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Autor: Hendry, Arnold W. / Jaeger, Leslie G.

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The Load Distribution in Interconnected Bridge Girders with Special Reference to Continuous Beams

Répartition de la charge dans les poutres de ponts associées entre elles, avec prise en considération particulière de la poutre continue

Die Lastverteilung in zusammenhängenden Brückenträgern unter besonderer Berücksichtigung des durchlaufenden Balkens

ARNOLD W. HENDRY D. Sc., Ph. D., M.I.C.E., M. I. Struct. E., and
LESLIE G. JAEGER M. A. Ph. D., University College of Khartoum

Several methods for the analysis of interconnected bridge girder systems have been formulated in recent years. Apart from relaxation or moment distribution [1], which have the disadvantage of not yielding general solutions, all methods entail the use of simplifying assumptions either as to the mode of deformation of the structure or as to its construction or both. One of the most successful solutions has been by the application of the theory of plates [2, 3] but recently the authors have developed a method in which the cross girders only are replaced by a continuous medium of total moment of inertia equal to that of the actual transverse system. The method has been discussed in detail elsewhere [4, 5]; it possesses a number of advantages as compared with plate theory in that its derivation is comparatively simple, that distribution coefficients of immediate physical significance and application are obtained, that greater accuracy is obtained in certain cases and that all the longitudinals need not have the same moment of inertia. Furthermore the treatment of transverse moments is believed to be more accurate.

The object of this paper is to give a brief résumé of the method as applied to simply supported spans and to discuss its application to the analysis of interconnected continuous beams. The analytical procedure depends on whether or not the longitudinals possess torsional stiffness and will be illustrated by considering the solution for a three girder bridge. A general solution for any degree of torsional stiffness is possible but in practice it is more convenient to obtain distribution coefficients for zero and for infinite torsional stiffness and to interpolate intermediate values by use of a suitable function.

The method is based on two simplifying assumptions only:

1. That the transverse members may be replaced by a continuous medium of the same total moment of inertia.
2. That the torsional stiffness of the transverse members may be neglected.

Assumption (1.) has been found to be valid for as few as three cross girders. Torsion of the transverse members can be taken into account, but its effect is usually very small.

Analysis of Three Girder Bridge: Zero Torsional Stiffness

Fig. 1 (a) shows a cross section of the bridge at distance x from mid-span. Suppose that the inner girder (2) is given a deflection $y_2 = a_2 \cos \frac{\pi x}{L}$ then the deflections of girders (1) and (3) will be $y_1 = y_3 = a_1 \cos \frac{\pi x}{L}$; a_1 and a_2 are of course the mid span deflections of the girders (1) and (2) respectively. The transverse medium receives only vertical forces from the longitudinals and

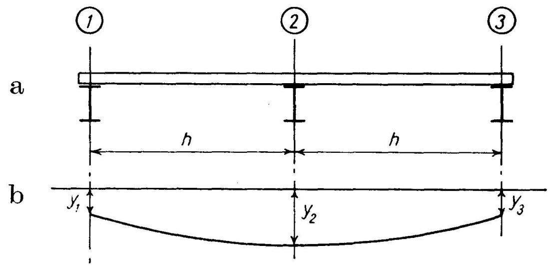

Fig. 1

these forces are proportional to the deflections of these girders. Thus the forces per unit length of the transverse medium are $K y_1$, $K y_2$ and $K y_3$; reversed in sign these forces are the loads per unit length applied to the longitudinals. Thus

$$E I \frac{d^4 y_1}{d x^4} = K y_1 \quad \text{and} \quad E I \frac{d^4 y_2}{d x^4} = K y_2$$

so that K is equal to $E I \frac{\pi^4}{L^4}$ on substitution for y_1 and y_2. The flexural rigidity per unit length of the transverse medium is $\frac{n E I_T}{L}$. The bending moment diagram is triangular so that by area moments:

$$\frac{K\,h^3\,L}{3\,n\,E\,I_T}\,y_1 = (y_2 - y_1)$$

Substituting $K = E\,I\,\frac{\pi^4}{L^4}$ and putting $\alpha = \frac{12}{\pi^4}\left(\frac{L}{h}\right)^3\frac{n\,E\,I_T}{E\,I}$ we have $\frac{4\,y_1}{\alpha} = y_2 - y_1$ i. e. $y_1 = y_2\left(\frac{\alpha}{4+\alpha}\right)$ or $a_1 = a_2\left(\frac{\alpha}{4+\alpha}\right)$. If we put $a_1 + a_2 + a_3 = A$ we then obtain

$$a_1 = \frac{\alpha}{4+3\,\alpha}\cdot A = \rho_1 A$$

$$a_2 = \frac{4+\alpha}{4+3\,\alpha}\cdot A = \rho_2 A$$

A is the amplitude of the first harmonic component of the "free deflection" curve which is the deflection curve of one of the longitudinals if it carried the entire loading on its own. ρ_1 and ρ_2 are distribution coefficients for the first harmonic of the free deflection curve. If the deflection of the loaded girder is $\cos p\,\frac{\pi\,x}{L}$ or $\sin p\,\frac{\pi\,x}{L}$ it is easily seen that the distribution coefficients will be obtained by substituting α/p^4 in place of α in the formulæ for the first harmonic coefficients.

If, therefore, the free deflection curve is analysed by Fourier's series into its harmonics, the deflection curve of each of the longitudinals can be found by applying the corresponding distribution coefficients to the component terms of the series i. e.

$$y_1 = \rho_1 A \cos\frac{\pi\,x}{L} + \rho_1{'''} A''' \cos\frac{3\,\pi\,x}{L} + \cdots + \rho_1{''} A'' \sin\frac{2\,\pi\,x}{L} + \cdots$$

$$y_2 = \rho_2 A \cos\frac{\pi\,x}{L} + \rho_2{'''} A''' \cos\frac{3\,\pi\,x}{L} + \cdots + \rho_2{''} A'' \sin\frac{2\,\pi\,x}{L} + \cdots$$

Exactly the same holds true for the bending moment curves so that the bending moment diagram for each longitudinal can be found by distributing the harmonics of the free bending moment diagram. In practice the first harmonic is always dominant and is frequently the only one which need be considered.

If the outer longitudinals are of different moment of inertia from the inner it is evident that we must write

$$E\,I_1\,\frac{d^4 y_1}{d\,x^4} = \eta\,K\,y_1 \quad \text{and} \quad E\,I_2\,\frac{d^4 y_2}{d\,x^4} = K\,y_2$$

where $\eta = \frac{I_1}{I_2}$ and $K = E\,I_2\,\frac{\pi^4}{L^4}$. On solution we find

$$\rho_1 = \frac{\alpha}{4\,\eta + \alpha\,(1+2\,\eta)} \quad \text{and} \quad \rho_2 = \frac{4\,\eta + \alpha}{4\,\eta + \alpha\,(1+2\,\eta)}$$

These are deflection distribution coefficients; bending moment distribution coefficients in this case are obtained by multiplying the deflection coefficients for the outers by η.

Analysis of Three Girder Bridge: Infinite Torsional Stiffness

In this analysis the longitudinals are assumed to be infinitely stiff tortionally and the cross girders are assumed to be rigidly connected to the main girders. Torques are thus transmitted into the longitudinals from the transverse medium and the longitudinals rotate as rigid bodies into positions of torsional equilibrium. To illustrate the method the analysis of a bridge having

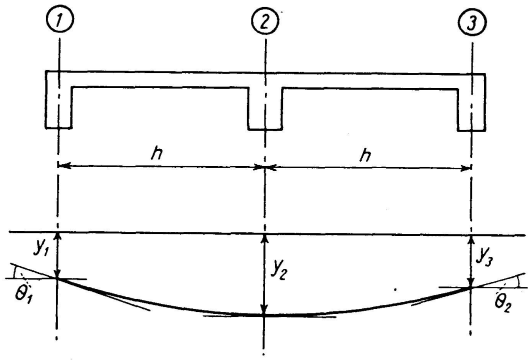

Fig. 2

three equal longitudinals, loaded on the centre one, will be worked out. Referring to fig. 2, suppose that the angle of rotation of each outer longitudinal is θ_1, then it is easily shown that the shear per unit length of the transverse medium at its connection with girder (1) is

$$s_1 = m\left[y_2 - y_1 - \frac{h\,\theta_1}{2}\right]$$

and that the bending moment per unit length in the transverse medium at the same place is

$$M_1 = \frac{6\,n\,E\,I_T}{L\,h^2}\left[y_2 - y_1 - \frac{2\,h\,\theta_1}{3}\right]$$

