Simplified calculus of the stability of multy-story frames

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Simplified Calculus of the Stability of Multi-Story Frames

Calcul simplifié de la stabilité des cadres multiples

Vereinfachte Stabilitätsberechnung von Stockwerksrahmen

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For multi-story frames of symmetrical design, subjected to vertical loads, there exist two different possible deformation patterns in crippling, as shown by figures 1 and 2. It is obvious, that the latter is the more critical, as it gives longer reduced crippling lengths; therefore it shall, alone, be treated here. Referring to an older publication[1]) in which the principle of the *Stiffness Method* was explained, the limit of stability is attained, when the resulting stiffness, in any node, reduces to zero. With symmetrical design as well as symmetrical loading conditions, the frame may be split into two halves, exerting no horizontal reaction upon each other. The static system then reduces to what is shown by fig. 3.

The "transmitted stiffness", at one end of a vertical bay, is now given by

$$k_2 = \frac{k_1 \cdot \varphi/3 \tan \varphi - k_0 \cdot \varphi^2/9}{k_1/k_0 + \varphi/3 \tan \varphi}$$

k_1 being the stiffness of the adjacent members at the other end, $k_0 = 3\,E\,I/l$ the "inherent stiffness" of the considered bay and $\varphi = \sqrt{P\,l^2/E\,I}$. In order to find the limit of stability, a first assumption has to be made for the load factor. A numerical example will best show how this method works; figures 2 and 3 give all necessary design data and the design loads in the three stories. Only the ratios of the stiffnesses are needed, so the factor "3 E" may be omitted and the following values are obtained:

Bay	AB	$3830:470 = 8.15$	BE	$2630:250 = 10.50$
	BC	$3830:470 = 8.15$	CF	$2630:250 = 10.50$
	CD	$1520:470 = 3.24$	DG	$1950:250 = 7.80$

Starting with an estimated load factor of "3", the "crippling angles" for the three bays will be

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[1]) KIRSTE: Momentenverteilungs- und Stabilitätsrechnung nach der Steifigkeitsmethode. Österr. Ingenieur-Archiv, Vol. I, 1/2.

$$\varphi_1 = \sqrt{\frac{3 \cdot 56 \cdot 470^2}{2100 \cdot 3830}} = 2.15; \qquad \varphi_2 = \sqrt{\frac{3 \cdot 35 \cdot 470^2}{2100 \cdot 3830}} = 1.70;$$

$$\varphi_3 = \sqrt{\frac{3 \cdot 12.8 \cdot 470^2}{2100 \cdot 1520}} = 1.63$$

and the transmitted stiffnesses, beginning at the node "A" and using the accompanying table for $\varphi/3 \tan\varphi$ and $\varphi^2/9$:

$$k_A = \infty; \qquad k_B = \frac{\infty \cdot (-0.469) - 8.15 \cdot 0.514}{\infty/8.15 - 0.469} + 10.50 = +6.68;$$

$$k_C = \frac{-6.68 \cdot 0.074 - 8.15 \cdot 0.321}{6.68/8.15 - 0.074} + 10.50 = +6.33;$$

$$k_D = \frac{-6.33 \cdot 0.032 - 3.24 \cdot 0.295}{6.33/3.24 - 0.032} + 7.80 = \underline{+7.20}.$$

The resulting value for the stiffness at "D" is strongly positive, indicating that a load factor of "3" is still below the critical. By "trial and error" one arrives at a critical load factor of 3.5, the crippling angles being multipled by $\sqrt{3.5/3}$, so that $\varphi_1 = 2.32$, $\varphi_2 = 1.84$, $\varphi_3 = 1.76$. Beginning again at "A", one has now:

$$k_A = \infty; \qquad k_B = -8.15 \cdot 0.719 + 10.50 = +4.65;$$

$$k_C = \frac{-4.65 \cdot 0.169 - 8.15 \cdot 0.376}{4.65/8.15 - 0.169} + 10.50 = 0.90;$$

$$k_D = \frac{-0.90 \cdot 0.112 - 3.24 \cdot 0.344}{0.90/3.24 - 0.112} + 7.80 \sim \underline{0}$$

