Method of successive approximations for design of continuous I beams submitted to torsion

Method of Successive Approximations for Design of Continuous I Beams Submitted to Torsion

Procédé d'approximations successives pour le calcul des poutres continues en double T soumises à la torsion

Verfahren für die Berechnung der auf Verdrehung beanspruchten durchlaufenden Doppel-T-Träger durch wiederholte Annäherungen

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A. Introduction. Definition of Problem

We consider an I beam which is prevented from torsion at its supports, see fig. 1. If this beam is submitted to a torsional load, then the moments M_{fl_1} to M_{fl_4}, which cause bending of the flanges, are produced on account of the prevention of warping at all those supports which are not hinged in a vertical direction. These moments can be determined directly by solving the fundamental equation of torsion of I beams so as to take account of the boundary

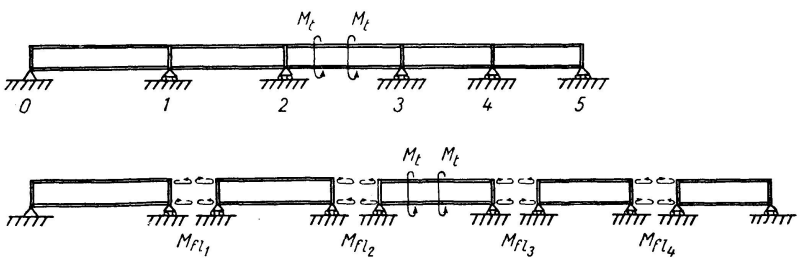

Fig. 1.

conditions and the conditions of continuity. However, except in highly idealised cases, this method results in time-wasting and intricate calculations. In order to save labour and at the same time to render the calculations more readily

intelligible, a method of successive approximation is advanced in this paper for the determination of the bending moments acting on the flanges. In principle, this method can be characterised as a generalised variant of the Cross method.

The process of calculation is described in what follows. In the first stage of calculation, all those ends of spans which are not free or not hinged in a vertical direction are assumed to be rigidly built in. The *basic bending moments acting on the flanges* which are produced on this assumption at the end points of the spans submitted to loads are calculated in this stage. In the second stage of calculation, those points of support which were assumed in the first stage to be rigidly built in are successively released from restraint one after another. Every time a point of support is released in this manner, *compensating bending moments acting on the flanges* are added in each flange at the point of support in question. These moments are chosen so that the point of support is kept in equilibrium by the combined effect of the compensating and basic bending moments acting on the flanges. The compensating bending moments are distributed over the beam spans which are adjacent to the point of support. After that, the *distributed bending moments acting on the flanges* which are obtained in this way are transferred to the opposite end points of the adjacent beam spans, where they give rise to *carry-over bending moments acting on the flanges*. As a next step, the point of support which has been released from restraint is assumed to be rigidly built in again, and the same calculation procedure is applied to the next point of support, and so forth. This procedure is repeated until the compensating bending moments, and hence also the distributed and the carry-over bending moments acting on the flanges, become so small as to be practically negligible. The resultant bending moments acting on the flanges at each support are then computed by the summation of the basic bending moments as well as the distributed and the carry-over bending moments acting on the flanges.

In the limit case where Saint Venant's torsional rigidity of the continuous I beam is equal to zero, the method of successive approximations described in the above becomes identical with the Cross method of fixed end moments[1]). The Author's method is illustrated in a more concrete manner by a numerical example in Chapter C. In order to adapt this method of successive approximations to practical calculations, the relations required for determining some basic quantities, viz., the basic bending moments acting on the flanges, the stiffness factor, as well as the distribution factor and the carry-over factor for the bending moments acting on the flanges, are deduced.

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[1]) H. Cross, Analysis of Continuous Frames by Distributing Fixed-End-Moments, Proc. A.S.C.E., Vol. 56 (1930), p. 919, 1747, 1913, 2029; Vol. 57 (1931), p. 119, 369, 725, 1059, 1354; Vol. 58 (1932), p. 95, 409, 559, 914.

H. Cross and N. D. Morgan, Continuous Frames of Reinforced Concrete, New York 1932.

B. Basic Quantities

To begin with, we shall study the elementary case of loading shown in fig. 2, which represents an I beam of monosymmetrical cross section hinged in a vertical direction at the supports and submitted to bending moments M_{fl} acting on the flanges at one of the supports.

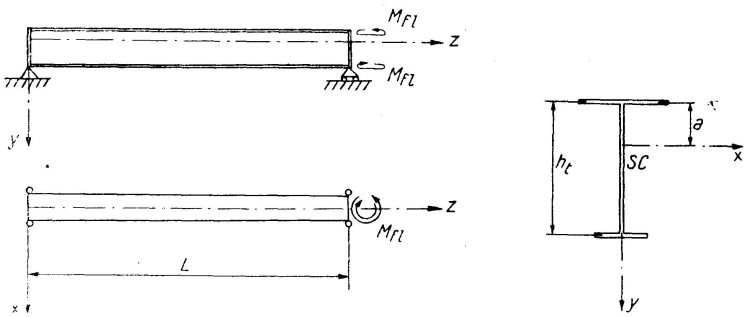

Fig. 2.

For further treatment, it is necessary to know the angle of rotation as well as its first and second derivatives. If a beam of open, thin-walled cross section is acted upon by a twisting moment M_t at the cross section z, then the angle of rotation φ is given by the basic equation[2])

$$C_w \frac{d^3\varphi}{dz^3} - C \frac{d\varphi}{dz} = M_t \tag{1}$$

where $C = G J_t$ = Saint Venant's torsional rigidity,
C_w = the warping rigidity of the cross section,
G = the modulus of elasticity in shear,
$J_t = \frac{C}{G}$ = the torsional moment of inertia of the cross section.

For I beams of monosymmetrical, ordinarily shaped cross section, the torsional moment of inertia J_t can to a close approximation be calculated from the relation[3])

$$J_t = \tfrac{1}{3} \varsigma \sum l \delta^3 \tag{2}$$

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[2]) Eq. (1) was first deduced for I beams of bisymmetrical cross section by S. TIMOSHENKO, Bull. Polyt. Inst. S. Petersburg, 1905. Analogous equations were subsequently deduced, for I beams of monosymmetrical cross section, by C. WEBER, Z. angew. Math. u. Mech., Vol. 6, 1926, p. 85, and, for beams of any arbitrary, thin-walled, open cross section, by H. WAGNER, 25th Anniversary Publication, Technische Hochschule, Danzig, 1929.