This bending moment acts as a torque on longitudinal (1) and since the ends of this longitudinal are not restrained we must have for equilibrium:

$$\int_{-L/2}^{L/2} M_1\,dx = 0$$

Taking $y_1 = a_1 \cos\frac{\pi x}{L}$, $y_2 = a_2 \cos\frac{\pi x}{L}$ and using the expression for M_1 given above, we find that

$$h\,\theta_1 = \frac{3}{\pi}\,(a_2 - a_1)$$

The characteristic equation for the deflection of longitudinal (1) is then

$$E\,I\,\frac{d^4 y_1}{d\,x^4} = m\,(y_2 - y_1) - \frac{m}{2}\,\frac{3}{\pi}\,(a_2 - a_1)$$

The expression for $h\,\theta_1$ is a constant and to obtain a first harmonic solution we replace $h\,\theta_1$ by its first harmonic component which is

$$\frac{4}{\pi}\,h\,\theta_1 \cos\frac{\pi x}{L}$$

Then $$E\,I\,\frac{\pi^4}{L^4}\,a_1 \cos\frac{\pi x}{L} = m\,(a_2 - a_1)\cos\frac{\pi x}{L} - \frac{6\,m}{\pi^2}\,(a_2 - a_1)\cos\frac{\pi x}{L}$$

whence $$a_1 = \alpha\,(a_2 - a_1)\left(1 - \frac{6}{\pi^2}\right)$$

Putting $$\alpha_1 = \alpha\left(1 - \frac{6}{\pi^2}\right):$$

$$a_1 = \alpha_1\,(a_2 - a_1)$$

and the distribution coefficients are

$$\rho_1 = \frac{\alpha_1}{1 + 3\,\alpha_1} \qquad \rho_2 = \frac{1 + \alpha_1}{1 + 3\,\alpha_1}$$

As in the previous case, distribution coefficients are easily obtained by the same procedure when the inner and outer girders are of different sections by the introduction of the ratio η.

In connection with calculations on interconnected continuous beams distribution coefficients are also required for the conditions that the longitudinals are of infinite torsional stiffness and do not rotate i.e. $\theta_1 = 0$ in the above example. These are very easily obtained and are

$$\rho_1 = \frac{\alpha}{1 + 3\,\alpha} \qquad \rho^2 = \frac{1 + \alpha}{1 + 3\,\alpha}$$

It is interesting to note that the form of ρ_1 is the same in each case, as the "no torsion" coefficients are obtained by putting $\alpha/4$ instead of α in the "no rotation" case just quoted and the "full torsion" coefficients by substituting $\alpha\left(1 - \frac{6}{\pi^2}\right)$.

Distribution coefficients for bridges having two, three, four, five and six longitudinals are given in appendix I for both full torsion and no torsion cases and for loads on the various longitudinals.

The Interpolation Function

In practice the cases of negligible and infinite torsional stiffness of the longitudinals are of greatest importance as it is found that on the one hand the torsional stiffness of structural steel girders of I section may be neglected and on the other that reinforced concrete beams are torsionally so stiff that they may be considered to be of infinite torsional stiffness. Only occasionally are structures encountered between those extremes. The criterion is established by a non-dimensional parameter:

$$\beta = \frac{\pi^2}{2n}\left(\frac{h}{L}\right)\left(\frac{C\,J}{E\,I_T}\right)$$

For negligible torsional resistance $\beta \to 0$. It has been found that if β exceeds about 1.25 the longitudinals can be considered infinitely stiff in torsion. Intermediate values of the distribution coefficients may be found by using the following interpolation function:

$$\rho_\beta = \rho_0 + (\rho_\infty - \rho_0)\sqrt{\frac{\beta\,\alpha}{3+\beta\,\alpha}}$$

where ρ_β is the required value and ρ_0 and ρ_∞ are the distribution coefficients corresponding to $\beta = 0$ and $\beta = \infty$ respectively. This function was obtained from consideration of the results of general analyses taking into account twist as well as rotation of the longitudinals.

Loads Acting Between the Longitudinals

In the analysis described above it has been assumed that the loads are applied directly to the longitudinals. If they are applied to the cross girders or the deck slab between the longitudinals they must be replaced by an equivalent system of loads acting on the longitudinals. The equivalent system is found by considering the cross girder or slab as a continuous beam simply supported at the longitudinals; the reactions of this beam are then the loads applied to the longitudinals and the moments and deflections arising from them are distributed by means of the distribution coefficients discussed above. The reason for this procedure may be appreciated by considering the longitudinals to be propped when the loads are applied; the simply supported continuous beam reactions will then be developed on the longitudinals and will be distributed through the system when the props are removed.

Transverse Moments

Expressions for transverse moments are obtained in the following manner. As an illustration, consider a three girder bridge with the loading on the outer girder.

a) *No torsion case:* The equation for bending moment in a strip of the transverse medium of unit width is

$$M_z = \eta K y_3 \cdot z + K y_2 [z-h]$$

There are no bending moments at girders (1) and (3); putting $z=h$ the transverse moment over the centre longitudinal is

$$-\frac{n E I_T}{L} M_{z_2} = \eta K y_3 h = K \rho_3 Y h$$

But $Y = A \cos \frac{\pi x}{L}$ and at mid span $M = E I \frac{\pi^2}{L^2} Y$ or $Y = \frac{L^2}{\pi^2 E I} \cdot M$. Therefore

$$M_{z_2} = \frac{\pi^2}{L^2} h \rho_3 \cdot M$$

or

$$\frac{M z_2}{M} \cdot \frac{L^2}{h} = \pi^2 \rho_3$$

b) *Infinite torsion case:* In the case under consideration

$$h\theta_1 = \frac{1}{2\pi}[5a_1 - 6a_2 + a_3], \quad h\theta_2 = \frac{1}{2\pi}[2a_1 - 2a_3], \quad h\theta_3 = \frac{1}{2\pi}[-a_1 + 6a_2 - 5a_3]$$

The bending moment per unit length in the transverse medium at its connection to the loaded girder is therefore:

$$M_{z_1} = \frac{6 n E I_T}{h^2 L}[y_1 - y_2 - \tfrac{2}{3} h\theta_1 - \tfrac{1}{3} h\theta_2]$$

whence:

$$M_{z_1} = \frac{6 n E I_T}{h^2 L}(a_1 - a_2)\left(\cos\frac{\pi x}{L} - \frac{2}{\pi}\right)$$

At midspan i.e. at $x=0$

$$M_{z_1} = \frac{6 n E I_T}{h^2 L}(a_1 - a_2)\left(1 - \frac{2}{\pi}\right) = \frac{6 L^2 n E I_T}{\pi^2 E I h^2 L}\left(\frac{M_{x_1}}{\eta} - M_{x_2}\right)\left(1 - \frac{2}{\pi}\right)$$

since the moments at midspan in the longitudinals M_{x_1}, and M_{x_2} are respectively

$$M_{x_1} = \eta \frac{\pi^2 E I}{L^2} a_1 \quad \text{and} \quad M_{x_2} = \frac{\pi^2 E I}{L^2} a_2$$

Substituting $M_{x_1} = \rho_1 M$ and $M_{x_2} = \rho_2 M$ where M is the free bending moment, and also

$$\frac{6 L^2 n E I_T}{\pi^2 E I h^2 L} = \frac{\pi^2}{2} \cdot \frac{h}{L^2} \alpha$$

we have

$$\frac{M_{z_1}}{M} = \frac{\pi^2}{2} \alpha \left(1 - \frac{2}{\pi}\right)(\rho_1 - \rho_2)\frac{h}{L^2}$$

or

$$\frac{M_{z_1}}{M} \cdot \frac{L^2}{h} = 1.79\,\alpha(\rho_1 - \rho_2)$$

Similar expressions for other loading cases and numbers of longitudinals up to four are tabulated in appendix II. For five and six longitudinals it is more convenient to evaluate the $h\theta$ terms numerically and to substitute into the slope deflection equations for transverse moments.

Solution of Continuous Girder Systems by Superposition

The above theory may be applied to the solution of continuous beam bridges and also to derive influence lines for the structure in the following manner: in essence, the bridge is treated as a single span between the first and last support with the intermediate supports removed and the applied loads are distributed in the usual manner. The support forces are introduced and their magnitudes are calculated from the requirement that the deflections at the support points must be zero. The support forces are then considered to be applied to a single span bridge and are distributed accordingly. Superposition of the effects of the loads and support forces then gives the solution for the continuous beam bridge.

