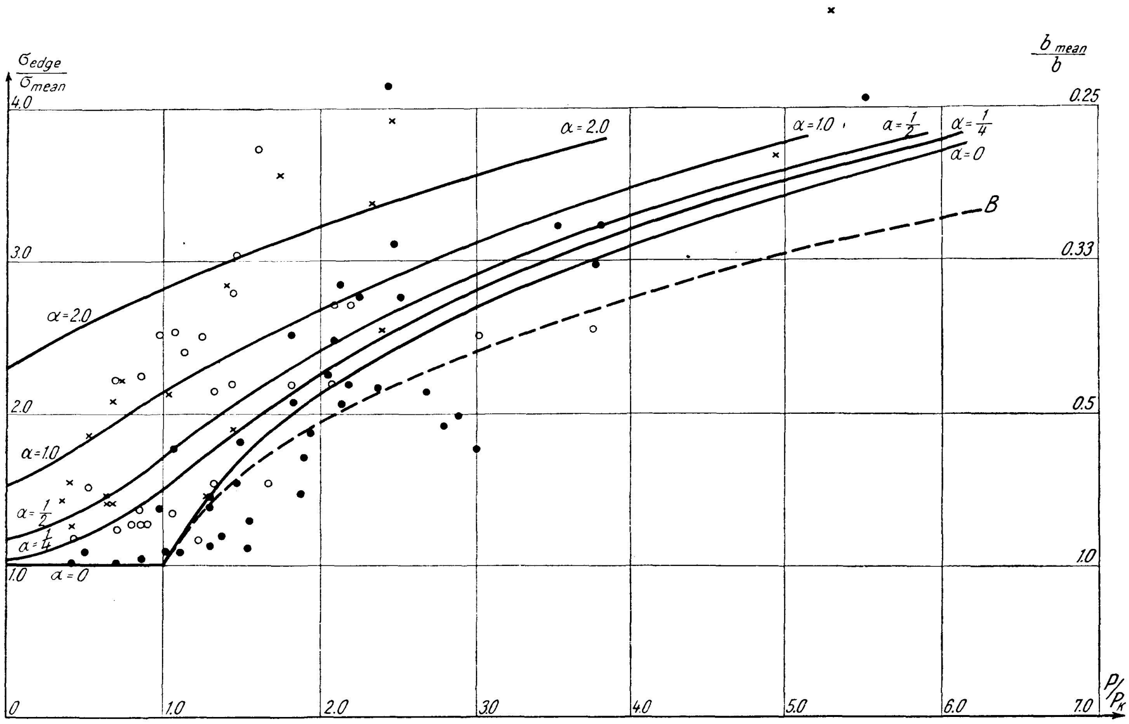

Fig. 10. Test values of $\frac{\sigma_{edge}}{\sigma_{mean}}\left(\frac{b_{mean}}{b}\right)$ plotted in relation to the curves for various values of the initial deflection.

● Test values given by G. WINTER in I.A.B.S.E. Preliminary Publication, 1948, p. 137.
o Test values given by G. WINTER in A.S.C.E. Proc., 1946, p. 199.
x Test values given by E. E. SECHLER in Publ. No. 27, Guggenheim Aeron. Lab., Cal. Inst. of Technology, Pasadena 1933.

Finally, it is to be borne in mind that the basic solution for the initially plane plate was approximate. Consequently, the curves for the initially deformed plate are also approximate. Therefore, Fig. 9 also shows curves which refer to boundary conditions other than those given in the legend. In addition, this diagram contains an estimated curve representing the relation between $\frac{\sigma_{mean}}{\sigma_{edge}}$ and $\frac{P}{P_k}$ for the boundary conditions in question. The tangent to this curve for $\frac{P}{P_k} = 1$ has been determined by MARGUERRE by means of an accurate solution. For the rest, the shape of this curve has been estimated on the basis of the second case of loading completely treated by Marguerre and with the aid of the author's curve. The estimated curve indicates the direction in which the curves for the initially deformed plate should be corrected.

Circular Plate

The fundamental equations given in the above refer to a rectangular co-ordinate system. These equations are also applicable to circular plates, but the solution of the problem can be considerably simplified in this case by using a polar co-ordinate system. However, the above reasoning, which states that the solution for the initially deflected plate whose initial deflection is affine to the additional deflection can be obtained from the solution for the initially plane plate, holds good irrespective of the co-ordinate system.[7]

Example No. 4

We shall deal with two cases of loading which have been calculated by KARL FEDERHOFER and HANS EGGER[8]) for the initially plane plate, viz.

a) Uniformly distributed load q. The plate is simply supported at the edge in a radial direction. The supports are freely movable in the plane of the plate.

b) Uniformly distributed load q. The plate is simply supported at the edge in a radial direction. The supports are fixed in the plane of the plate. Cf. Fig. 11.