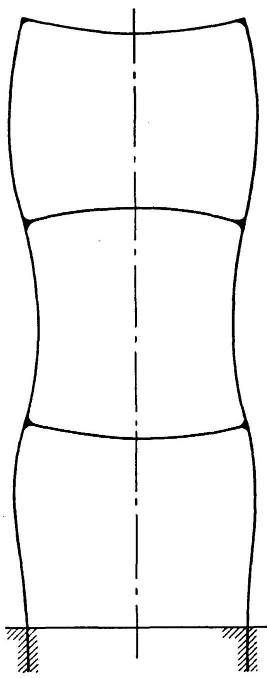
Fig. 1

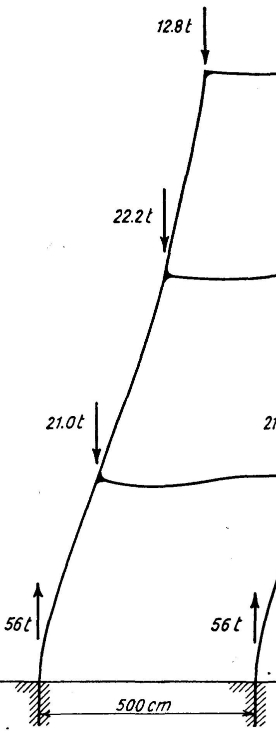

Fig. 2

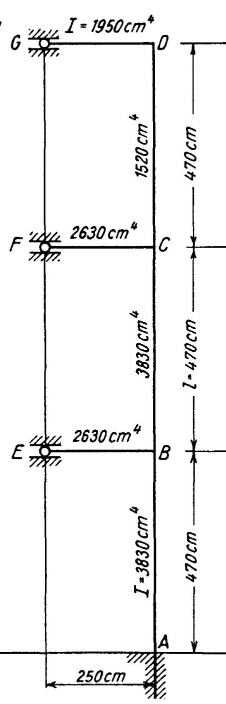

Fig. 3

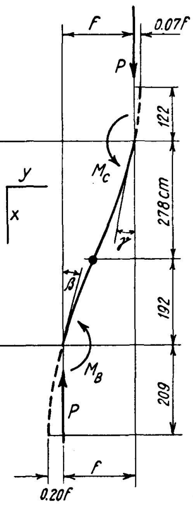

Fig. 4

and beginning at "D":

$$k_D = +7.80; \qquad k_C = \frac{-7.80\cdot 0.112 - 3.24\cdot 0.344}{7.80/3.24 - 0.112} + 10.50 = +9.62;$$

$$k_D = \frac{-9.62\cdot 0.169 - 8.15\cdot 0.376}{9.62/8.15 - 0.169} + 10.50 = +5.85;$$

$$k_A = \frac{-5.85\cdot 0.719 - 8.15\cdot 0.598}{5.85/8.15 - 0.719} + \infty = -\frac{9.07}{0} + \infty = \underline{0}.$$

Knowing the crippling angle and the stiffnesses from both sides, the elastic curve may be computed for each bay. With the notations of fig. 4 the equations for the equilibrium of moments in bay BC become:

$$M_x = -M_B + P\cdot y = -EI\cdot d^2y/dx^2; \qquad M_B + M_C = k_B\cdot\beta + k_C\cdot\gamma = \varphi^2 EI/l^2\cdot f$$

and the solution for the deflection y

$$y/f = \frac{\sin\varphi x/l\,(\varphi + 3\,r_B \tan\varphi x/2\,l)}{\sin\varphi\,(\varphi + 3\,r_B\tan\varphi/2)}.$$

This means a pure sine-curve, symmetrical to the X-axis. The slopes at B and C are

$$\beta = f/l\frac{\varphi^2/\sin\varphi}{\varphi + 3\,r_B\tan\varphi/2}; \qquad \gamma = f/l\frac{\varphi^2/\sin\varphi}{\varphi + 3\,r_C\tan\varphi/2}.$$