Calculations by this method are greatly facilitated by a suitable system of notation which will be clear from the following example. Consider a two span, three girder continuous beam with intermediate support distance r, and a concentrated load W distance a from the left hand end. Then, with origin at the left hand end, the Fourier series for the deflection of one of the longitudinals carrying the load by itself (i.e. the "free" deflection) with its centre support removed is:

$$Y_L = \frac{2\,W L^3}{\pi^4 E I}\left[\sin\frac{\pi a}{L}\sin\frac{\pi x}{L} + \frac{1}{2^4}\sin\frac{2\pi a}{L}\sin\frac{2\pi x}{L} + \frac{1}{3^4}\sin\frac{3\pi a}{L}\sin\frac{3\pi x}{L} + \cdots\right]$$

where L is the total length of the bridge.

Then considering this load to be distributed amongst the three longitudinals of the bridge treated as a single span L, the deflections of the three longitudinals are:

$$y_{1L} = \frac{2\,W L^3}{\pi^4 E I}\left[\rho_1 \sin\frac{\pi a}{L}\sin\frac{\pi x}{L} + \rho_1'' \cdot \frac{1}{2^4}\sin\frac{2\pi a}{L}\sin\frac{2\pi x}{L} + \right.$$
$$\left. + \rho_1''' \cdot \frac{1}{3^4}\sin\frac{3\pi a}{L}\sin\frac{3\pi x}{L} + \cdots\right] = \bar{\rho}_{12}\,[Y_L]$$

$$y_{2L} = \frac{2\,W L^3}{\pi^4 E I}\left[\rho_2 \sin\frac{\pi a}{L}\sin\frac{\pi x}{L} + \rho_2'' \cdot \frac{1}{2^4}\sin\frac{2\pi a}{L}\sin\frac{2\pi x}{L} + \right.$$
$$\left. + \rho_2''' \cdot \frac{1}{3^4}\sin\frac{3\pi a}{L}\sin\frac{3\pi x}{L} + \cdots\right] = \bar{\rho}_{22}\,[Y_L]$$

$$y_{3L} = \frac{2\,W\,L^3}{\pi^4\,E\,I}\left[\rho_3 \sin\frac{\pi\,a}{L}\sin\frac{\pi\,x}{L} + \rho_3'' \cdot \frac{1}{2^4}\sin\frac{2\,\pi\,a}{L}\sin\frac{2\,\pi\,x}{L} + + \rho_3''' \cdot \frac{1}{3^4}\sin\frac{3\,\pi\,a}{L}\sin\frac{3\,\pi\,x}{L} + \cdots\right] = \bar{\rho}_{32}\,[Y_L]$$

where $\bar{\rho}$ is used as an operator in the manner indicated. The suffixes are introduced to define the girder to which the coefficients apply and the girder on which the load acts. Thus $\bar{\rho}_{12}$ is the distribution operator for the first longitudinal for the load acting on the second. In the same way, the series for the free deflections which would result from application of unit load at the reaction point on any longitudinal is:

$$Y_R = \frac{2\,L^3}{\pi^4\,E\,I}\left[\sin\frac{\pi\,r}{L}\sin\frac{\pi\,x}{L} + \frac{1}{2^4}\sin\frac{2\,\pi\,r}{L}\sin\frac{2\,\pi\,x}{L} + \frac{1}{3^4}\sin\frac{3\,\pi\,r}{L}\sin\frac{3\,\pi\,x}{L} + \cdots\right]$$

Thus the deflections of the longitudinals for unit load applied to longitudinal (1) are:

$$y_{1R} = \bar{\rho}_{11}[Y_R], \quad y_{2R} = \bar{\rho}_{21}[Y_R], \quad y_{3R} = \bar{\rho}_{31}[Y_R]$$

and similarly for unit loads at the reaction points on the other two girders.

We may then write down the following equations to express the condition of zero deflection at the intermediate support point of each girder:

For longitudinals (1) and (3)

$$R_2 \cdot \bar{\rho}_{12}[Y_R]_{x=r} + R_1(\bar{\rho}_{11} + \bar{\rho}_{13})[Y_R]_{x=r} = \bar{\rho}_{12}[Y_L]_{x=r} \tag{1}$$

For longitudinal (2)

$$R_2 \cdot \bar{\rho}_{22}[Y_R]_{x=r} + 2\,R_1\,\bar{\rho}_{21}[Y_R]_{x=r} = \bar{\rho}_{22}[Y_L]_{x=r} \tag{2}$$

Solving (1) and (2) we obtain:

$$R_2 = \frac{2\,\bar{\rho}_{12}[Y_L]\,\bar{\rho}_{21}[Y_R] - \bar{\rho}_{22}[Y_L]\,(\bar{\rho}_{11} + \bar{\rho}_{13})\,[Y_R]}{2\,\bar{\rho}_{12}[Y_R]\,\bar{\rho}_{21}[Y_R] - \bar{\rho}_{22}[Y_R]\,(\bar{\rho}_{11} + \bar{\rho}_{13})\,[Y_R]}$$

and

$$R_1 = \frac{\bar{\rho}_{22}[Y_L]\,\bar{\rho}_{12}[Y_R] - \bar{\rho}_{12}[Y_L]\,\bar{\rho}_{22}[Y_R]}{2\,\bar{\rho}_{12}[Y_R]\,\bar{\rho}_{21}[Y_R] - \bar{\rho}_{22}[Y_R]\,(\bar{\rho}_{11} + \bar{\rho}_{13})\,[Y_R]}$$

assuming that x is put equal to r in each of the deflection series. Knowing R_1 and R_2 in terms of W the deflection curves for girders (1) and (2) with loads and reactions acting together are obtained immediately.

To obtain the bending moments in the continuous bridge, R_1 and R_2 are calculated as described and the bending moments to which they give rise are distributed through the single span bridge and superimposed on the bending moments due to the external loads similarly distributed.

As a simple example suppose that for a certain structure having torsionally stiff longitudinals and for which $\alpha = 22.2$ (taken over the whole length L) the load is applied at $a = L/4$ and the intermediate supports are located at $r = L/2$.

Then the distribution coefficients are as follows:

Load on Girder (1)	ρ_{11} or ρ_{33}	ρ_{21} or ρ_{23}	ρ_{31} or ρ_{13}
1st. Harmonic	0.435	0.321	0.244
2nd. Harmonic	0.575	0.268	0.156
3rd. Harmonic	0.816	0.150	0.033
Load on Girder (2)	ρ_{12}	ρ_{22}	ρ_{32}
1st. Harmonic	0.321	0.358	0.321
2nd. Harmonic	0.269	0.462	0.269
3rd. Harmonic	0.150	0.700	0.150

Substituting $a = L/4$ in the free deflection series:

$$Y_L = \frac{2\,W\,L^3}{\pi^4\,E\,I}\left[0.707\sin\frac{\pi x}{L} + \frac{1}{16}\sin\frac{2\pi x}{L} + \frac{0.707}{81}\sin\frac{3\pi x}{L}\right]$$

At $x = r = \frac{L}{2}, \quad Y_L = \frac{2\,W\,L^3}{\pi^4\,E\,I}\left[0.707\left(1 - \frac{1}{81}\right)\right]$

For the reactions $r = L/2$ so that:

$$Y_R = \frac{2\,L^3}{\pi^4\,E\,I}\left[\sin\frac{\pi x}{L} - \frac{1}{81}\sin\frac{3\pi x}{L}\right]$$

for unit load and at $x = r = L/2$

$$Y_R = \frac{2\,L^3}{\pi^4\,E\,I}\left[1 + \frac{1}{81}\right]$$

The figures within the square brackets must not, of course be added or subtracted before application of the distribution coefficients.

Substituting, we obtain (for R_2):

$$R_2 = \frac{2\cdot 0.707\left[0.321 - \frac{0.15}{81}\right]\left[0.321 + \frac{0.15}{81}\right] - 0.707\left[0.358 - \frac{0.700}{81}\right]\left[0.679 + \frac{0.849}{81}\right]}{2\left[0.321 + \frac{0.15}{81}\right]\left[0.321 + \frac{0.15}{81}\right] - \left[0.358 + \frac{0.700}{81}\right]\left[0.679 + \frac{0.849}{81}\right]}\,W$$

i.e. $R_2 = 0.560\,W$.