In both these cases, the boundary conditions are so formulated, that if they are fulfilled for the initially plane comparison plate, then they are also fulfilled for the initially deformed plate by virtue of the relations (4), (5), (12) and (13).

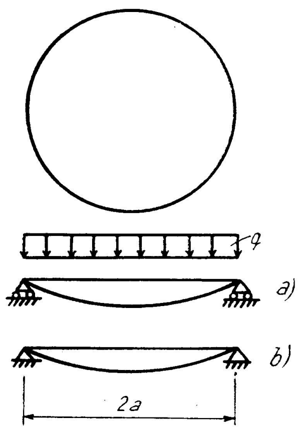

Fig. 11. Notations and co-ordinate system used in Example No. 4

Just as in Example No. 1, we have

$$\frac{q\,a^4}{E\,h^4} = \frac{q_0\,a^4}{E\,h_0{}^4}\,\frac{1}{\sqrt{(1+2\,k)\,(1+k)}} \qquad \text{(a)}$$

and

$$\frac{w}{h} = \frac{w_0}{h_0}\,\frac{1}{\sqrt{(1+2\,k)\,(1+k)}} \qquad \text{(b)}$$

Furthermore,

$$k = \alpha\,\frac{1}{\dfrac{w_{0\,max}}{h_0}}\,\sqrt{(1+2\,k)\,(1+k)} \qquad \text{(c)}$$

---

[7]) The fundamental equations can also be written in terms of polar co-ordinates, but this is unnecessary, at least for the examples treated below.

[8]) KARL FEDERHOFER and HANS EGGER, Berechnung der dünnen Kreisplatte mit großer Ausbiegung. Sitz.-Ber. d. Ak. Wiss. Wien, Math. Kl. Abt. IIa. 155, Bd. 1, und 2. Heft., 1946.

The deflections of the initially deformed plate have been calculated on the basis of the deflections of the initially plane plate computed by FEDERHOFER and EGGERT. The results are reproduced in Fig. 12. The tangent at the origin has been determined in the same manner as in Example No. 1.

In dimensionless representation, it is convenient to express *the membrane stresses* so that $\sigma_m \frac{1}{E} \frac{a^2}{h^2}$ is obtained as a function of $\frac{q a^4}{E h^4}$. The relation between the membrane stresses in the initially deformed plate and the initially plane plate is deduced as follows.

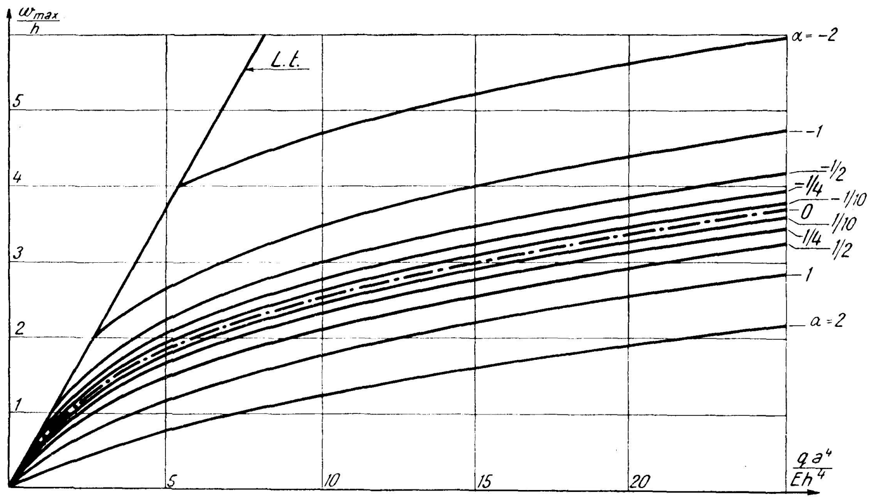

Fig. 12a. Deflection at the centre of the plate as a function of the load in Example No. 4a for various values of the initial deflection at the centre of the plate. *L.t.* = the deflection calculated in accordance with the linear theory of plates. The curves for negative values of α and for small values of $\frac{q a^4}{E h^4}$ are omitted in this diagram. The curve for the initially plane plate ($\alpha = 0$) has been calculated by FEDERHOFER and EGGER