[3]) A. FÖPPL, Sitzungsber. Bayr. Akad. Wiss., 19[illegible]. For a more accurate calculation of the torsional moment of inertia J_t, see e. g. [illegible]SE and B. G. JOHNSTON, Proc. A.S.C.E., Vol. 61, 1936.

where l and δ denote the length and the thickness, respectively, of each separate part of the cross section, while ς is a coefficient, which can be put equal to 1,15 for I beams with flanges of uniform thickness and to 1,30 for I beams of standard section.

For I beams of monosymmetrical cross section, the warping rigidity C_w can be computed from the relation

$$C_w = E \frac{J_{y_1} J_{y_2}}{J_{y_1} + J_{y_2}} h_t^2 \tag{3}$$

where E = Young's modulus of elasticity,
J_{y_1} = the moment of inertia of the top flange with respect to the y-axis,
J_{y_2} = the moment of inertia of the bottom flange with respect to the y-axis,
h_t = the distance between the centres of gravity of the flanges.

In the case of loading under consideration, which is characterised by a twisting moment M_t that is constant along the beam, the solution of the basic equation (1) is

$$\varphi = A_1 \cosh kz + A_2 \sinh kz + A_3 - \frac{M_t}{C} z \tag{4}$$

where A_1 to A_3 are constants of integration and k is a cross-sectional constant, which can be described by the abbreviated notation

$$k^2 = \frac{C}{C_w} \tag{5}$$

The constants of integration A_1 to A_3 and the twisting moment M_t are determined from the boundary conditions of the problem, which are stated in what follows.

1. and 2. $\varphi = 0$ for $z = 0$ and $z = L$. This condition expresses the prevention of torsion at the supports.
3. $\frac{d^2\varphi}{dz^2} = 0$ for $z = 0$. This condition expresses the freedom of warping at the left-hand support.
4. At the right-hand support, $z = L$, the boundary condition for the top flange of the I beam is

$$M_{fl} = -E J_{y_1} \frac{d^2 x_1}{dz^2} = -a E J_{y_1} \frac{d^2\varphi}{dz^2}$$

and the boundary condition for the bottom flange of the beam is

$$M_{fl} = -E J_{y_2} \frac{d^2 x_2}{dz^2} = -(h_t - a) E J_{y_2} \frac{d^2\varphi}{dz^2}$$

where x_1 and x_2 are the respective lateral deflections caused by torsion of the top flange and the bottom flange, while a is the distance from the centre of gravity of the top flange to the shear centre, SC, of the cross section.

From the boundary condition (4) we obtain the following relation, which is well known in the theory of torsion

$$a = \frac{J_{y_2}}{J_{y_1} + J_{y_2}} h_t \tag{6}$$

For the twisting moment M_t, the boundary conditions, in combination with eq. (4), yield the relation

$$M_t = -\frac{h_t}{L} M_{fl} \tag{7}$$

For the angle of rotation as well as for its first and second derivatives, we get the relations

$$\varphi = M_{fl} \frac{h_t}{C} \left(\frac{z}{L} - \frac{\sinh kz}{\sinh kL} \right) \tag{8}$$

$$\frac{d\varphi}{dz} = M_{fl} \frac{h_t}{CL} \left(1 - \frac{kL \cosh kz}{\sinh kL} \right) \tag{9}$$

$$\frac{d^2\varphi}{dz^2} = -M_{fl} \frac{h_t}{CL^2} (kL)^2 \frac{\sinh kz}{\sinh kL} \tag{10}$$

In the above we have carried out the solution of the elementary case of loading shown in fig. 2. From this solution we can directly deduce expressions for the basic quantities which are required for calculations made by means of the method of successive approximations, viz., the carry-over factor, the stiffness factor, and the distribution factor for the bending moments acting on the flanges.

1. *Carry-over factor*

We consider a single-span beam which is prevented from torsion at the supports, see fig. 3. We assume that bending moments acting on the flanges, M_{fl}, are applied at one support, B, of the beam. We define the carry-over

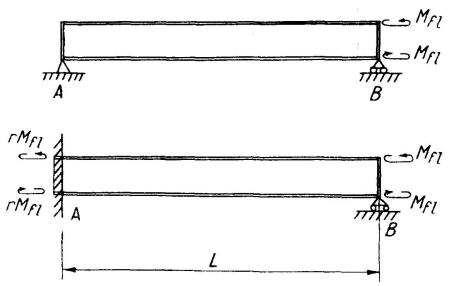

Fig. 3.

factor r as the ratio of the bending moments acting on the flanges which are produced in this case at the opposite end A of the beam to the bending moments acting on the flanges at the end B.

It follows directly from the above definition that $r = 0$ if the support at the end A is *hinged in a vertical direction.*

If the beam is *rigidly built in* at the end A, then eq. (9), in combination with the condition that $\frac{d\varphi}{dz}=0$ at the end A, yields the relation

$$\left(\frac{d\varphi}{dz}\right)_A = M_{fl}\frac{h_t}{CL}\left(1-\frac{kL}{\sinh kL}\right) - r M_{fl}\frac{h_t}{CL}\left(\frac{kL\cosh kL}{\sinh kL}-1\right) = 0$$

from which we calculate the expression for r

$$r = \frac{\sinh kL - kL}{kL\cosh kL - \sinh kL} \tag{11}$$

The functional relation for r is graphically represented in fig. 4. When the warping rigidity C_w is given, r increases, and hence the transferred bending moments acting on the flanges also become greater, as Saint Venant's torsional rigidity C decreases. The maximum value of r is equal to $\frac{1}{2}$, and is reached at

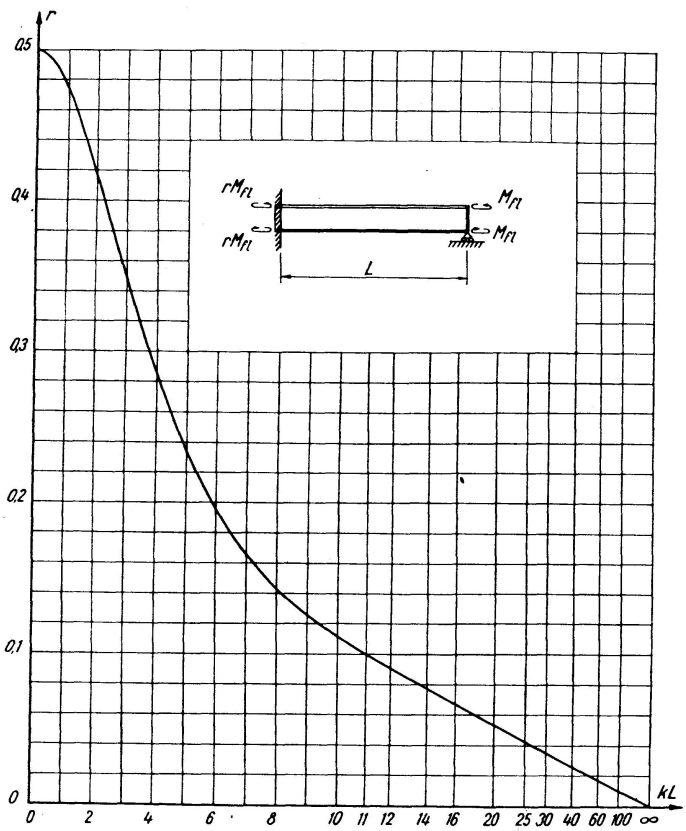

Fig. 4.