Similarly $R_1 = 0.065\,W$ and the deflection equations for the girders are

$$y_1 = \bar\rho_{12}[Y_L] - R_1(\bar\rho_{11} + \bar\rho_{13})[Y_R] - R_2\bar\rho_{12}[Y_R] =$$
$$= \frac{2\,W\,L^3}{\pi^4\,E\,I}\left[0.0065\sin\frac{\pi x}{L} + 0.0168\sin\frac{2\pi x}{L} + 0.0065\sin\frac{3\pi x}{L}\right]$$

$$y_2 = \bar\rho_{22}[Y_L] - 2R_1\bar\rho_{21}[Y_R] - R_2\bar\rho_{22}[Y_R] =$$
$$= \frac{2\,W\,L^3}{\pi^4\,E\,I}\left[0.0110\sin\frac{\pi x}{L} + 0.0289\sin\frac{2\pi x}{L} + 0.0110\sin\frac{3\pi x}{L}\right]$$

It will be observed that a check on the result is obtained since $y_1 = y_2 = 0$ at $x = L/2$ i.e. at the intermediate support points. The second harmonic term is in any case zero at $x = L/2$ and thus the coefficients of the first and third harmonics must be equal. Inspection will also show that the calculation of R_1 and R_2 is quite short as many of the terms appear more than once. The above is a simple example which has been solved in general terms; in more complicated cases the equations are easily written down and may be solved numerically. Due regard must of course be paid to numerical accuracy as the solution depends fundamentally on the differences of comparitively large numbers.

The method of deriving influence lines is based on Maxwell's reciprocal deflection theorem. If in the structure referred to in the previous example, the centre support of girder (2) is removed and unit vertical load applied at the reaction point, it is well known that the vertical deflection at any point is to some scale equal to the ordinate of the influence line for vertical reaction at the point of application of the unit load. The procedure is therefore as follows:

a) Assuming that all the intermediate supports are withdrawn find the deflection of each of those points resulting from the application of unit vertical load at the support for which the reaction influence line is required.

b) Calculate the forces which have to be applied at the other support points to bring them back to their original positions. Superposition of the deflection curves given by (a) and (b) will yield influence line curves for the support force to a certain scale; the scale may be corrected by adjusting the ordinate at the support removed to unity.

c) We may now obtain influence lines for bending moment at any point on the bridge, as indicated in fig. 3. First, draw the free influence line for bending moment at the required point for the bridge as a single span. Second, for several positions of the load on the span calculate the bending moments in the unloaded girders at this section and subtract their sum from the free bending moment influence line; this gives the influence line for the loaded girder treating the bridge as a single span interconnected system. Finally, superimpose on this influence line, the influence line for bending moment at the support point of the loaded beam resulting from the application of the intermediate support forces. The superimposed curves must obviously have the same ordinate at the support point as the ordinate of the bending moment influence line is there equal to zero.

Although this may appear to be rather lengthy, it is not in fact unduly so as may be seen from the following example. Taking the two equal span bridge of the previous example, to derive influence lines for the mid-point of the first span we draw first the "free" influence line diagram, the maximum ordinate of which is $0.1875\,L$ (see fig. 3a). The series for the free bending moment diagram for a unit load applied to a single span beam at distance a from the left hand end is:

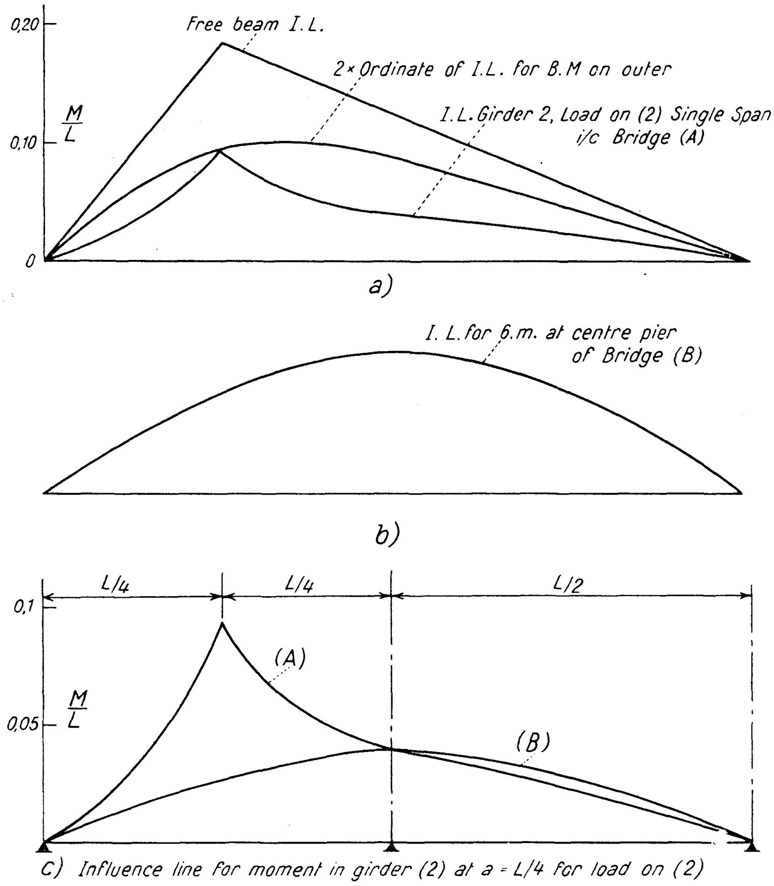

Fig. 3

$$M_x = \frac{2\,L}{\pi^2}\left(\sin\frac{\pi\,a}{L}\sin\frac{\pi\,x}{L} + \frac{1}{4}\sin\frac{2\,\pi\,a}{L}\sin\frac{2\,\pi\,x}{L} + \frac{1}{9}\sin\frac{3\,\pi\,a}{L}\sin\frac{3\,\pi\,x}{L} + \cdots\right)$$

Applying the appropriate distribution coefficients and substituting $x = L/4$ we obtain the influence line for this point on one of the outer girders with the load acting on the centre girder:

$$m_1 = \frac{2\,L}{\pi^2}\left(0.227\sin\frac{\pi\,a}{L} + 0.067\sin\frac{2\,\pi\,a}{L} + 0.0118\sin\frac{3\,\pi\,a}{L}\right)$$

taking the first three harmonics. On substituting several values of $\frac{a}{L}$, multiplying the resulting values by two and subtracting the ordinates from those of the free influence line we obtain the influence line for bending moment at the required section for the bridge simply supported on a single span, as shown in fig. 3a.

We must next obtain the influence line for the intermediate support force on girder (2) which is given by the deflection curve of that girder for unit load at its mid-point when the support points on the outers are brought back to their original levels. The restoring forces are obtained from the equation:

$$R_1 (\bar{\rho}_{11} + \bar{\rho}_{13}) [Y_R] = \bar{\rho}_{12} [Y_L]$$

i. e.

$$R_1 = \frac{\bar{\rho}_{12} [Y_L]}{(\bar{\rho}_{11} + \bar{\rho}_{13}) [Y_R]}$$

In this case Y_L and Y_R both refer to loads placed at the mid-point of the bridge and therefore

$$R_1 = \frac{\left[0.321 + \frac{0.15}{81}\right]}{\left[0.679 + \frac{0.849}{81}\right]} = 0.468$$

The upward deflection of girder (2) due to the forces R_1 on (1) and (3) is therefore:

$$k\left(0.468 \cdot 0.642 \sin \frac{\pi x}{L} - \frac{0.468}{81} \cdot 0.300 \sin \frac{3\pi x}{L}\right)$$

where k is some constant. The net deflection of (2) i. e. the influence line for R_2 is then

$$k\left[\left(0.358 \sin \frac{\pi x}{L} - \frac{0.700}{81} \sin \frac{3\pi x}{L}\right) - \left(0.3008 \sin \frac{\pi x}{L} - \frac{0.1404}{81} \sin \frac{3\pi x}{L}\right)\right]$$

i. e.