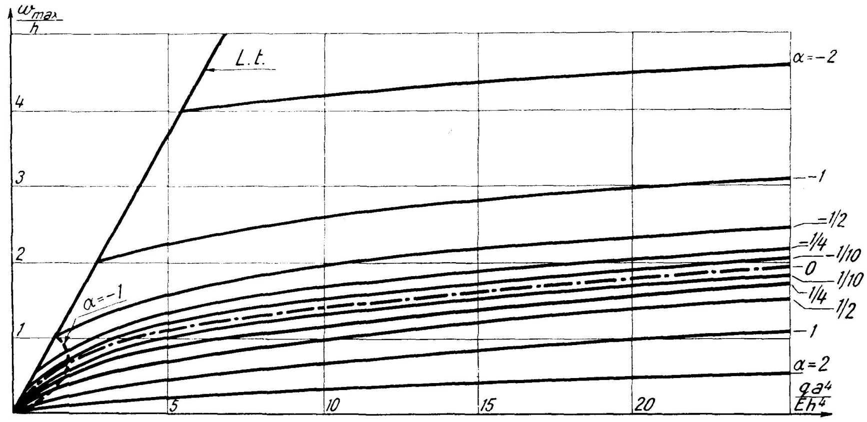

Fig. 12b. Deflection at the centre of the plate as a function of the load in Example No. 4b for various values of the initial deflection at the centre of the plate. *L.t.* = the deflection calculated in accordance with the linear theory of plates. For negative values of α and small values of $\frac{q a^4}{E h^4}$ a curve is given only for $\alpha = -1$

Eqs. (4) and (7) yield

$$\sigma_m \frac{1}{E} \frac{a^2}{h^2} = \sigma_{m_0} \frac{1}{E} \frac{a^2}{h_0^2} \frac{1}{1+k} \tag{d}$$

The curves for the initially deflected plate can be constructed from the curves for the initially plane plate by means of Eqs. (a), (c) and (d). The results for the case (b) are reproduced in Fig. 13. The procedure in the determination of stresses in the plate supported according to the case (a) is the same in principle. For lack of space, we shall not deal with this case.

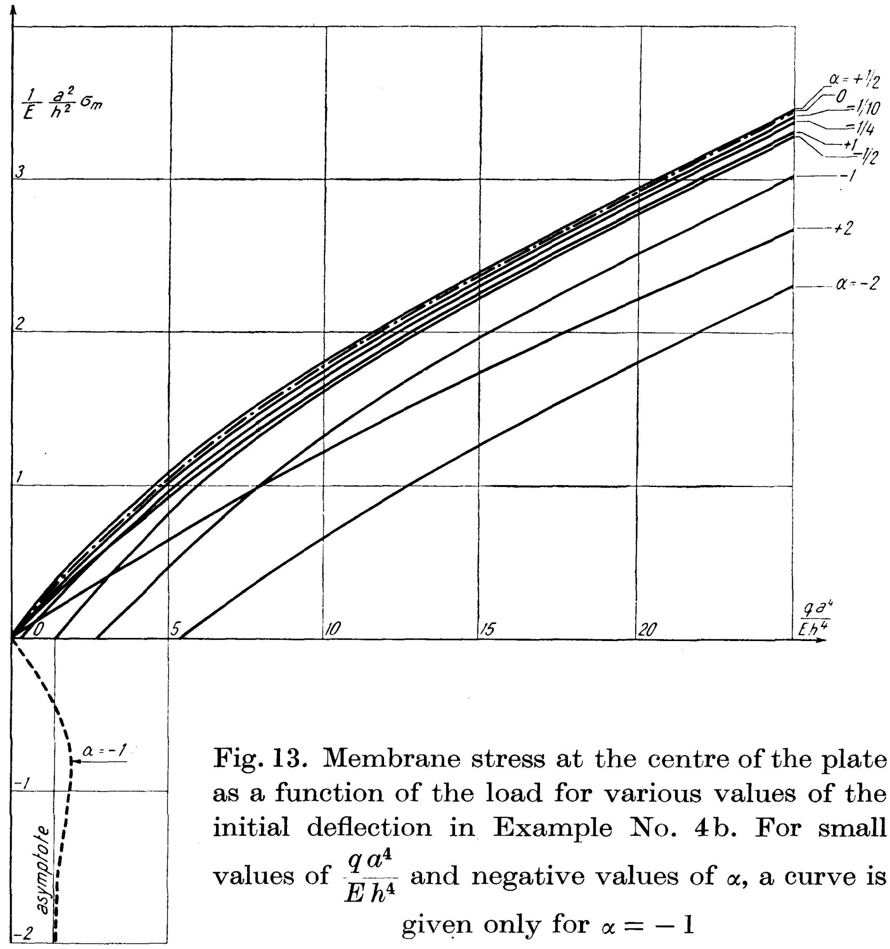

Fig. 13. Membrane stress at the centre of the plate as a function of the load for various values of the initial deflection in Example No. 4b. For small values of $\frac{q\,a^4}{E\,h^4}$ and negative values of α, a curve is given only for $\alpha = -1$