$kL=0$. This corresponds to an I beam which has no Saint Venant's torsional rigidity at all, and which therefore carries over the applied bending moments acting on the flanges by pure bending of the flanges.

2. *Stiffness factor and distribution factor*

We define the stiffness factor S as those bending moments acting on the flanges which must be applied at one support of a single-span beam prevented

from torsion at the supports in order to cause a specific change in angle $\frac{d\varphi}{dz} = 1$ at this support.

The stiffness factor S can be written in the form

$$S = \alpha \frac{C_w}{h_t L} \tag{12}$$

where α is a function of the quantity $k L$ alone. If the non-loaded end of the beam is *hinged in a vertical direction*, then eqs. (5) and (9) yield, for α, the relation

$$\alpha = \frac{(k L)^2 \operatorname{tgh} k L}{k L - \operatorname{tgh} k L} \tag{13}$$

On the other hand, if the non-loaded end of the beam is *rigidly built in*, then we obtain, for α, the relation

$$\alpha = \frac{(k L)^2 \sinh k L}{k L \cosh k L - \sinh k L - r (\sinh k L - k L)} \tag{14}$$

The variation in the α-coefficients with $k L$ is graphically represented in fig. 5. When the warping rigidity C_w is constant, the stiffness factors increase as Saint Venant's torsional rigidity C becomes greater. At the same time, the

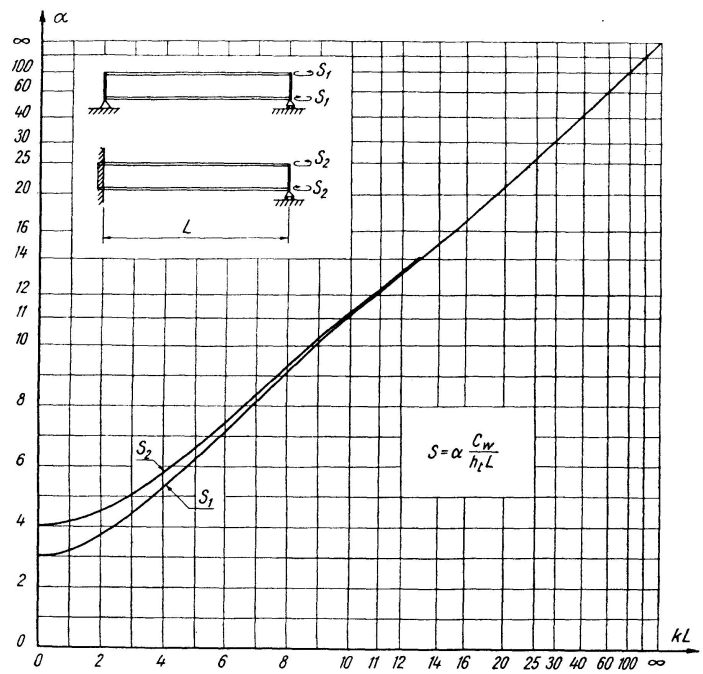

Fig. 5.

difference between the stiffness factor S_2 corresponding to a non-loaded end of the beam which is rigidly built in and the stiffness factor S_1 corresponding to a non-loaded end of the beam which is hinged in a vertical direction becomes smaller.

In order to elucidate the meaning of the concept "the distribution factor for bending moments acting on the flanges", we shall study the continuous beam shown in fig. 6. This beam is submitted to the external moments $M_{fl}=1$

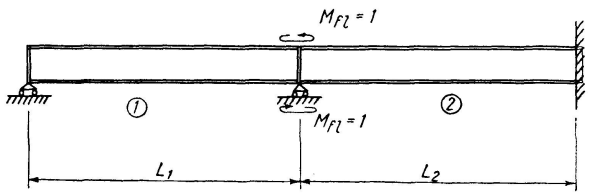

Fig. 6.

acting on the flanges at the intermediate support. Those portions of $M_{fl}=1$ which act upon the adjacent parts (1) and (2) of the beam are defined as distribution factors of these parts of the beam, and are denoted by s_1 and s_2 respectively.

For s_1 and s_2, we have the condition for equilibrium

$$s_1+s_2=1$$

and, the condition for continuity, which states that the values of $\frac{d\varphi}{dz}$ at the intermediate support shall be equal for the parts (1) and (2) of the beam, that is to say,

$$\frac{s_1}{S_1}=\frac{s_2}{S_2}$$

Hence we calculate, for the distribution factors s_1 and s_2, the relations

$$\left.\begin{aligned} s_1 &= \frac{S_1}{S_1+S_2} \\ s_2 &= \frac{S_2}{S_1+S_2} \end{aligned}\right\} \tag{15}$$

Generally, if n beams are adjacent to a support which is prevented from torsion, then the distribution factor s_μ of the μ-th beam can be expressed by the relation

$$s_\mu = \frac{S_\mu}{\sum\limits_{\mu=1}^{n} S_\mu} \tag{16}$$

3. *Basic bending moments acting on flanges in some characteristic cases of loading*

In the present section we shall deal with those restraining bending moments acting on the flanges and corresponding to the external torsional load which belong to the first stage of calculation in the method of successive approximations described in this paper, cf. p. 168.

We shall examine the following cases of loading.

a) I beam rigidly built in at one end and submitted to concentrated twisting moment M_t at any arbitrary cross section

The solution of the case of loading can be obtained by the superposition of two solutions, viz., the solution in the case of an I beam hinged in a vertical direction at both ends and submitted to a concentrated twisting moment at

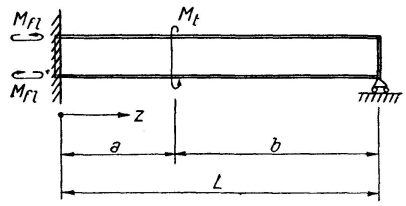

Fig. 7.

any arbitrary cross section, which solution is known from the literature dealing with torsion[4]), and the solution in the above-cited case of loading shown in fig. 2.