$$k\left[0.0472 \sin \frac{\pi x}{L} - 0.0069 \sin \frac{3\pi x}{L}\right]$$

The deflection curve for girder (2) for unit load at the mid-point of girder (1), the support there being removed, gives the influence line for R_1 with the load on girder (2). This curve is found in the same way as R_2 and is given by:

$$k\left[0.0065 \sin \frac{\pi x}{L} + 0.0065 \sin \frac{3\pi x}{L}\right]$$

It will be noted that we are concerned only with the form of the curves and therefore the constant k need not be evaluated. The forces R_1 and R_2 must now be considered to be applied simultaneously to the structure and their effect superimposed on the single span interconnected bridge. The upward bending moment due to R_1 and R_2 will therefore be proportional to

$$\bar{\rho}_{22} R_2 + 2\bar{\rho}_{21} R_1 \quad \text{i. e.} \quad \left(0.358 \cdot 0.0472 \sin \frac{\pi x}{L} - 0.700 \cdot 0.0069 \sin \frac{3\pi x}{L}\right) +$$

$$+ 2\left(0.321 \cdot 0.0065 \sin \frac{\pi x}{L} + 0.15 \cdot 0.0065 \sin \frac{3\pi x}{L}\right)$$

and the bending moment influence line is:

$$m_r = k'\left(0.0209 \sin \frac{\pi x}{L} - 0.0038 \sin \frac{3\pi x}{L}\right)$$

This curve is plotted to an arbitrary scale in fig. 3b and superimposed on the curve for the single span bridge in fig. 3c to give the bending moment influence line for the mid-point of the first span of the continuous beam bridge. Other bending moment influence lines are readily derived in the same manner.

Conclusion

Space does not permit the comparison of the theoretical results obtained by the method described in this paper with those obtained by experiment. A large number of comparisons have, however, been made both for single span and continuous beams and very satisfactory agreement has been obtained for zero and infinite torsional stiffness and also for intermediate cases. These results are being described elsewhere [4, 5].

The method is extremely convenient for single span bridges, particularly with design loadings when sufficient accuracy will be obtained by distributing only one or two harmonics of the bending moment diagram. Continuous beams with a limited number of supports and longitudinals can also be solved without difficulty for particular loading cases but for design purposes it may be found more convenient to derive influence lines for the various longitudinals.

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Notation

L Span of bridge.

h Transverse spacing of longitudinals.

n Number of cross girders.

$E\,I$ Flexural rigidity of longitudinals when all are of same dimensions.

$E\,I_1$ Flexural rigidity of outer longitudinals.

$E\,I_2$ Flexural rigidity of inner longitudinals.

$E\,I_T$ Flexural rigidity of one cross girder. In a beam and slab bridge: flexural rigidity per unit length of slab.

$C\,J$ Torsional rigidity of longitudinal (referred to inner when sections are different).

$m = \dfrac{12\,n\,E\,I_T}{h^3\,L}$ for a bridge with cross girders.

$\left(= \dfrac{12\,E\,I_T}{h^3} \quad \text{for a beam and slab bridge}\right)$

$\alpha = \dfrac{m\,L^4}{\pi^4\,E\,I} = \dfrac{12}{\pi^4}\left(\dfrac{L}{h}\right)^3 \dfrac{n\,E\,I_T}{E\,I} \left[= \dfrac{12}{\pi^4}\left(\dfrac{L}{h}\right)^3 \dfrac{L\,E\,I_T}{E\,I} \text{ for a beam and slab bridge}\right]$

$\alpha_1, \alpha_2 \cdots$ reduced α parameters as defined in Appendix I.

$\beta = \dfrac{\pi^2}{2\,n}\left(\dfrac{h}{L}\right)\left(\dfrac{C\,J}{E\,I_T}\right) \left[= \dfrac{\pi^2}{2\,L}\left(\dfrac{h}{L}\right)\left(\dfrac{C\,J}{E\,I_T}\right) \text{ for a beam and slab bridge}\right]$

$\eta = \dfrac{E\,I_1}{E\,I_2}$

ρ Distribution coefficient.

$\bar{\rho}$ Distribution coefficient operator.

x Distance measured in the longitudinal direction (from mid-span unless otherwise stated).

z Distance measured in the transverse direction (from left hand edge unless otherwise stated).

y Vertical deflection of the structure.

$s_1, s_2 \ldots$ Load per unit length transferred to longitudinals by the transverse medium.

$M_{12}, M_{21} \ldots$ Moments per unit length in the transverse medium at the ends of the intercepts between longitudinals 1, 2 etc.

Y Free deflection of single beam carrying total load on bridge (referred to inner girder if outers are of different section).

M Bending moment corresponding to Y i.e. total bending moment on the span.

A Amplitude of 1st. harmonic component of Y.

Appendix I

Table "A"

First harmonic bending moment distribution coefficients for interconnected bridge girders

$$\alpha = \frac{12}{\pi^4}\left(\frac{L}{h}\right)^3 \frac{n E I_T}{E I} \qquad \alpha_0 = \alpha\left(1-\frac{8}{\pi^2}\right) \qquad \alpha_1 = \alpha\left(1-\frac{6}{\pi^2}\right)$$

$$\alpha_2 = \alpha\left(1-\frac{36}{5\pi^2}\right) \qquad \alpha_3 = \alpha\left(1-\frac{20}{3\pi^2}\right) \qquad \alpha_4 = \alpha\left(1-\frac{4}{\pi^2}\right)$$

For deflection coefficients divide bending moment coefficients for outers by η.

No. of girders and load position	Coefficient	$\beta = 0$	$\beta = \infty$
2 Load on (1)	ρ_{11} ρ_{12}	1.0 0.0	$(1+\alpha_0)/(1+2\alpha_0)$ $\alpha_0/(1+2\alpha_0)$
3 Load on (1)	ρ_{11} ρ_{21} ρ_{31}	$[8\eta+\alpha(1+4\eta)]/D_1$ $2\alpha/D_1$ $-\alpha/D_1$ $D_1 = 8\eta+\alpha(2+4\eta)$	$\eta/2\,\{(1+2\alpha_1)/D_2+1/(\eta+\alpha_0)\}$ α_1/D_2 $\eta/2\,\{(1+2\alpha_1)/D_2-1/(\eta+\alpha_0)\}$ $D_2 = \eta+\alpha_1(1+2\eta)$
3 Uniformload covering bridge	ρ_{10} ρ_{20}	$(3+4\alpha)\eta/D_3$ $(10\eta+4\alpha)/D_3$ $D_3 = 4[4\eta+\alpha(1+2\eta)]$	$\eta(16\alpha_1+3)/16\alpha_1(1+2\eta)+16\eta$ $(16\alpha_1+10\eta)/16\alpha(1+2\eta)+16\eta$
4 Load on (1)	ρ_{11} ρ_{21} ρ_{31} ρ_{41}	$[60\eta^2+\alpha^2\eta(9+5\eta)+8\alpha\eta(1+12\eta)]/D_4$ $2\alpha[9\eta+\alpha+3\alpha\eta]/D_4$ $\alpha[-12\eta-\alpha+3\alpha\eta]/D_4$ $2\alpha\eta(1-2\alpha)/D_4$ $D_4 = [10\eta+\alpha(1+\eta)][6\eta+\alpha(1+9\eta)]$	$\eta/2\,\{(1+\alpha_2)/D_5+(1+3\alpha_4)/D_6\}$ $\frac{1}{2}\{\alpha_2/D_5+\alpha_4/D_6\}$ As ρ_{21} but with $-$ sign between terms As ρ_{11} but with $-$ sign between terms $D_5 = [\eta+\alpha_2(1+\eta)]$ $D_6 = [(\eta+\alpha_3)(1+3\alpha_4)-(\alpha_4)^2]$
4 Load on (2)	ρ_{12} ρ_{22} ρ_{32} ρ_{42}	$2\alpha\eta[9\eta+\alpha(1+3\eta)]/D_4$ $[60\eta^2+\alpha^2(1+5\eta)+16\alpha\eta(1+3\eta)]/D_4$ $2\alpha\eta(26\eta+2\alpha)/D_4$ $-\alpha\eta[12\eta+\alpha(1-3\eta)]/D_4$	$\eta/2\,\{\alpha_2/D_5+\alpha_4/D_6\}$ $\frac{1}{2}\{(\eta+\alpha_2)/D_5+(\eta+\alpha_3)/D_6\}$ As ρ_{22} but with $-$ sign between terms As ρ_{12} but with $-$ sign between terms
4 Uniformload covering bridge	$\rho_{10}=\rho_{40}$ $\rho_{20}=\rho_{30}$	$\eta(3\alpha+8)/D_1$ $(3\alpha+22\eta)/D_1$ $D_1 = 6\alpha(1+\eta)+60\eta$	$\eta(10\alpha_2+1)/D_8$ $(10\alpha_2+4\eta)/D_8$ $D_8 = 10\alpha_2(2+2\eta)+10\eta$
5 Load on (1) $\eta = 1$	ρ_{11} ρ_{21} ρ_{31} ρ_{41} ρ_{51}	$(224+654\alpha+324\alpha^2+15\alpha^3)/D_9$ $\alpha(68+101\alpha+10\alpha^2)/D_9$ $\alpha(\alpha-6)(5\alpha+8)/D_9$ $\alpha(12-35\alpha)/D_9$ $\alpha(-5\alpha^2+12\alpha-2)/D_9$ $D_9 = (5\alpha+8)(5\alpha^2+68\alpha+28)$	$(0.5+1.305\alpha+0.171\alpha^2)/D_{10}+$ $+(0.5+0.392\alpha)/D_{11}$ $(0.239\alpha+0.171\alpha^2)/D_{10}+0.196\alpha/D_{11}$ $(-0.174\alpha+0.171\alpha^2)/D_{10}$ As ρ_{21} but with $-$ sign between terms As ρ_{11} but with $-$ sign between terms $D_{10} = (1+2.914\alpha+0.855\alpha^2)$ $D_{11} = (1+1.07\alpha+0.076\alpha^2)$

