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Stresses in Non-Uniformly Supported Cylindrical Tanks

Contraintes dans les réservoirs cylindriques portés par des appuis de forme irrégulière

Spannungen in zylinderförmigen Behältern bei ungleichförmiger Abstützung

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Introduction

The design of vertical reinforced concrete liquid storage tanks occurs frequently in structural practice. Usually these structures possess complete rotational symmetry of both support and loading and the analysis of stress and deformation is readily carried out by the classical small displacement theory of thin elastic shells [1], [2] [1]). The corresponding equilibrium analysis has been completed for several types of circular shells in which the wall thickness is variable but still axi-symmetrical [3], [4], [5], [6]. Although such liquid containers are usually considered to deform in an axi-symmetrically manner uneven foundation settlement after the tank is in service can impart a state of deformation that deviates considerably from one of rotational symmetry. It is the purpose of this paper to examine the state of stress and deformation in a vertical cylindrical tank due to a non-uniform foundation settlement.

Consider a vertical right-circular cylindrical shell supported on a flat circular base slab. The upper extremity of the shell is subject to a vertical dead load distributed uniformly around the circumference of the shell along its middle surface and in addition the interior of the shell is filled with liquid.

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1) Numbers in [ ] refer to the Bibliography at the end of the paper.

The shell is prestressed by a wire winding that is applied to the external surface in such a manner that the circumferential tensile stress arising from the fluid pressure is everywhere annulled by a circumferential compressive stress of equal magnitude arising from the prestressing effect of the winding. Only the condition of the completely filled tank will be considered. If the base slab is only partially supported, for example if a sector-shaped region of the foundation beneath the slab has washed away, then axi-symmetrical deformation of the shell is no longer possible. Instead there exists a condition of symmetry of stress and deformation about a vertical plane through the diametral axis bisecting this sector. If the rigidity of the base slab is considered to be negligible then the equilibrium analysis of the tank reduces to the consideration of a thin cylindrical shell loaded by a uniform distribution of axial compressive forces at one extremity and by a non-uniform distribution of axial forces at the other. The procedure outlined in this paper may be applied to any assumed variation of the non-uniform foundation reaction that holds the dead load in equilibrium. For illustrative purposes a plausible form of reaction suggested by certain soil mechanics considerations is assumed and used throughout the numerical example presented.

The analysis of this paper is predicated upon the classical small deformation theory of thin elastic shells as presented by LOVE [1]. This theory assumes:

1. The shell is composed of a material which is elastically homogeneous and isotropic.
2. The material follows Hooke's law.
3. The thickness of the shell at any point is small compared to either of the principal radii of curvature at that point.
4. The normals to the middle surface of the shell before deformation also are normal to the middle surface after deformation.

Fundamental Equations

Let x, θ, and z denote the axial, circumferential, and radial coordinates of a point in the middle surface of the shells with positive directions as shown in fig. 1. The deformation is symmetrical about a vertical plane through the diameter from $\theta=0$ to $\theta=\pi$. Also, let u, v and w denote the corresponding components of displacement of a point in the middle surface of the shell. The three simultaneous differential equations of equilibrium of an element of the shell are (2):

$$\frac{\partial^2 u}{\partial x^2}+\frac{(1-\mu)}{2R^2}\cdot\frac{\partial^2 u}{\partial \theta^2}+\frac{1+\mu}{2R}\cdot\frac{\partial^2 v}{\partial x\,\partial \theta}-\frac{\mu}{R}\cdot\frac{\partial w}{\partial x}=0 \tag{1a}$$

$$\frac{(1+\mu)}{2}\cdot\frac{\partial^2 u}{\partial x\,\partial\theta}+R\,\frac{(1-\mu)}{2}\cdot\frac{\partial^2 v}{\partial x^2}+\frac{1}{R}\cdot\frac{\partial^2 v}{\partial\theta^2}-\frac{1}{R}\cdot\frac{\partial w}{\partial\theta}+$$
$$+\frac{t^2}{12\,R}\left(\frac{\partial^3 w}{\partial x^2\,\partial\theta}+\frac{\partial^3 w}{R^2\,\partial\theta^3}\right)+\frac{t^2}{12\,R}\left[(1-\mu)\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{R^2\,\partial\theta^2}\right]=0 \qquad (1\,\mathrm{b})$$

