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Autor(en): **Nowacki, Witold**

Objekttyp: **Article**

Zeitschrift: **IABSE publications = Mémoires AIPC = IVBH Abhandlungen**

Band (Jahr): **16 (1956)**

PDF erstellt am: **27.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-15073

The State of Stress in a Thin Plate Due to the Action of Sources of Heat

L'état de tension dans les plaques, causé par l'action des sources de chaleur

Spannungszustände, hervorgerufen in Scheiben infolge von Wärmequellen

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1.

It is known that the displacement equations of the theory of elasticity for a plane state of stress and a steady temperature field can be represented[1]), by the differential equation

$$\nabla^2 \nabla^2 \Phi = (1+\nu)\,\alpha\, \nabla^2 T, \tag{1.1}$$

where Φ is the so-called thermal potential of displacement, $T(x,y)$ the temperature, ν Poisson's ratio and α the coefficient of thermal expansion.

The temperature distribution in the plate is described by the differential equation

$$\nabla^2 T = \frac{W}{h\,k}. \tag{1.2}$$

W being the intensity of the source, k the coefficient of thermal conductivity and h the thickness of the plate.

The eqs. (1.1) and (1.2) can be replaced by the single equation

$$\nabla^2 \nabla^2 \Phi = -\frac{(1+\nu)\,\alpha\, W}{h\,k}. \tag{1.3}$$

The boundary conditions of the problem are as follows. The temperature at the edge is constant. We can assume, without limiting the generality, that $T=0$ at the edge. It follows that $\nabla^2\Phi = 0$ at the edge. The stresses due to the temperature field are related to the function Φ by the following equations, [1]:

$$\bar{\sigma}_x = -2\,G\frac{\partial^2\Phi}{\partial y^2}, \qquad \bar{\sigma}_y = -2\,G\frac{\partial^2\Phi}{\partial x^2}, \qquad \bar{\tau}_{xy} = 2\,G\frac{\partial^2\Phi}{\partial x\,\partial y}. \tag{1.4}$$

[1]) E. MELAN, H. PARCUS: Wärmespannungen stationärer Temperaturfelder, Wien 1953.

The second condition is that of the vanishing of the normal or shear stresses at the edge. Assuming $\Phi=0$ the vanishing stresses will be the normal stresses.

In order to eliminate the shear stresses at the edge the stresses

$$\bar{\bar{\sigma}}_x = \frac{\partial^2 F}{\partial y^2}, \qquad \bar{\bar{\sigma}}_y = \frac{\partial^2 F}{\partial x^2}, \qquad \bar{\bar{\tau}}_{xy} = -\frac{\partial^2 F}{\partial x \, \partial y}, \tag{1.5}$$

must be added to (1.4).

The function F should satisfy the differential equation

$$\nabla^2 \nabla^2 F = 0. \tag{1.6}$$

Solving the eq. (1.5) we assume that the normal stresses vanish at the edge and the shear stresses satisfy the boundary condition $\bar{\bar{\tau}}_{xy} = -\bar{\tau}_{xy}$.

The stresses due to the action of the heat are determined by the equations

$$\sigma_x = \bar{\sigma}_x + \bar{\bar{\sigma}}_x, \qquad \sigma_y = \bar{\sigma}_y + \bar{\bar{\sigma}}_y, \qquad \tau_{xy} = \bar{\tau}_{xy} + \bar{\bar{\tau}}_{xy}. \tag{1.7}$$

It should be noted that the differential eq. (1.3) with the boundary conditions $\Phi=0$, $\nabla^2\Phi=0$ is analogous to the differential equation of the deflection surface of a simply supported plate. We have

$$\nabla^2 \nabla^2 w = \frac{Q}{N} \tag{1.8}$$

the boundary conditions being $w=0$, $\nabla^2 w=0$.

In the eq. (1.3) W denotes the intensity of the source of heat, and it should therefore be considered as a function equal to zero outside the neighbourhood of the source. The function Q in the eq. (1.8) should have an analogous character. It should be regarded as the intensity of the external load of a plate, equal to zero outside the neighbourhood of the point being considered. Q can therefore be treated as a concentrated force.

In this paper the analogy between the eqs. (1.3) and (1.8) is used. The determination of the function Φ will be based on the known results of the theory of bending of plates. The principal problem will be that of determining the stress function F.

We shall confine our considerations to the state of stress due to the action of heat sources in an infinite and a semi-infinite strip of plate and in rectangular plate.

2. A Strip of Plate of Infinite Length

Let us consider a strip of plate of infinite length and of breadth a, having a source of heat located at the point $(\xi, 0)$. This strip will be replaced by a strip of plate of breadth a, simply supported at the edges and loaded with a concentrated force Q at the point $(\xi, 0)$. The deflection of the plate is expressed[2]), by the equation:

[2]) K. Girkmann: Flächentragwerke, Wien 1954, p. 179.

$$w = \frac{2\,Q}{a\,\pi\,N} \sum_{n=1}^{\infty} \sin\alpha_n\,\xi \sin\alpha_n\,x \int_0^{\infty} \frac{\cos\beta\,y\,d\beta}{(\alpha_n{}^2+\beta^2)^2}, \tag{2.1}$$

where $\alpha_n = n\,\pi/a$ and N is the flexural rigidity of the plate.