The tangent at the origin is obtained from

$$\left(\frac{\sigma_m \frac{1}{E}\frac{a^2}{h^2}}{\frac{q\,a^4}{E\,h^4}}\right)_{\frac{q\,a^4}{E\,h^4}=0} = \left(\frac{\sigma_{m_0}\frac{1}{E}\frac{a^2}{h_0^2}}{\frac{q_0\,a^4}{E\,h_0^4}} \; \frac{\frac{1}{1+k}}{\frac{1}{\sqrt{(1+2k)(1+k)}}}\right)_{k=\pm\infty} =$$

$$= \left(\pm \frac{\sigma_{m_0}\frac{1}{E}\frac{a^2}{h_0^2}}{\frac{q_0\,a^4}{E\,h_0^4}} \sqrt{2}\right)_{\frac{q_0\,a^4}{E\,h_0^4}} \text{ for } \frac{w_{0max}}{h_0} = \pm\alpha\sqrt{2} \tag{e}$$

The plus sign refers to the positive initial deflection and the minus sign refers to the negative initial deflection.

It follows from Eq. (d) that the membrane stresses are negative for $k < -1$. When $k \to -1$, the membrane stresses asymptotically tend towards $-\infty$.

The bending stresses σ_{b_x} are proportional to $\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) h$.

In dimensionless representation it is convenient to form the expression $\sigma_b \frac{1}{E} \frac{a^2}{h^2}$.

By virtue of Eqs. (5) and (7), we have

$$\frac{\sigma_{bx}}{\sigma_{b_{x0}}} = \frac{\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)}{\left(\frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2}\right)} \frac{h}{h_0} = \frac{1}{c_2} \sqrt{\frac{1+k}{c_1}} \tag{f}$$

The same relation between the bending stresses in the initially deflected plate and the initially plane plate is obtained for any arbitrary direction.

In view of Eqs. (f), (7) and (6) we obtain

$$\sigma_b \frac{1}{E} \frac{a^2}{h^2} = \sigma_{b0} \frac{1}{E} \frac{a^2}{h_0^2} \frac{1}{c_2} \sqrt{\frac{1+k}{c_1}} \frac{c_1}{1+k} = \sigma_{b0} \frac{1}{E} \frac{a^2}{h_0^2} \frac{1}{\sqrt{(1+2k)(1+k)}} \tag{g}$$

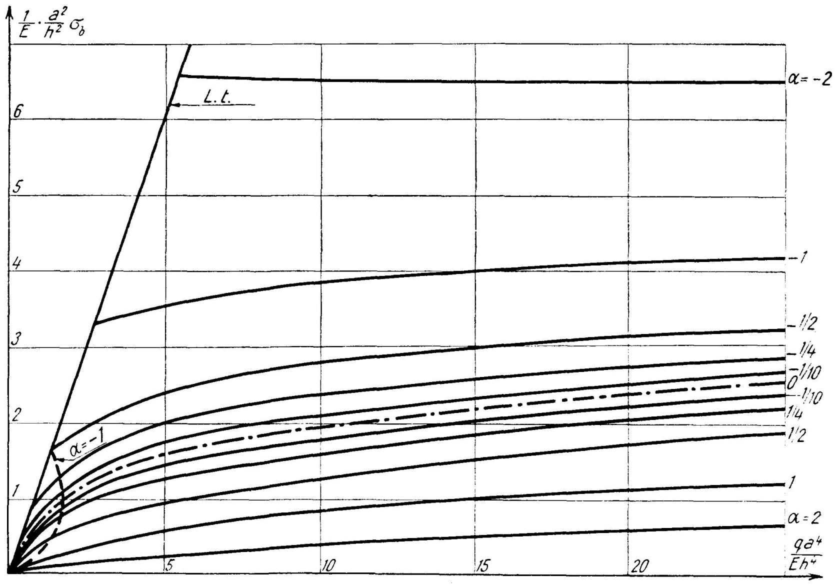

Fig. 14. Bending stress at the centre of the plate as a function of the load for various values of the initial deflection in Example No. 4b. For small values of $\frac{q a^4}{E h^4}$ and negative values of α, a curve is given only for $\alpha = -1$. *L.t.* = the bending stress calculated in accordance with the linear theory of plates

The curves for the initially deflected plate can now be constructed in a simple manner from the curve for the initially plane plate. Fig. 14 represents the bending stress at the centre of the plate for the method of support (b).