$$\mu\,\frac{\partial u}{\partial x}+\frac{1}{R}\cdot\frac{\partial v}{\partial\theta}-\frac{w}{R}-\frac{t^2}{12}\left(R\,\frac{\partial^4 w}{\partial x^4}+\frac{2}{R}\cdot\frac{\partial^4 w}{\partial x^2\,\partial\theta^2}+\frac{\partial^4 w}{R^3\,\partial\theta^4}\right)-$$
$$-\frac{t^2}{12}\left(\frac{2-\mu}{R}\cdot\frac{\partial^3 v}{\partial x^2\,\partial\theta}+\frac{\partial^3 v}{R^3\,\partial\theta^3}\right)=0 \qquad (1\,\mathrm{c})$$

where R denotes the mean radius of the shell, t the shell thickness, and μ represents Poisson's ratio. The solution to the problem consists of finding functions u, v, and w that satisfy these three equilibrium equations together with the boundary conditions pertaining to forces and displacements. Such a u, v, w would constitute an "exact" solution to the problem within the limits of linear small-deformation shell theory.

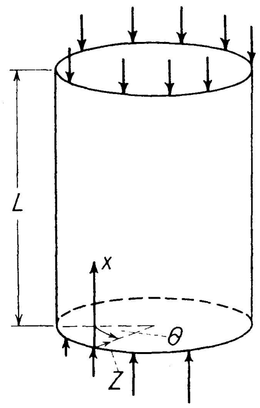

Fig. 1

An alternate method of solution that will yield results of reasonable accuracy is to assume a set of functions representing u, v, and w, determine as many as possible of the amplitude parameters in these functions from boundary conditions, and then determine the remainder of the parameters by the method of minimum potential energy of the structure. If infinite series are assumed for each of the functions u, v, and w then, presumably, the results obtained by this method would coincide with the "exact" solution. If only a finite number of terms are used the minimum energy criteria will yield values of the internal forces somewhat in excess of the "exact" values. The discrepancy between the two approaches decreases as more terms are used in the series representing the displacements.

Analysis

It is assumed that the portion of the lower extremity of the shell supported by the foundation remains plane during deformation. The unsupported portion undergoes some displacement in the axial direction and the amplitude of this deformation may be determined from minimum energy consideration. Further, it is assumed that the radial displacement as well as its derivative with respect to the axial coordinate vanishes at each end of the shell. The tangential displacement is also assumed to be zero at both extremities.

The functions u, v, and w may be expanded into double-Fourier series in the axial and circumferential directions. In this paper only finite numbers of terms are taken to represent the elastic curve in the axial direction. Let the following expressions be taken to represent the displacements of a point in the middle surface of the cylinder:

$$u = \left(1+\sin\frac{\pi x}{2L}\right)\left(\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}A_n\cos n\theta\right)+$$

$$+\left(\sin\frac{\pi x}{2L}+\sin\frac{3\pi x}{2L}\right)\left(\tfrac{1}{2}B_0+\sum_{n=1}^{\infty}B_n\cos n\theta\right)+k\left(\cos\frac{\pi x}{2L}-\cos\frac{3\pi x}{2L}\right) \quad (2a)$$

$$v = \left(1-\cos\frac{2\pi x}{L}\right)\sum_{n=1}^{\infty}C_n\sin n\theta \quad (2b)$$

$$w = \left(1-\cos\frac{2\pi x}{L}\right)\left[\tfrac{1}{2}D_0+\sum_{n=1}^{\infty}D_n\cos n\theta\right] \quad (2c)$$

The constants A_n, B_n, C_n, D_n and k must be determined so as to satisfy boundary conditions on forces and displacements as well as minimum energy of the structure. Let us begin the determination of these constants by assuming that the axial component of displacement, u, is zero in the supported region of the lower end of the shell ($x=0$), and varies as shown by the dotted line in

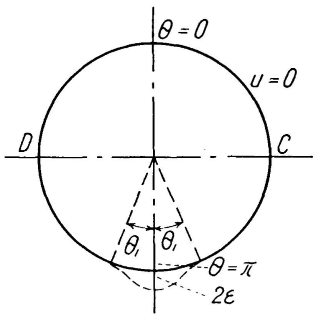

Fig. 2

fig. 2 in the unsupported region. The amplitude of u is denoted by 2ϵ and to impart a reasonable form of variation to u it is assumed the following relations hold:

For $\quad 0<\theta<(\pi-\theta_1): \quad u = 0 \quad$ (3a)