If the beam is hinged in a vertical direction at both ends, then the angle of rotation φ_1 and its second derivative $\frac{d^2\varphi_1}{dz^2}$, which determines the normal stresses caused by torsion, can be represented by the expressions

$$\varphi_1 = \frac{M_t}{kC}\left[kb\frac{z}{L} - \frac{\sinh kb}{\sinh kL}\sinh kz\right] \qquad \text{for} \quad 0 \leqslant z \leqslant a \qquad (17)$$

$$\varphi_1 = \frac{M_t}{kC}\left[ka\left(1-\frac{z}{L}\right) - \frac{\sinh ka}{\sinh kL}\sinh k(L-z)\right] \qquad \text{for} \quad a \leqslant z \leqslant L \qquad (17')$$

$$\frac{d^2\varphi_1}{dz^2} = -\frac{M_t}{C}k\frac{\sinh kb}{\sinh kL}\sinh kz \qquad \text{for} \quad 0 \leqslant z \leqslant a \qquad (18)$$

$$\frac{d^2\varphi_1}{dz^2} = -\frac{M_t}{C}k\frac{\sinh ka}{\sinh kL}\sinh k(L-z) \qquad \text{for} \quad a \leqslant z \leqslant L \qquad (18')$$

Now, in the case of loading shown in fig. 7, the condition that $\frac{d\varphi}{dz} = 0$ for $z = 0$, in combination with eqs. (9) and (17), yields, for the determination of the restraining bending moment M_{fl} acting on the flanges, the relation

$$\left(\frac{d\varphi}{dz}\right)_{z=0} = \frac{M_t}{C}\left[\frac{b}{L} - \frac{\sinh kb}{\sinh kL}\right] - M_{fl}\frac{h_t}{CL}\left[\frac{kL\cosh kL}{\sinh kL} - 1\right] = 0$$

Hence we obtain, for M_{fl}, the relation

$$M_{fl} = \beta\frac{M_t L}{h_t} \qquad (19)$$

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[4]) A comprehensive compilation of solutions relating to fundamental cases of torsional loading has been made by F. W. Bornscheuer, Stahlbau, 1953, No. 2, p. 32.

where

$$\beta = \frac{\frac{b}{L}\sinh kL - \sinh kb}{kL\cosh kL - \sinh kL} \tag{20}$$

The functional relation for the dimensionless β-coefficient, which is dependent on the cross section of load application $\frac{b}{L}$ and on the beam characteristic kL, is graphically represented by the curves shown in fig. 8.

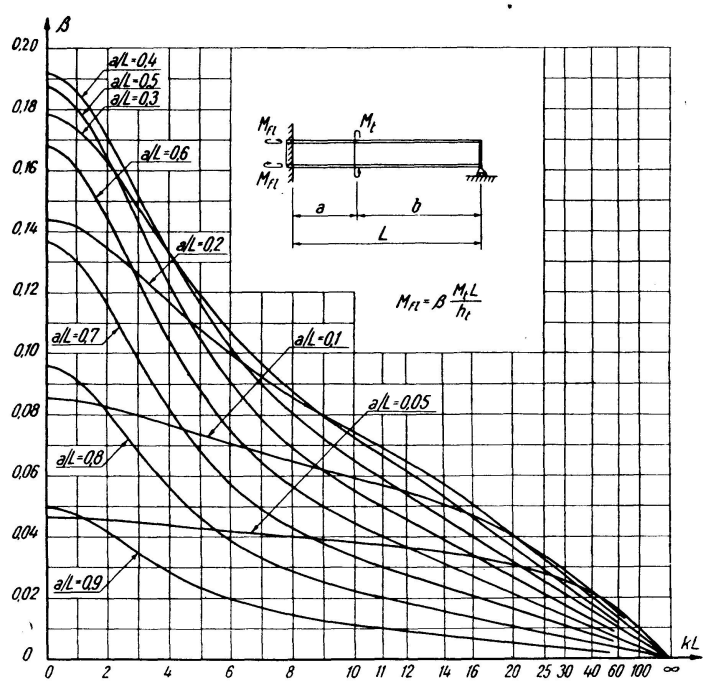

Fig. 8.

b) I beam rigidly built in at both ends and submitted to concentrated twisting moment M_t at any arbitrary cross section

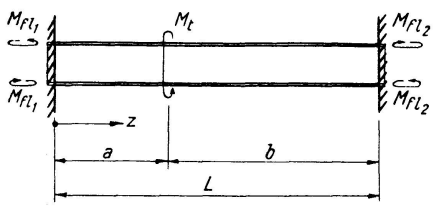

Fig. 9.

We find φ_1 and $\frac{d^2\varphi_1}{dz^2}$ from eqs. (17) and (18).

$$M_{fl_1} = \beta_1 \frac{M_t L}{h_t}; \quad M_{fl_2} = \beta_2 \frac{M_t L}{h_t} \tag{21}$$

$$\beta_1 = \frac{\left(\frac{b}{L}\sinh kL - \sinh kb\right)(kL\cosh kL - \sinh kL) - \left(\frac{a}{L}\sinh kL - \sinh ka\right)(\sinh kL - kL)}{(kL\cosh kL - \sinh kL)^2 - (\sinh kL - kL)^2} \tag{22}$$

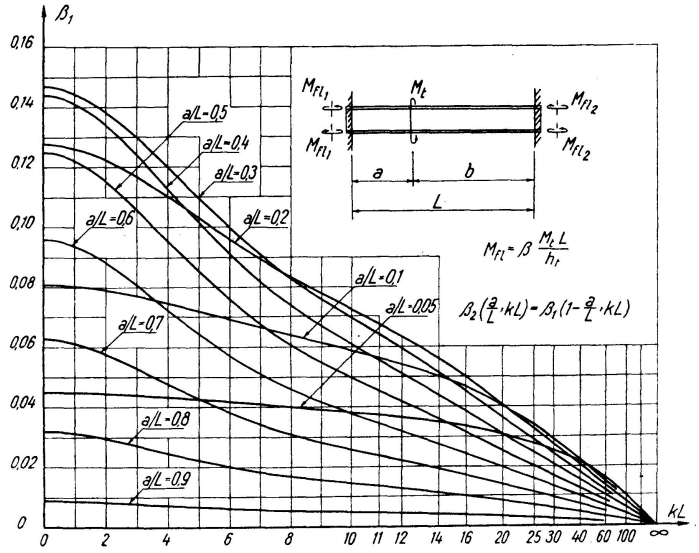

Fig. 10.