The expression for the tangent at the origin can be deduced by means of a method which is similar to that used in the case of the membrane stresses. We obtain

$$\left(\frac{\sigma_b \dfrac{1}{E}\,\dfrac{a^2}{h^2}}{\dfrac{q\,a^4}{E\,h^4}}\right)_{\frac{q\,a^4}{E\,h^4}=0} = \left(\frac{\sigma_{b_0} \dfrac{1}{E}\,\dfrac{a^2}{h_0^{\,2}}}{\dfrac{q_0\,a^4}{E\,h_0^{\,4}}}\right)_{\frac{q_0\,a^4}{E\,h_0^{\,4}}} \quad \text{for} \quad \frac{w_0}{h_0} = \pm\alpha\sqrt{2} \tag{h}$$

Example No. 5

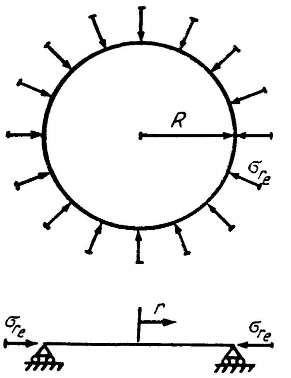

Fig. 15. Notations and co-ordinate system used in Example No. 5

Circular plate subjected to compression in a radial direction and simply supported at the circumference (Fig. 15).

If the boundary conditions are fulfilled for the initially plane plate, they are also fulfilled for the initially deflected plate.

We introduce the following notations:

σ_r = the radial membrane stress,
σ_θ = the circumferential membrane stress,
σ_b = the radial bending stress at the upper surface,
σ_{re}, $\sigma_{\theta e}$, σ_{be} = the values of stresses at the edge of the plate,
σ_k = the critical value of σ_{re}.

The quantities which are primarily of interest in the design are the maximum deflection w_{max}, which occurs at the centre of the plate, $\sigma_{\theta e}$, and $\sigma_{b\,max}$.

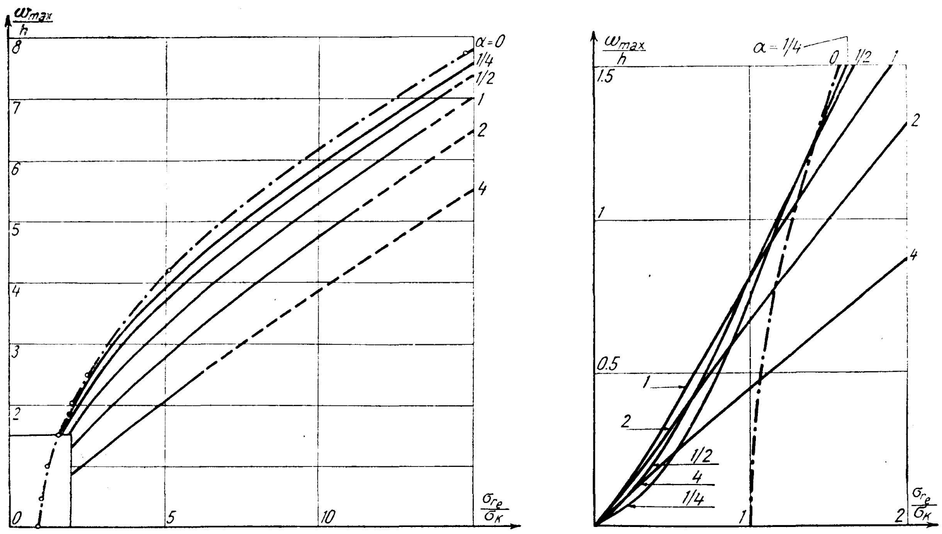

Fig. 16. Deflection at the centre of the plate as a function of the radial compressive stress applied along the external circumference in Example No. 5 for various values of the initial deflection. The curve for the initially plane plate has been calculated by Friedrichs and Stoker

The maximum value of σ_{θ_e} occurs at the edge of the plate. The fundamental solutions for the initially plate are those due to FRIEDRICHS and STOKER[9]. The value of POISSON's ratio is taken to be 0.318.

The methods of deducing the relations between the initially plane plate and the initially deformed plate are similar to those used in the foregoing example, and are omitted in what follows. The results are given in Figs. 16, 17 and 18.

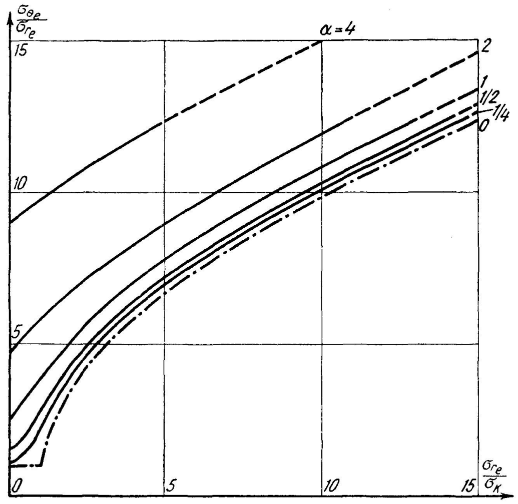

Fig. 17. Membrane stress in a tangential direction as a function of the external load at various values of the initial deflection in Example No. 5