For $\quad (\pi-\theta_1)<\theta<\pi: \quad u = \epsilon\left[1+\cos\left(\frac{\pi^2}{\theta_1}-\frac{\pi}{\theta_1}\theta\right)\right] \quad$ (3b)

where θ is the angular coordinate with origin at the opposite extremity of a diameter bisecting the unsupported portion of the shell. The angle θ_1 denotes the half-angle corresponding to the unsupported circumference. The function

u as defined by eqs. 3 may be expanded in a Fourier series which will contain only cosine terms because of the location of the axis of symmetry. Such a series may be written in the form

$$(u)_{x=0} = \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos n\,\theta \tag{4}$$

where the A_n are identical with those appearing in eq. 2a. The Fourier coefficients are readily found to be (7)

$$A_0 = \frac{2\,\epsilon\,\theta_1}{\pi} \tag{5a}$$

$$A_n = \frac{2\,\epsilon}{\pi}\,(-1)^n \sin n\,\theta_1 \left[\frac{\left(\frac{2\,\pi}{\theta_1} + n\right)}{2\,n\left(\frac{\pi}{\theta_1} + n\right)} - \frac{1}{2\left(\frac{\pi}{\theta_1} - n\right)} \right] \tag{5b}$$

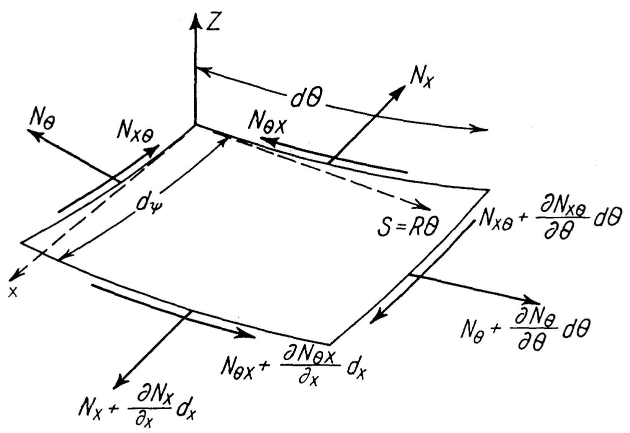

Fig. 3

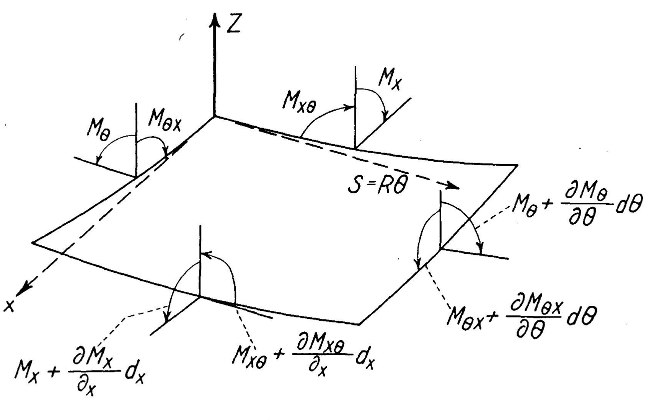

Fig. 4

An element cut from a stressed circular cylindrical shell is subject to the forces and moments (per unit length of the middle surface of the shell) shown in fig. 3 and 4 respectively. These various forces and moments may be expressed in terms of the displacement components u, v, and w in the following form (8):

$$N_x = \frac{Et}{(1-\mu^2)}\left[\frac{\partial u}{\partial x} + \mu\left(\frac{1}{R}\cdot\frac{\partial v}{\partial \theta} - \frac{w}{R}\right)\right] \tag{6a}$$

$$N_\theta = \frac{Et}{(1-\mu^2)}\left[\frac{1}{R}\cdot\frac{\partial v}{\partial \theta} - \frac{w}{R} + \mu\frac{\partial u}{\partial x}\right] \tag{6b}$$

$$N_{x\theta} = \frac{Et}{2(1+\mu)}\left[\frac{\partial v}{\partial x} + \frac{1}{R}\cdot\frac{\partial u}{\partial \theta}\right] \tag{6c}$$