$$\beta_2\left(\frac{a}{L};\, kL\right) = \beta_1\left(1 - \frac{a}{L};\, kL\right) \tag{23}$$

c) I beam rigidly built in at one end and submitted to twisting moment m_t uniformly distributed along the whole beam

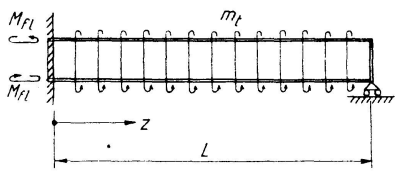

Fig. 11.

$$\varphi_1 = \frac{m_t}{k^2 C}\left[-1 + \tfrac{1}{2}(kL)^2 \frac{z}{L}\left(1 - \frac{z}{L}\right) + \frac{\cosh k\left(\frac{L}{2} - z\right)}{\cosh \frac{kL}{2}}\right] \tag{24}$$

$$\frac{d^2\varphi_1}{dz^2} = -\frac{m_t}{C}\left[1 - \frac{\cosh k\left(\frac{L}{2} - z\right)}{\cosh \frac{kL}{2}}\right] \tag{25}$$

$$M_{fl} = \beta \frac{m_t L^2}{h_t} \tag{26}$$

$$\beta = \frac{\left(\frac{kL}{2} - \operatorname{tgh}\frac{kL}{2}\right)\operatorname{tgh} kL}{kL\,(kL - \operatorname{tgh} kL)} \tag{27}$$

The coefficient β is graphically represented by the curve corresponding to $\frac{a}{L}=1$ in fig. 14.

d) I beam rigidly built in at both ends and submitted to twisting moment m_t uniformly distributed along the whole beam

We find φ_1 and $\frac{d^2\varphi_1}{dz^2}$ from eqs. (24) and (25).

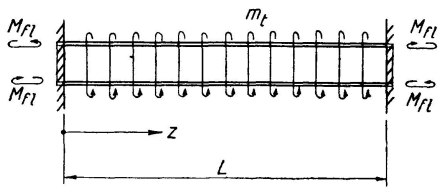

Fig. 12.

$$M_{fl} = \beta \frac{m_t L^2}{h_t} \tag{28}$$

$$\beta = \frac{\frac{kL}{2} - \operatorname{tgh}\frac{kL}{2}}{(kL)^2 \operatorname{tgh}\frac{kL}{2}} \tag{29}$$

The coefficient β is graphically represented by the curves corresponding to $\frac{a}{L}=1$ in figs. 16 and 17.

e) I beam rigidly built in at one end and submitted to twisting moment m_t uniformly distributed over length a

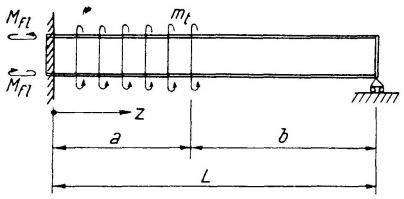

Fig. 13.

$$\varphi_1 = \frac{m_t}{k^2 C}\left\{-1 + (kL)^2 \frac{z}{L}\left[\frac{a}{L}\left(1-\frac{a}{2L}\right) - \frac{z}{2L}\right] + \frac{\sinh k(L-z) + \cosh kb \sinh kz}{\sinh kL}\right\}$$
$$\text{for} \quad 0 \leqslant z \leqslant a \tag{30}$$

$$\varphi_1 = \frac{m_t}{k^2 C}\left\{\tfrac{1}{2}(kL)^2\left(\frac{a}{L}\right)^2\left(1-\frac{z}{L}\right) - \frac{\cosh ka - 1}{\sinh kL}\sinh k(L-z)\right\}$$
$$\text{for} \quad a \leqslant z \leqslant L \tag{30'}$$

$$\frac{d^2\varphi_1}{dz^2} = -\frac{m_t}{C}\left[1 - \frac{\sinh k(L-z) + \cosh kb \sinh kz}{\sinh kL}\right] \quad \text{for} \quad 0 \leqslant z \leqslant a \tag{31}$$

$$\frac{d^2\varphi_1}{dz^2} = -\frac{m_t}{C}\cdot\frac{\cosh ka - 1}{\sinh kL}\sinh k(L-z) \quad \text{for} \quad a \leqslant z \leqslant L \tag{31'}$$

$$M_{fl} = \beta \frac{m_t L^2}{h_t} \tag{32}$$

$$\beta = \frac{\frac{a}{L}\left(1-\frac{a}{2L}\right)\sinh kL - \frac{1}{kL}(\cosh kL - \cos kb)}{kL\cosh kL - \sinh kL} \tag{33}$$

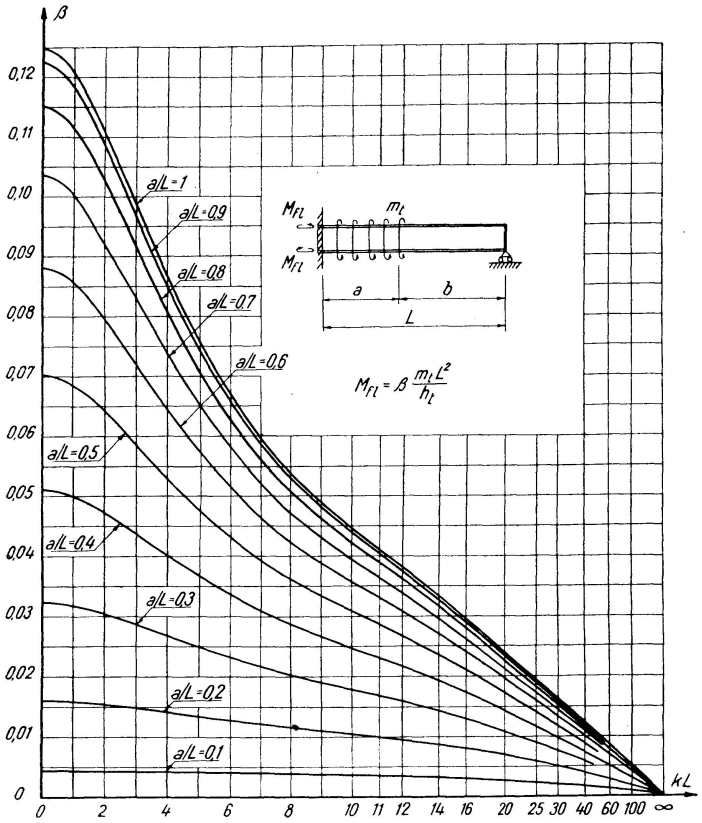

Fig. 14.

f) I beam rigidly built in at both ends and submitted to twisting moment m_t uniformly distributed over length a

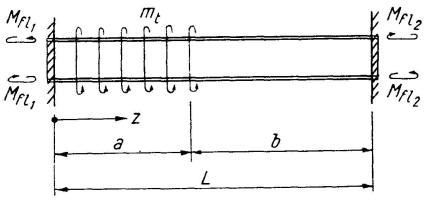

Fig. 15.