Taking $\quad \varphi_2 = 1.84 \qquad r_B = 4.65/8.15 = 0.517 \qquad r_C = 9.62/8.15 = 1.18$

gives $\quad \beta = 0.860\,f/l; \qquad \gamma = 0.542\,f/l.$

The positions of the summits of the sine-curve are determined by

$$dy/dx = 0:\ \tan\varphi x/l = -\varphi/3\,r_B = -1.075$$
$$x_1/l = -0.822; \qquad x_1 = -209 \text{ cm}$$
$$x_2/l = +2.320; \qquad x_2 = +592 \text{ cm}$$

Putting these values in the expression for y/f gives the deflections

$$y_1 = -0.20\,f \quad\text{and}\quad y_2 = +1.07\,f.$$

For the reason of continuity, the slope at any node, as computed from the two adjacent bays, must be the same. Considering bay $A\,B$, one has

$$\varphi_1 = 2.32; \qquad r_B = 5.85/8.15 = 0.718; \qquad \beta = 1.01\,f/l,$$

and bay CD:

$$\varphi_3 = 1.76; \qquad r_C = 0.90/3.24 = 0.278; \qquad \gamma = 1.10\,f/l.$$

Equalising the corresponding expressions yields a relation between the values of f in the three bays:

$$1.01\,f_1 = 0.860\,f_2; \qquad 0.542\,f_2 = 1.10\,f_3; \qquad f_1 : f_2 : f_3 = 1 : 1.157 : 0.578.$$

With these data, the elastic curves of the uprights can be drawn, the absolute values of the ordinates remaining indeterminate, provided that they are "small"; the cross-members, having no axial load, deform into parabolas of the third order.

Using the diagram fig. 5 which gives the critical value of φ as a function of the relative stiffnesses r_1 and r_2 of the cross-members, it is even possible to find the limit of stability almost without any calculation, by splitting the multi-story frame into simple rectangular frames. The cutting of the cross-members must be done in such a way that the sum of the moments of inertia of the two parts equals the moment of inertia of the original cross-member, and that the critical values of φ in all bays are in the proportion of the square roots of the axial loads.

Taking the same numerical example as before, the stiffness 10.50 of the cross-member BE should be split into 5.96 and 4.54, for the lower and the upper frame, and the same stiffness of CF into 10.00 and 0.50. The relative stiffnesses of the uprights will be now:

in the lowest frame $r_A = \infty$, $r_B = 5.96/8.15 = 0.73$, for which the diagramm gives a critical value of $\varphi_1 = 2.32$;

in the middle frame $r_B = 4.54/8.15 = 0.56$, $r_C = 10.00/8.15 = 1.22$, $\varphi_2 = 1.84$;

in the top frame $r_C = 0.50/3.24 = 0.16$, $r_D = 7.80/3.24 = 2.41$, $\varphi_3 = 1.76$.