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \frac{\mu}{R^2}\left(\frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2}\right)\right] \tag{6d}$$

$$M_\theta = -D\left[\frac{1}{R^2}\left(\frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2}\right) + \mu\frac{\partial^2 w}{\partial x^2}\right] \tag{6e}$$

$$M_{x\theta} = D(1-\mu)\frac{1}{R}\left[\frac{1}{2}\cdot\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x\,\partial \theta}\right] \tag{6f}$$

where
$$D = \frac{Et^3}{12(1-\mu^2)}$$

The assumed uniform distribution of normal force N_x (dead load) along a generator at the upper end of the shell, $x = L$, will next be utilized to determine the constant k in eq. 2a. With the assumed u, v, and w substituted in eq. 6a, N_x is given by

$$\begin{aligned} N_x = \frac{Et}{(1-\mu^2)}\Bigg[&\frac{\pi}{2L}\cos\frac{\pi x}{2L}\left(\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta\right) + \\ &+\left(\frac{\pi}{2L}\cos\frac{\pi x}{2L} + \frac{3\pi}{2L}\cos\frac{3\pi x}{2L}\right)\left(\frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos n\theta\right) + \\ &+ k\left(-\frac{\pi}{2L}\sin\frac{\pi x}{2L} + \frac{3\pi}{2L}\sin\frac{3\pi x}{2L}\right) + \\ &+\frac{\mu}{R}\left(1-\cos\frac{2\pi x}{L}\right)\left(-\frac{1}{2}D_0 + \sum_{n=1}^{\infty} nC_n\cos n\theta - \sum_{n=1}^{\infty} D_n \cos n\theta\right)\Bigg] \end{aligned} \tag{7}$$

At the upper extremity, $x = L$, this reduces to

$$(N_x)_{x=L} = -\frac{Et}{(1-\mu^2)}\cdot\frac{2\pi k}{L} \tag{8}$$

If the resultant axial force (dead load) at the upper end is denoted by P then we have

$$2\pi R\,(N_x)_{x=L} = -P \tag{9}$$

Substituting N_x from eq. 8 in eq. 9 and solving for k we obtain

$$k = \frac{PL(1-\mu^2)}{4\pi^2 ERt} \tag{10}$$

Let us now examine the forces acting at the lower end of the cylinder. From eq. 7 we have

$$(N_x)_{x=0} = \frac{E\,t}{(1-\mu^2)}\left[\frac{\pi}{2\,L}\left(\frac{1}{2}A_0+\sum_{n=1}^{\infty}A_n\cos n\,\theta\right)+\frac{4\,\pi}{2\,L}\left(\frac{1}{2}B_0+\sum_{n=1}^{\infty}B_n\cos n\,\theta\right)\right] \quad (11)$$

To evaluate the B_n it will be necessary to assume the form of variation of the contact pressure exerted by the supporting earth on the lower extremity of the cylinder. Analogous to the contact pressure distribution existing under the loaded portion of the boundary of a semi-infinite plane loaded by a normal force distributed over a region in which the deflection is constant (9), we assume[2])

For $\quad 0<\theta<(\pi-\theta_1)$

$$(N_x)_{x=0} = C_1\left(\frac{\theta}{\pi-\theta_1}\right)^4+C_2 \quad (12a)$$

For $\quad (\pi-\theta_1)<\theta<\pi$

$$(N_x)_{x=0} = 0 \quad (12b)$$

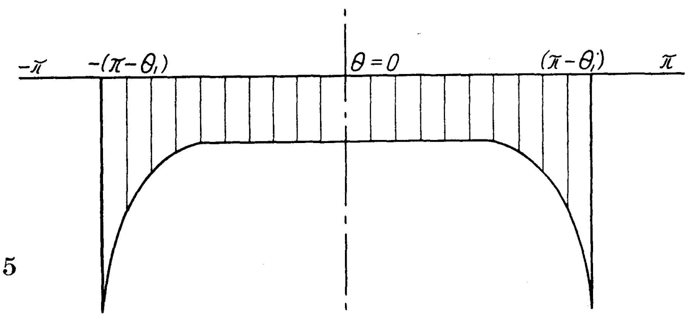

Fig. 5

This pressure distribution is illustrated in fig. 5 where the shell has been developed onto a plane for simplicity in depicting the variation of contact pressure. The constants C_1 and C_2 are determined from the equations

$$\sum F_V = 0 \quad (13a)$$

and

$$\sum M_{CD} = 0 \quad (13b)$$

where CD is the diameter illustrated in fig. 2. Applying these conditions to eqs. 12 we find

$$C_1 = \frac{P}{2R(\pi-\theta_1)K} \quad (14a)$$

$$C_2 = \frac{P}{2R(\pi-\theta_1)}\left(1-\frac{1}{5K}\right) \quad (14b)$$

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[2]) The use of additional terms involving the second, sixth and higher powers of the angular coordinate would undoubtedly improve the accuracy of the final result. The constants accompanying each of these additional terms would be determined by the method of minimum potential energy, to be discussed later.