We find φ_1 and $\frac{d^2 \varphi_1}{d z^2}$ from eqs. (30) and (31).

$$M_{fl_1} = \beta_1 \frac{m_t L^2}{h_t}; \quad M_{fl_2} = \beta_2 \frac{m_t L^2}{h_t} \tag{34}$$

$$\beta_1 = \frac{1}{(k L \cosh k L - \sinh k L)^2 - (\sinh k L - k L)^2} \cdot$$
$$\cdot \left\{ \left[\frac{a}{L} \left(1 - \frac{a}{2L}\right) \sinh k L - \frac{1}{k L} (\cosh k L - \cosh k b) \right] \cdot (k L \cosh k L - \sinh k L) - \right.$$
$$\left. - \left[\frac{a^2}{2 L^2} \sinh k L - \frac{1}{k L} (\cosh k a - 1) \right] (\sinh k L - k L) \right\} \tag{35}$$

$$\beta_2 = \frac{1}{(k\,L\cosh k\,L - \sinh k\,L)^2 - (\sinh k\,L - k\,L)^2}\cdot$$
$$\cdot\left\{\left[\frac{a^2}{2\,L^2}\sinh k\,L - \frac{1}{k\,L}(\cosh k\,a - 1)\right](k\,L\cosh k\,L - \sinh k\,L) -\right.$$
$$\left. - \left[\frac{a}{L}\left(1 - \frac{a}{2\,L}\right)\sinh k\,L - \frac{1}{k\,L}(\cosh k\,L - \cosh k\,b)\right](\sinh k\,L - k\,L)\right\} \quad (36)$$

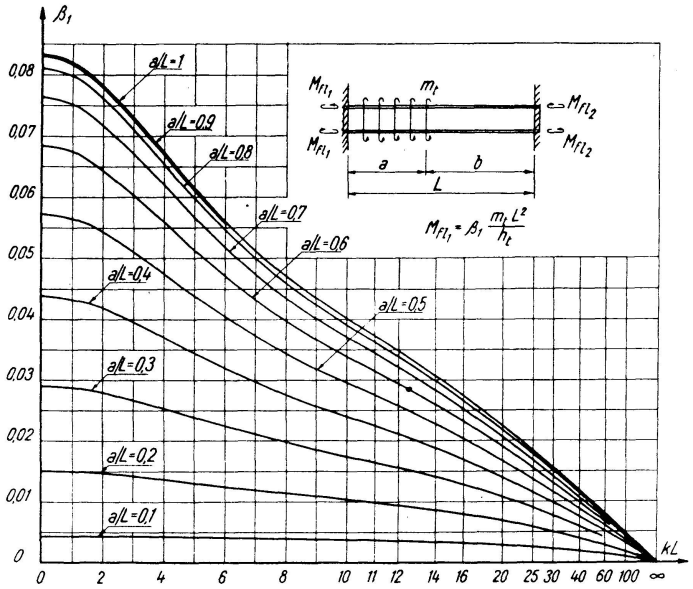

Fig. 16.

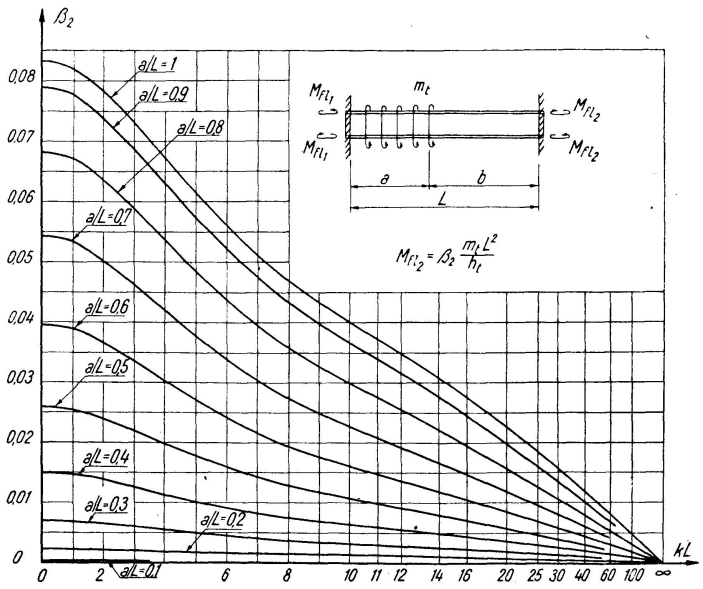

Fig. 17.

C. Numerical Example

In order to illustrate the application of the method of successive approximations described in the above, we shall study a continuous I beam prevented from torsion at its supports. The shape of the beam and the loading are shown

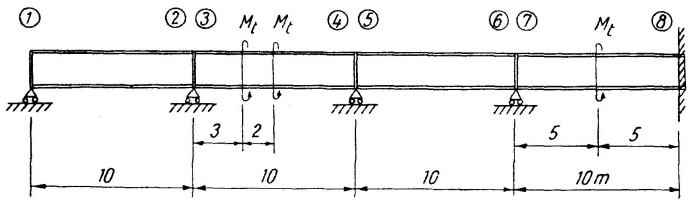

Fig. 18.

in fig. 18. The beam is a structural steel shape Dip 85, for which $E = 2{,}1 \cdot 10^6$ kp per cm² and $G = 0{,}81 \cdot 10^6$ kp per cm².

The problem to be solved is to calculate the bending moments M_{fl} acting on the flanges at the supports, and the variation in the angle of rotation φ and in its second derivative $\frac{d^2\varphi}{dz^2}$ along the beam, which determines the normal stresses caused by torsion in the longitudinal direction of the beam.