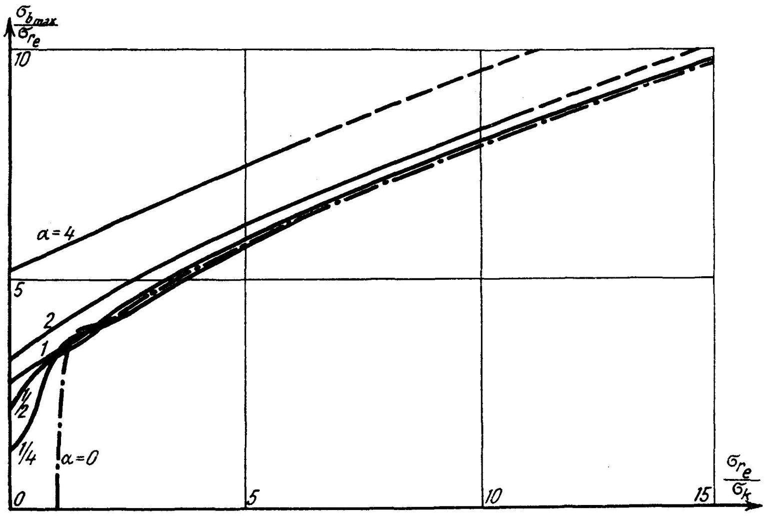

Fig. 18. Maximum bending stress occurring in the plate as a function of the external load at various values of the initial deflection in Example No. 5

Discussion of Results

In the discussion of the basic results obtained in this paper, the examples can be conveniently classified in two groups as follows.

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[9]) K. O. FRIEDRICHS and J. J. STOKER, Buckling of the Circular Plate Beyond the Critical Thrust. Journal of Appl. Mech., March 1942. — CH. MASSONET has studied a radially compressed clamped plate with an initial deflection of a certain definite form, see CH. MASSONET: Buckling of Plates. Final report of Third Congress, Int. Ass. for Bridge and Struct. Eng. 1948.

Group No. 1 comprising Examples Nos. 1 and 4, in which the external load consists of a transverse load acting at right angles to the plane of the plate.

Group No. 2 comprising Examples Nos. 2, 3, and 5, in which the external load consists of forces acting in the plane of the plate.

In Group No. 1 we can distinguish between two principal cases, viz., positive and negative initial deflection, i. e. the initial deflection and the additional deflection having the same or the opposite directions respectively.

When the initial deflection is positive, the stiffness of the plate increases as the initial deflection becomes greater (see Figs. 3a, 3b, 12a and 12b). It is of interest to note that the stiffness of the initially deflected plate is greater than that of the initially plane plate also at small values of the transverse load (the slope of the tangents at the origin in the above-mentioned diagrams decreases as the value of α becomes greater). It is quite natural that the smallest increase in the stiffness of the plate is to be observed in that example (Fig. 12a) where the deformation in the plane of the plate was not prevented at the supports.

In Example No. 4, Case (b), we have determined the stresses at the centre of the plate. The results given in Figs. 13 and 14 show that there is a very large decrease in the bending stresses at the centre of the plate when the initial deflection is positive. This statement holds at small loads too. On the other hand, the membrane stresses are relatively insensitive to the initial deflection. Even though the maximum stresses at higher loads do not occur at the centre of the plate, a comparison with the stresses calculated from the linear theory shows that there are extensive possibilities of utilising the effect of the initial deflection in various structures, e. g. in roof slabs on circular cylindrical containers, by designing the plate so as to obtain an initial deflection.

When the initial deflection is negative, the stiffness of the plate is reduced by the initial deflection at high values of q (cf. Figs. 3a, 3b, 12a and 12b). This reduction is connected with an increase in the bending stresses (cf. Fig. 14), whereas the membrane stresses are relatively insensitive to the initial deflection, just as in the case of the positive initial deflection (cf. Fig. 13).

When the value of q is small and the initial deflection is negative, the problem is relatively intricate. The results are given only for $\alpha = -1$ in Example No. 4. It follows from the dash-line curve in Fig. 12b that the problem is not unambiguous within a given region, if we consistently adhere to the assumption that the initial deformation and the additional deformation are affine. For $\alpha = -1$, the linear theory provides a solution at the points $\frac{w_{max}}{h} = 1{,}0$ and $\frac{w_{max}}{h} = 2{,}0$. A corresponding statement can be made about the bending stresses (Fig. 14), whereas the membrane stresses are negative at small values of $\frac{q\,a^4}{E\,h^4}$, and tend towards $-\infty$ at that point which corresponds to $\frac{w_{max}}{h} = 1{,}0$. This phenomenon can probably be attributed to the effect of the assumed affinity.