where

$$K = \left[\frac{1}{5} - \frac{1}{(\pi-\theta_1)^4}\right]\left[(\pi-\theta_1)^4 - 4\{[3(\pi-\theta_1)^2-6] - [(\pi-\theta_1)^3 - 6(\pi-\theta_1)]\cot(\pi-\theta_1)\}\right]$$

The $(N_x)_{x=0}$ represented by eqs. 12 may be expanded in a Fourier series of the form

$$(N_x)_{x=0} = \frac{P}{2R(\pi-\theta_1)^5 K}\left\{\frac{1}{5\pi}(\pi-\theta_1)^5 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(\pi-\theta_1)^4}{n}\sin n(\pi-\theta_1) + \right.\right.$$

$$+\frac{4(\pi-\theta_1)^3}{n^2}\cos n(\pi-\theta_1) - \frac{12(\pi-\theta_1)^2}{n^3}\sin n(\pi-\theta_1) - \frac{24(\pi-\theta_1)}{n^4}\cos n(\pi-\theta_1) +$$

$$\left.\left.+\frac{24}{n^5}\sin n(\pi-\theta_1)\right]\cos n\theta\right\} + \frac{P}{2R(\pi-\theta_1)}\left(1-\frac{1}{5K}\right) \tag{15}$$

Equating coefficients of corresponding terms in eqs. 11 and 15 we find the following expressions for the B_n:

$$B_0 = -\frac{A_0}{4} + \frac{L(1-\mu^2)}{Et\pi}\left[\frac{P}{10\pi RK} + \frac{P}{2R(\pi-\theta_1)}\left(1-\frac{1}{5K}\right)\right] \tag{16a}$$

$$B_n = -\frac{A_n}{4} + \frac{PL(1-\mu^2)}{2Et\pi^2 RK(\pi-\theta_1)^5}\left[\frac{(\pi-\theta_1)^4}{n}\sin n(\pi-\theta_1) + \frac{4(\pi-\theta_1)^3}{n^2}\cos n(\pi-\theta_1) - \right.$$

$$\left. - \frac{12(\pi-\theta_1)^2}{n^3}\sin n(\pi-\theta_1) - \frac{24(\pi-\theta_1)}{n^4}\cos n(\pi-\theta_1) + \frac{24}{n^5}\sin n(\pi-\theta_1)\right] \tag{16b}$$

To this point, for a given set of geometric and elastic constants of the shell together with specified values of the load and the angle θ_1 characterizing the unsupported region, the parameters k, A_n, and B_n have been determined, the A_n and B_n still being functions of the unknown deflection parameter ϵ. It remains to determine the C_n and D_n so as to minimize the total potential energy.

The principle of conservation of energy applied to the loaded tank states that the potential energy of the system must remain constant. But this energy consists of two parts, namely, the internal strain energy of the shell (U) and also the potential energy of the external loads (V). If U and V are each expressed in terms of displacement components, then the sum of U and V when differentiated partially with respect to any displacement component vanishes. This is the principle of minimum potential energy (10).

The strain energy U in a thin cylindrical shell may be considered to consist of membrane energy due to stretching of the middle surface, energy of shear, and bending energy. The membrane energy is given by (11)

$$U_m = \frac{Et}{2R(1-\mu^2)}\int_0^L\int_0^{2\pi}[R^2 u_x{}^2 + (v_\theta - w)^2 + 2R\mu u_x(v_\theta - w)]\,dx\,d\theta \tag{17a}$$

where the subscripts x, θ denote partial differentiation. The shear energy is (11)

$$U_s = \frac{E\,t\,R}{4\,(1+\mu)} \int_0^L \int_0^{2\pi} \left(\frac{1}{R}\, u_\theta + v_x\right)^2 dx\, d\theta \tag{17b}$$

The bending energy is given by (11)

$$U_b = \frac{D}{2\,R} \int_0^L \int_0^{2\pi} \Big[R^2 w_{xx}^2 + \frac{1}{R^2}\,(w_{\theta\theta}+w)^2 + 2\,\mu\, w_{xx}\,(w_{\theta\theta}+w) + \\ + 2\,(1-\mu)\left(w_{x\theta} + \tfrac{1}{2}\, v_x - \frac{1}{2\,R}\, u_\theta\right)^2\Big]\, dx\, d\theta \tag{17c}$$

Consequently the internal strain energy of the cylinder

$$U = U_m + U_s + U_b$$

may be expressed as a function of the A_n, B_n, C_n, and D_n.