No. of girders and load position	Coeffi-cient	$\beta = 0$	$\beta = \infty$
5 Load on (2) $\eta = 1$	ρ_{12}	$\alpha\,(68 + 101\,\alpha + 10\,\alpha^2)/D_9$	$(0.239\,\alpha + 0.171\,\alpha^2)/D_{10} + 0.196\,\alpha/D_{11}$
	ρ_{22}	$(224 + 500\,\alpha + 152\,\alpha^2 + 7.5\,\alpha^3)/D_9$	$(0.5 + 0.805\,\alpha + 0.171\,\alpha^2)/D_{10} +$ $+ (0.5 + 0.145\,\alpha)/D_{11}$
	ρ_{32}	$\alpha\,(22 + \alpha)\,(5\,\alpha + 8)/D_9$	$(0.826\,\alpha + 0.171\,\alpha^2)/D_{10}$
	ρ_{42}	$\alpha\,(-72 + 44\,\alpha + 2.5\,\alpha^2)/D_9$	As ρ_{22} but with − sign between terms
	ρ_{52}	$\alpha\,(12 - 35\,\alpha)/D_9$	As ρ_{12} but with − sign between terms
5 Load on (3) $\eta = 1$	$\rho_{13} = \rho_{53}$	$\alpha\,(\alpha - 6)\,(5\,\alpha + 8)/D_9$	$(-0.174\,\alpha + 0.171\,\alpha^2)/D_{10}$
	$\rho_{23} = \rho_{43}$	$\alpha\,(22 + \alpha)\,(5\,\alpha + 8)/D_9$	$(0.826\,\alpha + 0.171\,\alpha^2)/D_{10}$
	ρ_{33}	$(28 + 36\,\alpha + \alpha^2)\,(5\,\alpha + 8)/D_9$	$(1 + 1.61\,\alpha + 0.171\,\alpha^2)/D_{10}$
6 Load on (1) $\eta = 1$	ρ_{11}	$(76+78\,\alpha+\alpha^2)/D_{12}+(44+130\,\alpha+25\,\alpha^2)/D_{13}$	$\frac{1}{2}\{(1 + 1.4\,\alpha + 0.109\,\alpha^2)/D_{14} +$ $+ (1 + 3.34\,\alpha + 1.25\,\alpha^2)/D_{15}\}$
	ρ_{21}	$\alpha\,(\alpha + 22)/D_{12} + \alpha\,(15\,\alpha + 14)/D_{13}$	$\frac{1}{2}\{(0.424\,\alpha + 0.109\,\alpha^2)/D_{14} +$ $+ (0.448\,\alpha + 0.75\,\alpha^2)/D_{15}\}$
	ρ_{31}	$\alpha\,(\alpha - 12)/D_{12} + \alpha\,(5\,\alpha - 12)/D_{13}$	$\frac{1}{2}\{(-0.128\,\alpha + 0.109\,\alpha^2)/D_{14} +$ $+ (-0.221\,\alpha + 0.25\,\alpha^2)/D_{15}\}$
	ρ_{41}	As ρ_{31} but with − sign between terms	As ρ_{31} but with − sign between terms
	ρ_{51}	As ρ_{21} but with − sign between terms	As ρ_{21} but with − sign between terms
	ρ_{61}	As ρ_{11} but with − sign between terms	As ρ_{11} but with − sign between terms
		$D_{12} = (152+176\alpha+6\alpha^2)$, $D_{13} = (88+272\,\alpha+$ $+70\,\alpha^2)$,	$D_{14} = (1+1.696\,\alpha+0.327\,\alpha^2)$ $D_{15} = (1+3.64\,\alpha+2\,\alpha^2+0.094\,\alpha^3)$
6 Load on (2) $\eta = 1$	ρ_{12}	$\alpha\,(\alpha + 22)/D_{12} + \alpha\,(15\,\alpha + 14)/D_{13}$	$\frac{1}{2}\{(0.424\,\alpha + 0.109\,\alpha^2)/D_{14} +$ $+ (0.448\,\alpha + 0.75\,\alpha^2)/D_{15}\}$
	ρ_{22}	$(76+32\,\alpha+\alpha^2)/D_{12}+(44+96\,\alpha+9\,\alpha^2)/D_{13}$	$\frac{1}{2}\{(1 + 0.72\,\alpha + 0.109\,\alpha^2)/D_{14} +$ $(1 + 2.53\,\alpha + 0.62\,\alpha^2)/D_{15}\}$
	ρ_{32}	$\alpha\,(\alpha + 34)/D_{12} + \alpha\,(3\,\alpha + 50)/D_{13}$	$\frac{1}{2}\{(0.552\,\alpha + 0.109\,\alpha^2)/D_{14} +$ $+ (1.115\,\alpha + 0.235\,\alpha^2)/D_{15}\}$
	ρ_{42}	As ρ_{32} but with − sign between terms	As ρ_{32} but with − sign between terms
	ρ_{52}	As ρ_{22} but with − sign between terms	As ρ_{22} but with − sign between terms
	ρ_{62}	As ρ_{12} but with − sign between terms	As ρ_{12} but with − sign between terms
6 Load on (3) $\eta = 1$	ρ_{13}	$\alpha\,(\alpha - 12)/D_{12} + \alpha\,(5\,\alpha - 12)/D_{13}$	$\frac{1}{2}\{(-0.128\,\alpha + 0.109\,\alpha^2)/D_{14} +$ $+ (-0.221\,\alpha + 0.25\,\alpha^2)/D_{15}\}$
	ρ_{23}	$\alpha\,(\alpha + 34)/D_{12} + \alpha\,(3\,\alpha + 50)/D_{13}$	$\frac{1}{2}\{(0.552\,\alpha + 0.109\,\alpha^2)/D_{14} +$ $+ (1.115\,\alpha + 0.235\,\alpha^2)/D_{15}\}$
	ρ_{33}	$(76+66\,\alpha+\alpha^2)/D_{12}+(44+46\,\alpha+\alpha^2)/D_{13}$	$\frac{1}{2}\{(1 + 1.272\,\alpha + 0.109\,\alpha^2)/D_{14} +$ $+ (1 + 1.415\,\alpha + 0.14\,\alpha^2)/D_{15}\}$
	ρ_{43}	As ρ_{33} but with − sign between terms	As ρ_{33} but with − sign between terms
	ρ_{53}	As ρ_{23} but with − sign between terms	As ρ_{23} but with − sign between terms
	ρ_{63}	As ρ_{13} but with − sign between terms	As ρ_{13} but with − sign between terms

Table "B"
Bending moment distribution coefficients for higher harmonics

For $\beta=0$: Use coefficients of table "A" but replace α by α/p^4
For $\beta=\infty$: Use coefficients of table "B" but replace α by α/p^4