For all spans of the beam, we compute from tables for Dip structural shapes as well as from eqs. (2), (3), and (5)

$$C = 1{,}035 \cdot 10^9 \text{ kp cm}^2$$
$$C_w = 0{,}567 \cdot 10^{14} \text{ kp cm}^4$$
$$k = 0{,}427 \cdot 10^{-2} \text{ 1/cm}; \; kL = 4{,}27$$

Carry-over factor, r

For the transfer of the bending moments acting on the flanges from (2) to (1), we have $r = 0$. For all other spans of the continuous I beam, cf. fig. 4, we have

$$r = 0{,}270$$

Stiffness factor, S, and distribution factor, s

For $kL = 4{,}27$, we obtain from eq. (12) and fig. 5

$$S_2 = 5.58 \frac{C_w}{h_t L}$$

$$S_3 = S_4 = S_5 = S_6 = S_7 = 6.01 \frac{C_w}{h_t L}$$

For the distribution factors s, eq. (15) yields

$$s_2 = \frac{S_2}{S_2+S_3} = 0.482; \quad s_3 = 1-s_2 = 0.518$$

$$s_4 = s_5 = s_6 = s_7 = \tfrac{1}{2}$$

Basic bending moments acting on flanges, M^i_{fl}

The bending moments acting on the top flange and the bottom flange of the beam are numerically equal but opposite in direction in all spans. In the determination of the bending moments acting on the flanges, it is therefore sufficient to study one flange of the I beam. In the following treatment, we shall study the top flange, and define as positive those moments acting on the beam, which tend to cause clockwise rotation when seen from above.

For the basic bending moments acting on the flanges, we obtain from eq. (21) and fig. 10

$$M^i_{fl_3} = -(0.1170+0.0923)\,\frac{M_t L}{h_t} = -2.46\,M_t$$

$$M^i_{fl_4} = +(0.0460+0.0923)\,\frac{M_t L}{h_t} = +1.63\,M_t$$

$$M^i_{fl_7} = -0.0923\,\frac{M_t L}{h_t} = -1.09\,M_t = -M^i_{fl_8}$$

Successive compensation of basic bending moments acting on flanges

The basic bending moments acting on the flanges, are entered uppermost in the calculation pattern shown in fig. 19. After that, the supports which were assumed to be rigidly built in for the calculation of the moments M^i_{fl} are

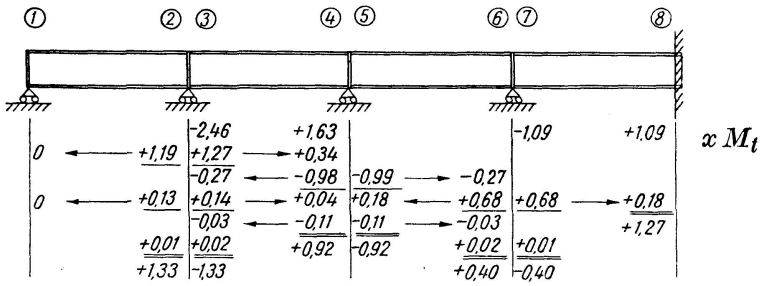

Fig. 19.

successively released from restraint one after another. At the same time, we introduce the compensating moments which are required for equilibrium of the points of support. In order that the convergency shall be as rapid as possible in the successive compensation of the basic bending moments acting on

the flanges, the restraint should always be released in the first place at that point of support whose equilibrium requires the greatest compensating moment.

In the example under consideration, this implies that the successive release from the assumed rigid restraint is started at the point of support (2)—(3). After release from rigid restraint, the beam is submitted at this point of support to the bending moment $-2{,}46\,M_t$ acting on the flange. In order to ensure equilibrium, it is therefore necessary to add at this point of support a compensating moment $+2{,}46\,M_t$, which is distributed over the parts of the beam which are adjacent to this point of support in proportion to the distribution factors of these parts of the beam. This gives the distributed bending moment acting on the flange $+0{,}482 \cdot 2{,}46\,M_t = +1{,}19\,M_t$ for the part of the beam (2)—(1) and the distributed bending moment acting on the flange $+0{,}518 \cdot 2{,}46\,M_t = +1{,}27\,M_t$ for the part of the beam (3)—(4). At the opposite ends of the spans, the distributed bending moments acting on the flange give rise to carry-over bending moments acting on the flange, which are determined by the carry-over factors r. At the cross section (1), this causes a carry-over bending moment which is equal to zero, while the carry-over bending moment produced at the cross section (4) is equal to $+0{,}270 \cdot 1{,}27\,M_t = +0{,}34\,M_t$. No further transfer of the bending moments acting on the flange will take place, since the support (1) is hinged in a vertical direction and the support (4)—(5) is assumed to be rigidly built in. The distributed and the carry-over bending moments acting on the flange which have been calculated in the above are entered in the calculation pattern in the horizontal row directly below the basic bending moments acting on the flange. Thus we have taken account of the whole effect produced by the release from rigid restraint at the support (2)—(3). For further treatment, we shall assume that this support is rigidly built in again. This is indicated in the calculation pattern by the horizontal lines drawn below the last-computed bending moments acting on the flange at the support in question.

The same procedure is repeated for the support (4)—(5), at which the compensating bending moment acting on the flange $= -1{,}63\,M_t - 0{,}34\,M_t = -1{,}97\,M_t$ must be introduced for ensuring equilibrium, then for the support (6)—(7), and so on, until the compensating bending moments required for equilibrium become so small as to be practically negligible.

The summation of the basic bending moments acting on the flange and the distributed and the carry-over moments obtained from the successive compensation of moments yields the resultant bending moments acting on the flange

$$
\begin{aligned}
M_{fl_1} &= 0 \\
M_{fl_2} &= -M_{fl_3} = +1{,}33\,M_t \\
M_{fl_4} &= -M_{fl_5} = +0{,}92\,M_t \\
M_{fl_6} &= -M_{fl_7} = +0{,}40\,M_t \\
M_{fl_8} &= +1{,}27\,M_t
\end{aligned}
$$

Determination of angle of rotation φ and its second derivate $\frac{d^2\varphi}{dz^2}$