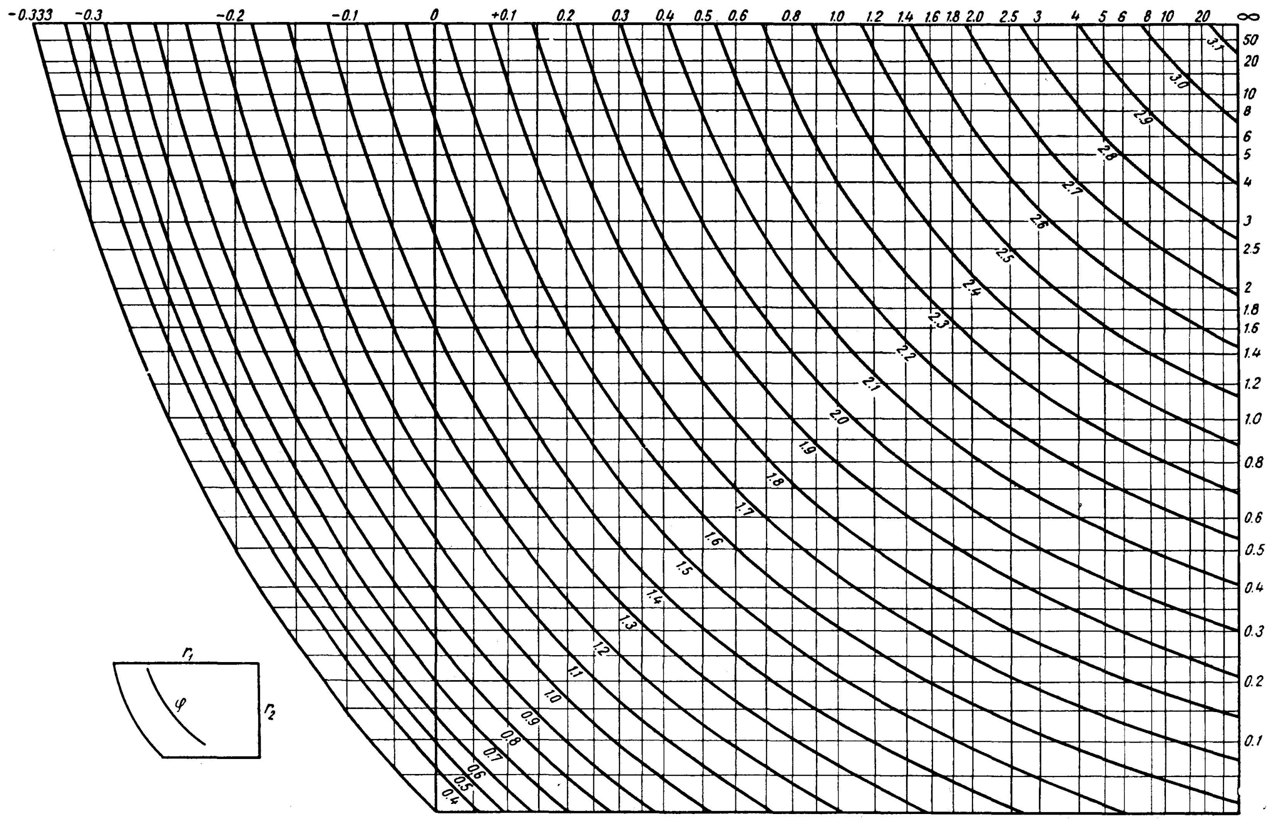

Fig. 5. Simplified calculus of the stability of multi-story frames.

The negative values of r_1 in the diagram are necessary, when φ is very small in comparison with the value of the adjacent bay. Then, it may be convenient to distribute the stiffnesses so that one part will be more than the total amount, and the other a negative figure.

φ	$\varphi^2/9$	$\varphi/3\,\mathrm{tg}\varphi$	φ	$\varphi^2/9$	$\varphi/3\,\mathrm{tg}\varphi$	φ	$\varphi^2/9$	$\varphi/3\,\mathrm{tg}\varphi$
0.02	+0.00004	+0.3333	0.86	+0.0822	+0.2468	1.70	+0.3211	−0.0736
4	00018	3332	88	0860	2425	72	3287	0862
6	00040	3329	0.90	0900	2381	74	3364	0991
8	00071	3326	92	0940	2335	76	3442	1123
0.10	00111	3322	94	0982	2288	78	3520	1260
12	00160	3317	96	1024	2240	1.80	3600	1400
14	00218	3312	98	1067	2191	82	3680	1544
16	00285	3305	1.00	1111	2140	84	3762	1692
18	00360	3297	02	1156	2088	86	3844	1845
0.20	00444	3289	04	1202	2035	88	3927	2002
22	00537	3279	06	1248	1980	1.90	4011	2164
24	00640	3269	08	1296	1924	92	4096	2330
26	00752	3258	1.10	1344	1866	94	4182	2502
28	00871	3246	12	1394	1807	96	4268	2680
0.30	0100	3233	14	1444	1746	98	4356	2862
32	0114	3219	16	1495	1684	2.00	4444	3051
34	0128	3204	18	1547	1621	02	4534	3246
36	0144	3188	1.20	1600	1555	04	4624	3447
38	0160	3171	22	1654	1488	06	4715	3656
0.40	0178	3154	24	1708	1419	08	4807	3871
42	0196	3135	26	1764	1349	2.10	4900	4094
44	0215	3115	28	1820	1277	12	4994	4325
46	0235	3095	1.30	1878	1203	14	5088	4564
48	0256	3073	32	1936	1127	16	5184	4813
0.50	0278	3051	34	1995	1050	18	5280	5070
52	0300	3027	36	2055	0970	2.20	5378	5338
54	0324	3003	38	2116	0888	22	5476	5616
56	0348	2977	1.40	2178	0805	24	5575	5906
58	0374	2951	42	2240	0719	26	5675	6207
0.60	0400	2923	44	2304	0631	28	5776	6522
62	0427	2895	46	2368	0541	2.30	5878	6850
64	0455	2865	48	2434	0449	32	5980	7193
66	0484	2835	1.50	2500	0355	34	6084	7551
68	0514	2803	52	2567	0258	36	6188	7927
0.70	0544	2770	54	2635	0158	38	6294	8320
72	0576	2736	56	2704	+0.00561	2.40	6400	8734
74	0608	2701	58	2774	−0.00485	42	6507	9168
76	0642	2665	1.60	2844	0156	44	6615	−0.9625
78	0676	2628	62	2916	0266	46	6724	−1.0107
0.80	0711	2590	64	2988	0379	48	6834	−1.0617
82	0747	2550	66	3062	0495	2.50	6944	−1.1156
84	0784	2510	68	3136	0614	52	7056	−1.1727