It is obvious that due regard must be paid to the initial deflection in the interpretation of test results. At small initial deflections and high values of $\frac{q a^4}{E h^4}$, the stiffness of the initially plane plate can be determined by making loading tests in both directions on the initially deformed plate, as may be seen from Figs. 3a, 3b, 12a, and 12b.

In Examples Nos. 2, 3 and 5, Group No. 2, the initially plane plate remains plane when the values of the external load are smaller than the critical buckling force. On the other hand, the deflection of the initially deformed plate increases from the very moment of application of external load. Figs. 5, 7 and 16 shows the deflections of the initially deformed plates. When the values of the external load are small, the plates subjected to the greatest initial deformation exhibit the greatest deflections, whereas the reverse is the case at heavy loads.

The curves indicate how the results obtained from tests made at known initial deformations can be interpreted in respect of the magnitude of the critical load.

As regards the stresses, it is the membrane stresses that are of interest in the first place. Fig. 8 shows the effect of the initial deflection on the greatest edge stress in Example No. 3, and Fig. 17 gives the membrane stress in a tangential direction at the support in Example No. 5. The membrane stresses shown in these diagrams represent the maximum values of these stresses occurıing in both examples. It is seen that the initial deflection has an extremely great influence on the magnitude of these stresses. The increase in stresses due to the initial deflection is particularly marked in the region $P < P_k$. In both examples, the membrane stresses are concentrated in the neighbourhood of the edges.

In Example No. 3, $\frac{\sigma_{edge}}{\sigma_{mean}}$ expresses the inverse value of the ratio of the effective width to the total width of the plate. The strong influence of the initial deflection on the effective width is therefore clearly seen from Fig. 8.

The stresses due to the bending of the plate (Figs. 9 and 18) do not increase to the same degree as the membrane stresses on account of the initial deflection.

The fact that the membrane stresses are so strongly influenced by the initial deflection, particularly in the subcritical region ($P < P_k$), but also in the supercritical region ($P > P_k$), shows that due regard should be paid to the effect of possible initial deflections in drawing up design rules for similar cases of loading.

Summary

This paper deals with thin plates subjected to an initial deflection which is of the same order of magnitude as the thickness of the plate, and is affine to the additional deflection.

By comparing the fundamental equations of an initially plane plate and an initially deformed plate, which have been deduced by von Kármán and

further developed by MARGUERRE in accordance with the non-linear theory of plates, it is demonstrated that the solution for the initially deformed plate can be obtained from the solution for the initially plane plate if the initial deflection is assumed to be affine to the additional deflection. The reasoning is as follows. The condition that the initial deflection shall be affine to the additional deflection ($W = k \cdot w$) is satisfied by the solution of Eqs. (1′) and (2′). For a certain definite k, the solution is single-valued. The fundamental equations (1′) and (2′) for the initially deflected plate and the fundamental equations (1‴) and (2‴) for the initially plane plate are identical if the conditions in Eqs. (6), (7) and (8) are fulfilled. In this manner the solution of the equations of the initially deflected plate can be obtained from the solution of the equations of the initially plane plate.

Since the fulfilment of the fundamental equations is an expression of the fact that the conditions of equilibrium and continuity are satisfied, but does not affect the boundary conditions, it is necessary to make sure that the boundary conditions are fulfilled in each individual example.

The examples adduced in this paper show, among other things, how that form of the initial deflection which complies with the condition that the additional deflection shall be affine to the initial deflection can be determined from the solutions for the initially plane plate. (For instance, in Example No. 1, it is found to be given by that deflection form of the initially plane plate which corresponds to the value of $\frac{w_0}{h_0}$ determined by Eqs. (3), (d) and (f)).

Furthermore, a comparison of the fundamental equations shows that the membrane stresses in the initially deformed plate and its deflection at a known load are uniform enlargements of the membrane stresses in the initially plane reference plate and its deflection at another known load.

In applying the rules deduced in this paper, it is to be observed that the boundary conditions shall be fulfilled.

Five selected examples demonstrate the procedure of deducing the solutions for the initially deflected plate from previously known solutions for the initially plane plate. The results are discussed.

Zusammenfassung

Die vorliegende Arbeit behandelt die dünne Platte mit anfänglicher Verformung, die von gleicher Größenordnung wie die Dicke der Platte und zu der zusätzlichen Verbiegung affin ist.