No. of girders and load position	Coefficient	$\beta=0$
2 Load on (1)	ρ_{11} ρ_{21}	$(1+\alpha)/(1+2\alpha)$ $\alpha/(1+2\alpha)$
3 Load on (1)	ρ_{11} ρ_{21} ρ_{31}	$\frac{1}{2}\{1/D_1+(1+2\alpha)/D_2\}$ α/D_2 As ρ_{11} but with − sign between terms $D_1=(\eta+\alpha)$ $D_2=\eta+\alpha(1+2\eta)$
3 Load on (2)	$\rho_{12}=\rho_{32}$ ρ_{22}	$\eta\alpha/D_2$ $(\eta+\alpha)/D_2$
4 Load on (1)	ρ_{11} ρ_{21} ρ_{31} ρ_{41}	$\eta/2\{(1+\alpha)/D_3+(1+3\alpha)/D_4\}$ $\frac{1}{2}\{\alpha/D_3+\alpha/D_4\}$ As ρ_{21} but with − sign between terms As ρ_{11} but with − sign between terms $D_3=\eta+\alpha(1+\eta)$ $D_4=\eta+\alpha(1+3\eta)+2\alpha^2$
4 Load on (2)	ρ_{12} ρ_{22} ρ_{32} ρ_{42}	$\eta/2\{\alpha/D_3+\alpha/D_4\}$ $\frac{1}{2}\{(\eta+\alpha)/D_3+(\eta+\alpha)/D_4\}$ As ρ_{22} but with − sign between terms As ρ_{12} but with − sign between terms
5 Load on (1) $\eta=1$	ρ_{11} ρ_{21} ρ_{31} ρ_{41} ρ_{51}	$\frac{1}{2}\{(1+4\alpha+2\alpha^2)/D_5+(1+2\alpha)/D_6\}$ $\frac{1}{2}\{\alpha(1+2\alpha)/D_5+\alpha/D_6\}$ α^2/D_5 As ρ_{21} but with − sign between terms As ρ_{11} but with − sign between terms $D_5=(1+5\alpha+5\alpha^2)$ $D_6=(1+3\alpha+\alpha^2)$

No. of girders and load position	Coefficient	$\beta=\infty$
5 Load on (2) $\eta=1$	ρ_{12} ρ_{22} ρ_{32} ρ_{42} ρ_{52}	$\frac{1}{2}\{\alpha(1+2\alpha)/D_5+\alpha/D_6\}$ $\frac{1}{2}\{(1+\alpha)(1+2\alpha)/D_5+(1+\alpha)/D_6\}$ $\alpha(1+\alpha)/D_5$ As ρ_{22} but with − sign between terms As ρ_{12} but with − sign between terms
5 Load on (3)	$\rho_{13}=\rho_{53}$ $\rho_{23}=\rho_{43}$ ρ_{33}	α^2/D_5 $\alpha(1+\alpha)/D_5$ $(1+3\alpha+\alpha^2)/D_5$
6 Load on (1) $\eta=1$	ρ_{11} ρ_{21} ρ_{31} ρ_{41} ρ_{51} ρ_{61}	$\frac{1}{2}\{(1+3\alpha+\alpha^2)/D_7+(1+5\alpha+5\alpha^2)/D_8\}$ $\frac{1}{2}\{\alpha(1+\alpha)/D_7+\alpha(1+3\alpha)/D_8\}$ $\frac{1}{2}\{\alpha^2/D_7+\alpha^2/D_8\}$ As ρ_{31} but with − sign between terms As ρ_{21} but with − sign between terms As ρ_{11} but with − sign between terms $D_7=(1+\alpha)(1+3\alpha)$ $D_8=(1+6\alpha+9\alpha^2+2\alpha^3)$
6 Load on (2) $\eta=1$	ρ_{12} ρ_{22} ρ_{32} ρ_{42} ρ_{52} ρ_{62}	$\frac{1}{2}\{\alpha(1+\alpha)/D_7+\alpha(1+3\alpha)/D_8\}$ $\frac{1}{2}\{(1+\alpha)^2/D_7+(1+\alpha)(1+3\alpha)/D_8\}$ $\frac{1}{2}\{\alpha(1+\alpha)/D_7+\alpha(1+\alpha)/D_8\}$ As ρ_{32} but with − sign between terms As ρ_{22} but with − sign between terms As ρ_{12} but with − sign between terms
6 Load on (3) $\eta=1$	ρ_{13} ρ_{23} ρ_{33} ρ_{43} ρ_{53} ρ_{63}	$\frac{1}{2}\{\alpha^2/D_7+\alpha^2/D_8\}$ $\frac{1}{2}\{\alpha(1+\alpha)/D_7+\alpha(1+\alpha)/D_8\}$ $\frac{1}{2}\{(1+3\alpha+\alpha^2)/D_7+(1+3\alpha+\alpha^2)/D_8\}$ As ρ_{33} but with − sign between terms As ρ_{23} but with − sign between terms As ρ_{13} but with − sign between terms

Appendix II

Transverse moments

Transverse moments are per unit length of transverse medium at mid-span.
M is the "free" longitudinal bending moment.
Distribution coefficients in formulæ are for bending moment.

No. of girders and load position	$\beta=0$	$\beta=\infty$
2 Load on (1)		$M_{12}=M_{21}=1.79\,\frac{h}{L^2}\,\alpha\,(\rho_1-\rho_2)\,M$
3 Load on (1)	$M_{12}=M_{32}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\cdot h\cdot\rho_3\cdot M$	$M_{12}=1.79\,\frac{h}{L^2}\,\alpha\,(\rho_1-\rho_2)\cdot M$ $M_{21}=4.94\,\frac{h}{L^2}\,\alpha\,[(\rho_2-\rho_1)+0.16\,(3\,\rho_1-2\,\rho_2-\rho_3)]\cdot M$ $M_{23}=4.94\,\frac{h}{L^2}\,\alpha\,[(\rho_2-\rho_3)+0.16\,(3\,\rho_3-2\,\rho_2-\rho_1)]\cdot M$ $M_{32}=1.79\,\frac{h}{L^2}\,\alpha\,(\rho_3-\rho_2)\cdot M$
3 Load on (2)	$M_{12}=M_{32}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\cdot h\cdot\rho_1\cdot M$	$M_{12}=M_{32}=1.79\,\frac{h}{L_2}\,\alpha\,(\rho_1-\rho_2)\cdot M$ $M_{21}=M_{23}=3.37\,\frac{h}{L^2}\,\alpha\,(\rho_2-\rho_1)\cdot M$
3 Uniform load over bridge	$M_{12}=M_{32}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\cdot h\cdot\rho_1\cdot M-\frac{wh^2}{2}$	$M_{12}=M_{32}=1.79\,\frac{h}{L^2}\,\alpha\,(\rho_1-\rho_2)\cdot M$ $M_{21}=M_{23}=3.37\,\frac{h}{L^2}\,\alpha\,(\rho_2-\rho_1)\cdot M-\frac{wh^2}{8}$
4 Load on (1)	$M_{12}=M_{43}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\cdot h\,(2\,\rho_4+\rho_3)\cdot M$ $M_{32}=M_{34}=\frac{\pi^2}{L^2}\cdot h\cdot\rho_4\cdot M$	$M_{12}=1.79\,\frac{h}{L^2}\,\alpha\,(\rho_1-\rho_2)\cdot M$ $M_{21}=4.94\,\frac{h}{L^2}\,\alpha\,[(\rho_2-\rho_1)-0.044\,(-11\,\rho_1+6\,\rho_2+6\,\rho_3-\rho_4)]\cdot M$ $M_{23}=4.94\,\frac{h}{L^2}\,\alpha\,[(\rho_2-\rho_1)-0.044\,(-4\,\rho_1-6\,\rho_2+9\,\rho_3+\rho_4)]\cdot M$ $M_{32}=4.94\,\frac{h}{L^2}\,\alpha\,[(\rho_3-\rho_4)-0.044\,(-4\,\rho_4-6\,\rho_3+9\,\rho_2+\rho_1)]\cdot M$ $M_{34}=4.94\,\frac{h}{L^2}\,\alpha\,[(\rho_3-\rho_4)-0.044\,(-11\,\rho_4+6\,\rho_3+6\,\rho_2-\rho_1)]\cdot M$ $M_{43}=1.79\,\frac{h}{L^2}\,\alpha\,(\rho_4-\rho_3)\cdot M$
4 Load on (2)	$M_{12}=M_{43}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\cdot h\cdot\rho_1\cdot M$ $M_{32}=M_{34}=\frac{\pi^2}{L^2}\cdot h\cdot\rho_4\cdot M$	As for load on (1)
4 Uniform load over bridge	$M_{12}=M_{43}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\cdot h\cdot\rho_1\cdot M-\frac{wh^2}{2}$ $M_{32}=M_{34}=$ do.	$M_{12}=M_{43}=1.79\,\frac{h}{L^2}\,\alpha\,(\rho_1-\rho_2)\cdot M$ $M_{21}=M_{34}=2.43\,\frac{h}{L^2}\,\alpha\,(\rho_2-\rho_1)\cdot M-\frac{wh^2}{10}$ $M_{23}=M_{32}=0.628\,\frac{h}{L^2}\,\alpha\,(\rho_2-\rho_1)\cdot M-\frac{w\,h^2}{10}$