The potential energy of the external forces, V, may be written in the form

$$V = -\int_0^{2\pi} (N_x)_{x=L}\,(u)_{x=L}\, R\, d\theta = P A_0 = \frac{2\,P\,\epsilon\,\theta_1}{\pi} \tag{17d}$$

In writing the final form of the energy it is convenient to introduce a number of auxiliary parameters which are functions of the geometry of the shell and its elastic constants. These constants K_i are defined as follows:

$$\begin{aligned}
K_1 &= \frac{E\,t}{2\,R\,(1-\mu^2)} & K_{12} &= \frac{E\,t}{4\,R\,(1+\mu)} & K_{23} &= \frac{2\,\pi^3}{L} \\
K_2 &= \frac{\pi^3 R^2}{8\,L} & K_{13} &= \frac{3\,\pi\,L}{2} - 4\,L & K_{24} &= \frac{\pi^3}{2\,L} \\
K_3 &= \frac{10\,k^2 R^2 \pi^3}{4\,L} & K_{14} &= \pi\,L & K_{25} &= \frac{1}{4\,R^2}\left(\frac{3\,\pi\,L}{2} - 4\,L\right) \\
K_4 &= \frac{5\,\pi^3 R^2}{4\,L} & K_{15} &= \frac{4\,\pi\,R}{L^2}\,(30\,\pi^2 + 16\,L^2) & K_{26} &= \frac{L\,\pi}{4\,R^2} \\
K_5 &= \frac{k\,R^2 \pi^2}{L} & K_{16} &= \frac{8\,R\,\pi}{105} & K_{27} &= 2\,\pi^3 \\
K_6 &= \frac{R^2 \pi^3}{4\,L} & K_{17} &= \frac{2\,\pi^3 R^2}{L} & K_{28} &= \frac{32\,\pi}{15\,R\,L} \\
K_7 &= \frac{8\,k\,R^2 \pi^2}{2\,L} & K_{18} &= \frac{E\,t^3}{24\,R\,(1-\mu^2)} & K_{29} &= \frac{16\,\pi}{105\,R} \\
K_8 &= \frac{3\,\pi\,L}{2} & K_{19} &= \frac{8\,\pi^5 R^2}{L^3} & K_{30} &= \frac{16\,\pi}{15\,R}
\end{aligned} \tag{18}$$

$$K_9 = \frac{32 R \mu \pi}{15} \qquad K_{20} = \frac{3 \pi L}{2 R^2} \qquad K_{31} = \frac{8 \pi}{105 R}$$

$$K_{10} = \frac{256 k \mu R \pi}{105} \qquad K_{21} = \frac{4 \mu \pi^3}{L} \qquad K_{32} = \frac{L \pi}{4 R^2} \tag{18}$$

$$K_{11} = \frac{256 \mu R \pi}{105} \qquad K_{22} = \frac{E t^3}{12 R (1+\mu)}.$$

The strain energies may then be written in the forms:

$$\begin{aligned} U_m = K_1 \Big[& K_2 (\tfrac{1}{2} A_0^2 + \sum_{n=1}^{\infty} A_n^2) + K_3 + K_4 (\tfrac{1}{2} B_0^2 + \sum_{n=1}^{\infty} B_n^2) - K_5 A_0 - \\ & - K_6 (\tfrac{1}{2} A_0 B_0 + \sum_{n=1}^{\infty} A_n B_n) + K_7 B_0 + K_8 (\sum_{n=1}^{\infty} n^2 C_n^2 + \tfrac{1}{2} D_0^2 + \sum_{n=1}^{\infty} D_n^2 - 2 \sum_{n=1}^{\infty} n C_n D_n) + \\ & + K_9 \{ -\tfrac{1}{2} A_0 D_0 + \sum_{n=1}^{\infty} A_n (n C_n - D_n) \} - K_{10} D_0 - \\ & - K_{11} \{ -\tfrac{1}{2} B_0 D_0 + \sum_{n=1}^{\infty} B_n (n C_n - D_n) \} \Big] \end{aligned} \tag{19a}$$

$$\begin{aligned} U_s = K_{12} \Big[& K_{13} (\sum_{n=1}^{\infty} n^2 A_n^2) + K_{14} (\sum_{n=1}^{\infty} n^2 B_n^2) - K_{14} (\sum_{n=1}^{\infty} n^2 A_n B_n) - \\ & - K_{15} (\sum_{n=1}^{\infty} n A_n C_n) - K_{16} (\sum_{n=1}^{\infty} n B_n C_n) + K_{17} (\sum_{n=1}^{\infty} C_n^2) \Big] \end{aligned} \tag{19b}$$