Durch Vergleich der Grundgleichungen einer anfänglich ebenen Platte mit denen einer anfänglich gekrümmten Platte, welche von KÁRMÁN abgeleitet und von MARGUERRE in Übereinstimmung mit der nichtlinearen Plattentheorie weiter entwickelt worden sind, wird gezeigt, daß die Lösung für eine anfänglich gekrümmte Platte aus der Lösung der ebenen Platte gefunden werden kann,

wenn die anfängliche Verformung zu der zusätzlichen affin angenommen wird. Die Überlegung geht wie folgt: Der Bedingung, daß die anfängliche Durchbiegung affin zur zusätzlichen sein soll ($W = k \cdot w$) genügen die Gleichungen (1′) und (2′). Für ein bestimmtes k ist die Lösung eindeutig. Die Grundgleichungen (1′) und (2′) für die anfänglich gekrümmte Platte und die Grundgleichungen (1‴) und (2‴) für die ebene Platte sind identisch, wenn die Bedingungen in den Gleichungen (6), (7) und (8) erfüllt sind. Auf diese Weise kann die Lösung der anfänglich gekrümmten Platte aus der Lösung der ebenen Platte erhalten werden.

Da die Erfüllung der Grundgleichungen die Tatsache ausdrückt, daß das Gleichgewicht und der Zusammenhang gewährleistet sind, über die Randbedingungen aber nichts aussagt, ist es notwendig, daß die Randbedingungen in jedem einzelnen Falle erfüllt werden.

Die in dieser Arbeit angeführten Beispiele zeigen u. a., wie die Form der anfänglichen Durchbiegung, welche affin zu der zusätzlichen Durchbiegung sein soll, aus der Lösung der ebenen Platte bestimmt werden kann. (Z.B. in Beispiel No. 1 ist sie gegeben durch die Biegefläche der ebenen Platte, welche dem Wert wo/ho, bestimmt durch Gleichung (3), (d) und (f), entspricht.)

Weiter zeigt ein Vergleich der Grundgleichungen, daß die Membrankräfte in der anfänglich gekrümmten Platte und ihre Durchbiegung unter einer gegebenen Last ähnliche Vergrößerungen der Membrankräfte und Durchbiegungen einer ebenen Platte unter einer anderen bekannten Last sind.

Bei der Anwendung der abgeleiteten Regeln ist zu beachten, daß die Randbedingungen erfüllt sein müssen.

Fünf ausgesuchte Beispiele zeigen das Vorgehen zur Ableitung von Lösungen für die anfänglich gekrümmte Platte aus vorher bekannten Lösungen der ebenen Platte. Die Ergebnisse werden besprochen.

Résumé

Le présent mémoire porte sur le cas d'une plaque mince comportant une déformation initiale du même ordre de grandeur que sa propre épaisseur et affine du fléchissement ultérieur.

Par comparaison entre les équations de base d'une plaque initialement plane et d'une plaque initialement incurvée, telles qu'elles ont été établies par KÁRMÁN et développées ultérieurement par MARGUERRE en concordance avec la théorie non linéaire des dalles, l'auteur montre que la solution relative à une plaque initialement incurvée peut être obtenue à partir de la solution relative à une plaque plane, lorsque l'on peut admettre que la déformation initiale est affine de la déformation ultérieure. Le raisonnement est le suivant. La condition pour que le fléchissement initial soit affin du fléchissement ultérieur ($W = k \cdot w$) est satisfaite par les équations (1′) et (2′). Pour une valeur

déterminée de k, la solution est parfaitement déterminée. Les équations de base ($1'$) et ($2'$) pour la plaque initialement incurvée et les équations de base ($1'''$) et ($2'''$) pour la plaque plane sont identiques lorsque les conditions des équations (6), (7) et (8) sont remplies. Dans ces conditions, la solution relative à la plaque initialement incurvée peut être obtenue à partir de celle de la plaque plane.

Les équations de base exprimant le fait que l'équilibre et la cohésion sont assurés, mais ne donnant aucune indication au sujet des conditions marginales, il est nécessaire que ces dernières soient remplies dans chaque cas particulier.

Les exemples cités dans le présent mémoire montrent, en particulier, comment la forme du fléchissement initial, qui doit être affine de celle du fléchissement additionnel, peut être déduite de la solution relative à la plaque plane. (C'est ainsi que dans l'exemple 1, elle est donnée par la surface de flexion de la plaque plane, qui correspond à la valeur $\frac{w_0}{h_0}$ déterminée par l'équation (3), (d) et (f).

Une comparaison des équations de base montre en outre que les efforts de membrane dans la plaque initialement incurvée et son fléchissement sous une charge donnée représentent des accroissements analogues des efforts de membrane et des fléchissements respectifs d'une plaque plane, sous l'action d'une autre charge connue.

Dans l'application des règles établies, il faut noter que les conditions marginales doivent être remplies.

Cinq exemples mettent en évidence la marche à suivre pour arriver aux solutions relatives à la plaque initialement incurvée, à partir des solutions déjà connues de la plaque plane. L'auteur discute les résultats ainsi obtenus.