φ	$\varphi^2/9$	$\varphi/3\,\mathrm{tg}\,\varphi$	φ	$\varphi^2/9$	$\varphi/3\,\mathrm{tg}\,\varphi$	φ	$\varphi^2/9$	$\varphi/3\,\mathrm{tg}\,\varphi$
2.54	+0.7168	−1.2333	2.76	+0.8464	−2.2928	2.98	+0.9867	−6.0896
56	7282	−1.2980	78	8587	−2.4501	3.00	+1.0000	−7.0147
58	7396	−1.3669	2.80	8711	−2.6252	02	1.0134	−8.2379
2.60	7511	−1.4406	82	8836	−2.8215	04	1.0268	−9.9392
62	7627	−1.5197	84	8962	−3.0432	06	1.0404	−12.4724
64	7744	−1.6048	86	9088	−3.2955	08	1.0540	−16.6428
66	7862	−1.6965	88	9216	−3.5857	3.10	1.0678	−24.8272
68	7980	−1.7959	2.90	9344	−3.9230	12	1.0816	−48.1336
2.70	8100	−1.9038	92	9474	−4.3204	3.14	1.0955	$-\infty$
72	8220	−2.0217	94	9604	−4.7985			
74	8342	−2.1507	96	9735	−5.3734			

Summary

Classical methods lead to an intricate determinant, which is not practicable for numerical computation. In the case of symmetrical design and loading, the "stiffness method" gives the critical value of axial compression in the uprights, or their "reduced crippling length", by calculating the "transmitted stiffnesses", proceeding from one node to the next: "Zero stiffness" indicates that the limit of stability is attained.

Résumé

Les méthodes classiques conduisent à un déterminant complexe, qui ne se prête guère à un calcul numérique. S'il y a symétrie, aussi bien au point de vue de la construction que du chargement, la ,,méthode de la rigidité" donne la valeur critique de la compression dans les montants ou bien leur ,,longueur réduite de flambage", en calculant la ,,rigidité transmise", d'un nœud au suivant: Une ,,rigidité nulle" indique que la limite de stabilité est atteinte.

Zusammenfassung

Klassische Methoden führen auf eine komplizierte Determinante, die sich nicht zur praktischen Auswertung eignet. Für den Fall symmetrischer Belastung und Konstruktion gibt die ,,Steifigkeitsmethode" durch Ermittlung der ,,weitergeleiteten Steifigkeiten", von einem Knoten zum nächsten fortschreitend, die kritische Belastung der Ständer oder ihre ,,reduzierte Knicklänge": Eine ,,Steifigkeit null" zeigt an, daß die Stabilitätsgrenze erreicht ist.