$$\begin{aligned} U_b = K_{18} \Big[& K_{19} (\tfrac{1}{2} D_0^2 + \sum_{n=1}^{\infty} D_n^2) + K_{20} \{ \tfrac{1}{2} D_0^2 + \sum_{n=1}^{\infty} (n+1)^2 D_n^2 \} \\ & K_{21} \{ -\tfrac{1}{2} D_0^2 + \sum_{n=1}^{\infty} (n^2-1) D_n^2 \} \Big] + K_{22} \Big[K_{23} (\sum_{n=1}^{\infty} n^2 D_n^2) \\ & K_{22} K_{23} (\sum_{n=1}^{\infty} n^2 D_n^2) + K_{24} (\sum_{n=1}^{\infty} C_n^2) + K_{25} (\sum_{n=1}^{\infty} n^2 A_n^2) \\ & K_{26} (\sum_{n=1}^{\infty} n^2 B_n^2) - K_{27} (\sum_{n=1}^{\infty} n D_n C_n) + K_{28} (\sum_{n=1}^{\infty} n^2 A_n D_n) \\ & - K_{29} (\sum_{n=1}^{\infty} n^2 B_n D_n) + K_{30} (\sum_{n=1}^{\infty} n A_n C_n) + K_{31} (\sum_{n=1}^{\infty} n B_n C_n) - \\ & - K_{32} (\sum_{n=1}^{\infty} n^2 A_n B_n) \Big] \end{aligned} \tag{19c}$$

It is evident that eqs. 5 and 16 determine A_n and B_n as functions of the amplitude parameter ϵ. Consequently, the total potential energy $(U+V)$ may be expressed as a function of C_n, D_n, and ϵ and then to satisfy the condition of minimum potential energy we have

$$\frac{\partial (U+V)}{\partial C_n} = 0 \qquad \frac{\partial (U+V)}{\partial D_n} = 0 \qquad \frac{\partial (U+V)}{\partial \epsilon} = 0 \tag{20}$$

The procedure is evident: The total potential energy $(U+V)$ is minimized and the resulting equations give C_n and D_n as functions of ϵ. These values are substituted into the equation for $(U+V)$ so as to yield an equation which is a function of ϵ only. The total potential energy is then minimized with respect to this variable and thus a numerical value of ϵ is determined. Knowing ϵ the A_n, B_n, C_n, and D_n are immediately known. It would be possible to present an explicit algebraic expression for ϵ but it would be extremely cumbersome and consequently it is recommended that the solution of any problem be carried no farther than eqs. 19 before substituting numerical values. The application of this technique to be solution of an actual problem is illustrated in the following section.

Numerical Example

Let us consider a vertical cylindrical tank having the following dimensions:

$$R = 754 \text{ inches}$$
$$L = 264 \text{ inches}$$
$$t = 8 \text{ inches}$$

In addition, the elastic constants are taken to be $E = 3{,}900{,}000$ lb/in.^{2} and $\mu = 0.2$. The cylinder is subject to a resultant vertical force of 1,520,000 lb applied uniformly around the circumference at its upper extremity. Further, there is an 80 foot length of arc that is unsupported at the lower extremity of the cylinder. This corresponds to a value of $\theta_1 = 36° 28'$ in fig. 2.

The first step in the solution is to obtain numerical values of the K_i defined by eqs. 18. However it is most convenient to delay the substitution of the K_i until later rather than expressing the total strain energy in terms of them immediately.

The total potential energy $(U+V)$ must now be minimized with respect to each of the coefficients C_n and D_n. In the case of the C_1 and D_1 for example this is accomplished by taking the partial derivatives of $(U+V)$ with respect to C_1 and D_1. This leads to the following two equations:

$$2(K_1K_8+K_{12}K_{17}+K_{22}K_{24})C_1-(2K_1K_8+K_{22}K_{27})D_1 = (K_1K_9+K_{12}K_{15}-K_{22}K_{30})A_1+(K_1K_{11}+K_{12}K_{16}-K_{22}K_{31})B_1 \quad (21a)$$

$$-(2K_1K_8+K_{22}K_{27})C_1+2(K_1K_8+K_{18}K_{19}+K_{22}K_{23})D_1 = (-K_1K_9+K_{22}K_{28})A_1+(-K_1K_{11}+K_{22}K_{29})B_1 \quad (21b)$$