When the bending moments acting on the flanges at the supports are known as a result of the above calculations, the angle of rotation φ and its derivatives in each span of the continuous beam can be determined by the superposition of two solutions, viz., first, the angle of rotation φ_1 and its derivatives for the beam which is hinged in a vertical direction at both ends, and second, the solution in the case of loading shown in fig. 2. For example, for the angle of rotation φ and its second derivative $\frac{d^2\varphi}{dz^2}$ at the centre of the span (3)—(4), we obtain from eqs. (8), (10), (17), and (18) the values

$$\varphi = \frac{M_t}{kC}\left[0.15\,kL - \frac{\sinh 0.3\,kL}{2\cosh\frac{kL}{2}} + 0.25\,kL - \tfrac{1}{2}\,\mathrm{tgh}\,\frac{kL}{2}\right] +$$

$$+(M_{fl_3}-M_{fl_4})\,\frac{h_t}{C}\left(\tfrac{1}{2}-\frac{1}{2\cosh\frac{kL}{2}}\right) = 2.326\cdot 10^{-7}\,M_t - 0.678\cdot 10^{-7}\,M_t = 1.648\cdot 10^{-7}\,M_t$$

radians

$$\frac{d^2\varphi}{dz^2} = -\frac{M_t}{C}\,k\left[\frac{\sinh 0.3\,kL}{2\cosh\frac{kL}{2}} + \tfrac{1}{2}\,\mathrm{tgh}\,\frac{kL}{2}\right] - (M_{fl_3}-M_{fl_4})\,\frac{h_t}{CL^2}\,(kL)^2\,\frac{1}{2\cosh\frac{kL}{2}} =$$

$$= -2.805\cdot 10^{-12}\,M_t + 0.376\cdot 10^{-12}\,M_t = -2.429\cdot 10^{-12}\,M_t \qquad 1/\mathrm{cm}^2$$

if M_t is expressed in kp cm.

The variation in the angle of rotation and in its second derivative along the continuous I beam is graphically represented by the curves in fig. 20.

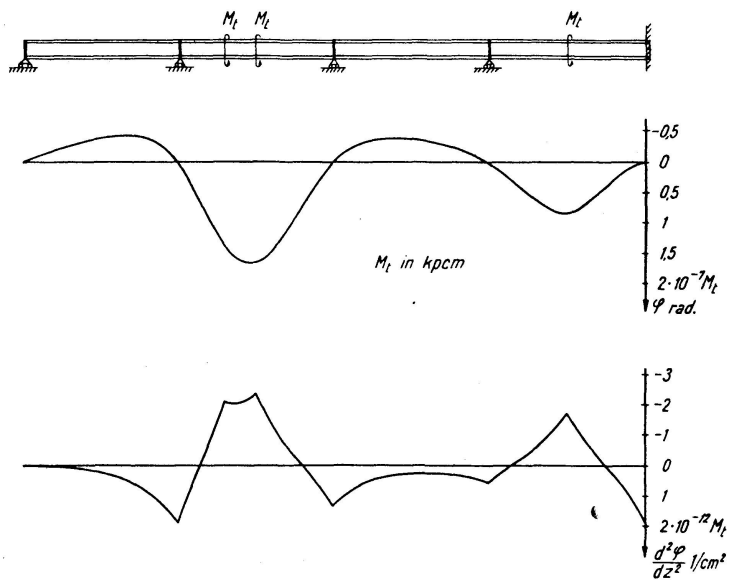

Fig. 20.

Summary

In this paper, a method of successive approximations is advanced for the calculation of the bending moments acting on the flanges produced at those supports of a continuous I beam in torsion which are not hinged in a vertical direction. In principle, this method can be characterised as a generalised variant of the Cross method. When Saint Venant's torsional rigidity of an I beam is equal to zero, and when the torsional load therefore produces pure bending of the flanges, the method evolved by the Author becomes identical with the method of fixed end moments devised by Cross.

In order to adapt this method of successive approximations to practical calculations, the Author deduces the relations required for the determination of some basic quantities, viz., the carry-over factor, the stiffness factor, and the distribution factor for the bending moments acting on the flanges as well as the basic bending moments acting on the flanges in some common cases of torsional loading. Furthermore, these basic quantities are graphically represented in diagrams. The application of the method of successive approximations is illustrated by a numerical example.

Résumé

Dans le présent rapport, l'auteur décrit un procédé d'approximations successives pour le calcul des moments fléchissants qui agissent sur les ailes d'une poutre continue en double T soumise à la torsion aux appuis qui ne sont pas articulés dans le sens vertical. En principe, on peut caractériser ce procédé comme une variante généralisée du procédé Cross pour la compensation successive des moments. En effet, quand la rigidité de Saint Venant d'une poutre en double T soumise à la torsion est égale à zéro et quand, par conséquent, la charge de torsion cause une flexion pure des ailes, le procédé imaginé par l'auteur et le procédé de Cross pour la compensation successive des moments deviennent identiques.

Afin d'adapter ce procédé d'approximations successives aux calculs pratiques, l'auteur déduit les expressions requises pour la détermination de quelques grandeurs fondamentales, à savoir: le facteur de transmission, le facteur de rigidité et le facteur de distribution pour les moments fléchissants qui agissent sur les ailes ainsi que les moments fléchissants fondamentaux qui agissent sur les ailes dans quelques cas ordinaires de charges de torsion. En outre, ces grandeurs fondamentales sont représentées graphiquement dans des diagrammes. Un exemple numérique sert a illustrer l'application du procédé d'approximations successives.

Zusammenfassung

In diesem Bericht entwickelt der Verfasser ein auf wiederholten Annäherungen fußendes Verfahren für die Berechnung der Flanschbiegemomente, die an den nicht in senkrechter Richtung gelenkig gelagerten Stützen eines auf Verdrehung beanspruchten durchlaufenden Doppel-T-Trägers auftreten. Dieses Verfahren kann grundsätzlich als ein verallgemeinertes Croßsches Momentausgleichsverfahren gekennzeichnet werden. Wenn ein Doppel-T-Träger gar keine Saint Venantsche Drehsteifigkeit besitzt und die Drehlast daher durch reine Flanschbiegung aufnimmt, geht nämlich das vom Verfasser ersonnene Verfahren in das Croßsche Momentausgleichsverfahren über.

Um das geschilderte Verfahren der wiederholten Annäherungen den praktischen Berechnungsbedürfnissen anzupassen, werden die für die Bestimmung einiger Grundgrößen erforderlichen Beziehungen im Bericht aufgestellt. Diese Größen sind der Übertragungsfaktor, der Steifigkeitsfaktor und die Verteilungszahl für die Flanschbiegemomente und die in einigen gewöhnlichen Drehbelastungsfällen vorkommenden Grundflanschbiegemomente. Außerdem sind diese Grundgrößen auch in Schaubildern graphisch dargestellt. Um die Anwendung des Verfahrens der wiederholten Annäherungen zu veranschaulichen, wird ein Zahlenbeispiel durchgerechnet.