Transverse moments in bridges having more than four torsionally stiff longitudinals (i. e. $\beta=\infty$) are most conveniently obtained by substitution in the general equations:

$$M_{12}=\frac{6\,n\,E\,I_T}{L\,h^2}\,[y_2-y_1-\tfrac{2}{3}h\,\theta_1-\tfrac{1}{3}h\,\theta_2] \qquad M_{21}=\frac{6\,n\,E\,I_T}{L\,h^2}\,[y_2-y_1-\tfrac{1}{3}h\,\theta_1-\tfrac{2}{3}h\,\theta_2]$$

$$M_{23}=\frac{6\,n\,E\,I_T}{L\,h^2}\,[y_3-y_2-\tfrac{2}{3}h\,\theta_1-\tfrac{1}{3}h\,\theta_3] \quad \text{etc.}$$

Values of the $h\,\theta$ terms are tabulated below. For this case the solutions are in two parts a) a symmetrical system: $\downarrow\frac{w}{2}\ \downarrow\frac{w}{2}$ and b) a skew symmetrical system: $\downarrow\frac{w}{2}\ \uparrow\frac{w}{2}$. The distribution coefficients resulting from these two parts are respectively the first and second terms in the expressions for distribution coefficients in Appendix I. The corresponding $h\,\theta$ values are quoted separately in the following table.

No. of girders and load position	$\beta=0$	$\beta=\infty$
5 Load on (1)	$M_{12}=M_{54}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\,h\,(3\,\rho_5+2\,\rho_4+\rho_3)\,M$ $M_{32}=M_{34}=\frac{\pi^2}{L^2}\,h\,(2\,\rho_5+\rho_4)\,M$ $M_{43}=M_{45}=\frac{\pi^2}{L^2}\,h\,\rho_5\,M$	a) $h\,\theta_1=\frac{6}{7\,\pi}\,(-3\,a_1+4\,a_2-a_3)$ $h\,\theta_2=\frac{6}{7\,\pi}\,(-a_1-a_2+2\,a_3)$ $h\,\theta_3=0$ b) $h\,\theta_1=\frac{1}{2\,\pi}\,(-5\,a_1+6\,a_2)$ $h\,\theta_2=\frac{1}{2\,\pi}\,(-2\,a_1)$ $h\,\theta_3=\frac{1}{2\,\pi}\,(a_1-6\,a_2)$
5 Load on (2)	$M_{12}=M_{54}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\,h\,\rho_1\,M$ $M_{32}=M_{34}=\frac{\pi^2}{L^2}\,h\,(2\,\rho_5+\rho_4)\,M$ $M_{43}=M_{45}=\frac{\pi^2}{L^2}\,h\,\rho_5\,M$	
5 Load on (3)	$M_{12}=M_{54}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\,h\,\rho_1\,M=M_{43}=M_{45}$ $M_{32}=M_{34}=\frac{\pi^2}{L^2}\,h\,(2\,\rho_1+\rho_2)\,M$	
6 Load on (1)	$M_{12}=M_{65}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\,h\,(4\,\rho_6+3\,\rho_5+2\,\rho_4+\rho_3)\,M$ $M_{32}=M_{34}=\frac{\pi^2}{L^2}\,h\,(3\,\rho_6+2\,\rho_5+\rho_4)\,M$ $M_{43}=M_{45}=\frac{\pi^2}{L^2}\,h\,(2\,\rho_6+\rho_5)\,M$ $M_{54}=M_{56}=\frac{\pi^2}{L^2}\,h\,\rho_6\,M$	a) $h\,\theta_1=\frac{6}{19\,\pi}\,(-8\,a_1+10\,a_2-2\,a_3)$ $h\,\theta_2=\frac{6}{19\,\pi}\,(-3\,a_1-a_2+4\,a_3)$ $h\,\theta_3=\frac{6}{19\,\pi}\,(a_1-6\,a_2+5\,a_3)$ b) $h\,\theta_1=\frac{2}{11\,\pi}\,(-14\,a_1+18\,a_2-6\,a_3)$ $h\,\theta_2=\frac{2}{11\,\pi}\,(-5\,a_1-3\,a_2+12\,a_3)$ $h\,\theta_3=\frac{2}{11\,\pi}\,(a_1-6\,a_2-9\,a_3)$
6 Load on (2)	$M_{12}=M_{65}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\,h\,\rho_1\,M$ $M_{32}=M_{34}=\frac{\pi^2}{L^2}\,h\,(3\,\rho_6+2\,\rho_5+\rho_4)\,M$ $M_{43}=M_{45}=\frac{\pi^2}{L^2}\,h\,(2\,\rho_6+\rho_5)\,M$ $M_{54}=M_{56}=\frac{\pi^2}{L^2}\,h\,\rho_6\,M$	
6 Load on (3)	$M_{12}=M_{65}=0$ $M_{21}=M_{23}=\frac{\pi^2}{L^2}\,h\,\rho_1\,M$ $M_{32}=M_{34}=\frac{\pi^2}{L^2}\,h\,(2\,\rho_1+\rho_2)\,M$ $M_{43}=M_{45}=\frac{\pi^2}{L^2}\,h\,(2\,\rho_6+\rho_5)\,M$ $M_{54}=M_{56}=\frac{\pi^2}{L^2}\,h\,\rho_6\,M$	

Summary

The method outlined is for the analysis of interconnected bridge girders having any degree of torsional rigidity and is based on two assumptions viz. that the transverse members can be replaced by a continuous medium and that torsion of these members can be neglected. The solution is reached by harmonic analysis and distribution coefficients are tabulated for single span bridges having from two to six main girders for all harmonics of the bending moment and deflection curves for the span. The application of the method to continuous beam systems by superposition is demonstrated; this is greatly facilitated by the use of an operational system of notation. A method for the derivation of influence lines for bending moments in the longitudinals of continuous bridges is also developed.

Résumé

La méthode ici indiquée est destinée au calcul des poutres de ponts associés entre elles et présentant une rigidité de torsion arbitraire. Cette méthode repose sur deux hypothèses, à savoir que les éléments transversaux sont remplacés par une liaison continue et que la torsion de ces éléments peut être négligée. L'étude de la solution conduit à une analyse harmonique, c'est-à-dire au développement de la flexion sous la forme d'une somme trigonométrique; les coefficients de répartition relatifs aux ponts à poutres simples comportant deux à six poutres principales sont indiqués dans des tableaux pour tous les termes de la série du moment fléchissant et de la courbe de flexion. En outre, les auteurs exposent les conditions de l'application de la méthode aux poutres continues avec superposition; l'emploi du système de travail relatif aux désignations facilite notablement l'étude. Les auteurs indiquent enfin une méthode pour l'obtention des lignes d'influence pour les moments fléchissants, dans les poutres longitudinales continues des ponts.

Zusammenfassung

Die angeführte Methode ist für die Berechnung von miteinander verbundenen Brückenträgern mit beliebiger Torsionssteifigkeit bestimmt. Sie beruht auf den zwei Annahmen, daß die Querglieder durch eine kontinuierliche Verbindung ersetzt und die Torsion dieser Glieder vernachlässigt werden kann. Der Lösungsweg führt über eine harmonische Analyse, d. h. eine Entwicklung der Durchbiegung in eine trigonometrische Summe, und die Verteilungskoeffizienten für einfache Balkenbrücken mit zwei bis sechs Hauptträgern sind für

alle Reihenglieder des Biegungsmoments und der Durchbiegungskurve tabelliert. Ferner wird die Anwendung der Methode auf Durchlaufträger mit Superposition gezeigt; die Verwendung eines Arbeitssystems für die Bezeichnungen erleichtert dabei dieses Vorgehen wesentlich. Dann folgt noch eine Methode für die Herleitung von Einflußlinien für die Biegungsmomente in durchlaufenden Brückenlängsträgern.