Similarly other pairs of simultaneous algebraic equations must be formed, the number depending upon the accuracy of solution desired and the number deemed necessary to represent the foundation reaction satisfactorily. It is to be noted that one of the advantages of the technique presented in the paper is that in the n-th pair of equations only the coefficients C_n and D_n occur, hence it is only necessary at each step to solve a pair of equations containing two unknowns. Eqs. 21 are solved for the C_1 and D_1 in terms of the A_1 and B_1 using the numerical values of the K_i already tabulated. The solution is found to be

$$\begin{aligned} C_1 &= (93026\,A_1 + 261.9\,B_1)/162{,}384 \\ &= -0.171630\,\epsilon - 0.0000000461797 \end{aligned} \tag{22a}$$

$$\begin{aligned} D_1 &= (38{,}328\,A_1 - 62{,}266.8\,B_1)/178{,}601 \\ &= -0.0381540\,\epsilon + 0.00000998232 \end{aligned} \tag{22b}$$

After analogous solutions for all of the desired C_n and D_n as functions of ϵ have been determined they, together with the A_n and B_n may be substituted in the equation for $(U+V)$ thus giving $(U+V)$ as a function of the single variable ϵ. In accordance with eq. 20 the total potential energy is now minimized with respect to ϵ, this process leading to a single linear equation in ϵ and from this equation the numerical value of ϵ may be determined. In the problem the result of solving this equation is

$$\epsilon = -0.0002533 \text{ in.} \tag{23}$$

Knowing ϵ, the numerical values of the A_n and B_n may now be determined from eqs. 5 and 16 respectively. The first few values were found to be

$$\begin{aligned} A_0 &= 0.000103204 \\ B_0 &= 0.00103004 \\ A_1 &= 0.0000758337 \\ B_1 &= -0.00000967410 \end{aligned}$$

Use of eqs. 22 yields numerical values of the C_1 and D_1. They are

$$\begin{aligned} C_1 &= 0.0000434277 \\ D_1 &= 0.0000196467 \end{aligned}$$

In a similar manner additional coefficients through $n = 6$ were calculated and thus a reasonable estimate of the deflection surface given by eqs. 2 was obtained. Knowing u, v, and w the internal forces and moments are readily found from eqs. 6. In this manner these quantities may be evaluated at any point having coordinates x, θ.

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Summary

An equilibrium analysis is presented for the problem of a vertical cylindrical tank subject to an axial loading distributed uniformly around the circumference of the shell at its upper extremity and supported non-uniformly by the foundation at its lower extremity. Infinite series expressions, compatible with boundary conditions, are assumed for the three orthogonal displacement components and the principle of minimum potential energy is employed to determine the various coefficients of the series. From these displacements the normal and shearing stresses in the shell may be determined to any desired degree of accuracy by the usual stress-displacement relations for thin shells.

Résumé

L'auteur étudie les conditions d'équilibre d'un réservoir cylindrique vertical soumis à sa partie supérieure a une charge axiale répartie uniformément sur la périphérie, mais dont le fond est porté d'une manière irrégulière, par suite de la position et de la forme des fondations. En tenant compte des conditions aux limites, il obtient des expressions comportant des séries infinies pour les trois composantes orthogonales de la déformation, en employant la méthode de l'énergie potentielle minimum pour la détermination des différents coefficients de ces séries. A partir de ces déformations, il est possible de déterminer avec une précision arbitraire les contraintes normales et de cisaillement de la paroi mince, en faisant intervenir les relations habituelles contraintes-déformations relatives aux parois minces.

Zusammenfassung

Es werden die Gleichgewichtsbedingungen in einem senkrecht stehenden zylindrischen Behälter untersucht, welcher im obern Teil unter einer längs des Umfanges gleichförmigen, axialen Belastung steht, der am Boden aber wegen der Lage und Form der Fundamente ungleichmäßig unterstützt ist. Unter Berücksichtigung der Randbedingungen ergeben sich Ausdrücke mit unendlichen Reihen für die 3 orthogonalen Verschiebungskomponenten, wobei zur Bestimmung der verschiedenen Reihenkoeffizienten die Methode der kleinsten potentiellen Energie angewendet wird. Aus diesen Verschiebungen lassen sich die Normal- und Schubspannungen der Schale unter Verwendung der üblichen Spannungs-Formänderungsbeziehungen dünner Schalen mit beliebiger Genauigkeit ermitteln.