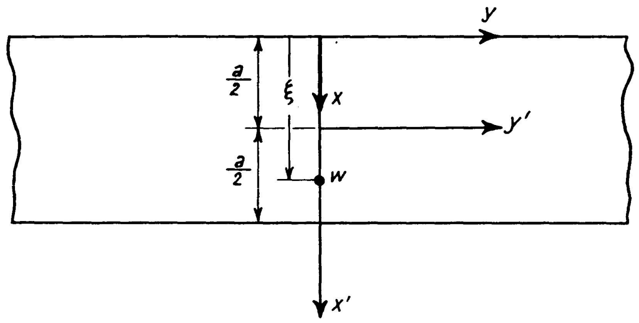

Fig. 1

From the analogy between the differential eq. (1.3) and (1.8) and from similar boundary conditions it follows that

$$\Phi = -\frac{2\,K}{a\,\pi\,h} \sum_{n=1}^{\infty} \sin\alpha_n\,\xi \sin\alpha_n\,x \int_0^{\infty} \frac{\cos\beta\,y\,d\beta}{(\alpha_n{}^2+\beta^2)^2}, \tag{2.2}$$

where
$$K = \frac{(1+\nu)\,\alpha\,W}{K}.$$

Using the eqs. (1.4) and bearing in mind that the expression (2.2) can be represented by the series

$$\Phi = -\frac{K\,a^2}{2\,\pi^3\,h} \sum_{n=1,2,\ldots}^{\infty} \frac{e^{-\alpha_n y}}{n^3}(1+\alpha_n\,y) \sin\alpha_n\,\xi \sin\alpha_n\,x \quad \text{for} \quad y \geqq 0 \tag{2.3}$$

we can calculate successively

$$\begin{aligned}
\bar{\sigma}_x &= -2\,G\frac{\partial^2\Phi}{\partial\,y^2} = -\frac{K\,G}{\pi\,h}\sum_{n=1}^{\infty}\frac{e^{-\alpha_n y}}{n}(1-\alpha_n\,y)\sin\alpha_n\,\xi\sin\alpha_n\,x,\\
\bar{\sigma}_y &= -2\,G\frac{\partial^2\Phi}{\partial\,x^2} = -\frac{K\,G}{\pi\,h}\sum_{n=1}^{\infty}\frac{e^{-\alpha_n y}}{n}(1+\alpha_n\,y)\sin\alpha_n\,\xi\sin\alpha_n\,x,\\
\bar{\tau}_{xy} &= 2\,G\frac{\partial^2\Phi}{\partial\,x\,\partial\,y} = \frac{K\,G}{a\,h}\sum_{n=1}^{\infty}e^{-\alpha_n y}\sin\alpha_n\,\xi\cos\alpha_n\,x.
\end{aligned} \tag{2.4}$$

The eqs. (2.4) are valid for $y \geqq 0$.

The series in the eqs. (2.4) being slowly convergent and for $y=0$, $x=\xi$ divergent, it will be convenient to represent them in closed forms

$$\bar{\sigma}_x = \frac{K\,G}{h}\left(\varphi + y\frac{\partial\varphi}{\partial\,y}\right), \qquad \bar{\sigma}_y = \frac{K\,G}{h}\left(\varphi - y\frac{\partial\varphi}{\partial\,y}\right), \qquad \bar{\tau}_{xy} = -\frac{K\,G}{h}\,y\frac{\partial\varphi}{\partial\,x}, \tag{2.5}$$

where

$$\varphi = -\frac{1}{\pi} \sum_{n=1,2,\ldots}^{\infty} \frac{e^{-\alpha_n y}}{n} \sin \alpha_n \xi \sin \alpha_n x ,$$

$$= \frac{1}{4\pi} \ln \frac{\cosh \frac{\pi y}{a} - \cos \frac{\pi}{a}(x-\xi)}{\cosh \frac{\pi y}{a} - \cos \frac{\pi}{a}(x+\xi)} . \tag{2.5'}$$

It is evident that the stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$ vanish at the edge and that all stresses vanish for $y \to \infty$. In the neighbourhood of the source the stresses tend to infinity in a logarithmic manner.

For subsequent considerations it will be convenient to represent the stresses τ_{xy} in the form

$$\bar{\tau}_{xy} = \frac{4KG}{a\pi h} \int_0^{\infty} \beta \sin \beta y \left(\sum_{n=1}^{\infty} \frac{\alpha_n \sin \alpha_n \xi \cos \alpha_n x}{(\alpha_n{}^2 + \beta^2)^2} \right) d\beta \tag{2.6}$$

following directly from the eq. (2.2).

Bearing in mind that

$$\sum_{n=1}^{\infty} \frac{\alpha_n \sin \alpha_n \xi}{(\alpha_n{}^2 + \beta^2)^2} = \frac{a^3}{4} \eta_1 (\xi, \beta) ,$$

$$\sum_{n=1}^{\infty} \frac{\alpha_n (-1)^n \sin \alpha_n \xi}{(\alpha_n{}^2 + \beta^2)^2} = \frac{a^3}{4} \eta_2 (\xi, \beta) , \tag{2.7}$$

where

$$\eta_1 (\xi, \beta) = \frac{\beta \xi \sinh \lambda \cosh \beta (a-\xi) - \lambda \sinh \beta \xi}{\lambda^2 \sinh^2 \lambda} ,$$

$$\eta_2 (\xi, \beta) = \frac{\beta \xi \sinh \lambda \cosh \beta \xi - \lambda \cosh \lambda \sinh \beta \xi}{\lambda^2 \sinh^2 \lambda} , \tag{2.7'}$$

$$\lambda = \beta a ,$$

we have

$$\bar{\tau}_{xy}\big|_{x=0} = \frac{KGa^2}{\pi h} \int_0^{\infty} \beta \eta_1 (\xi, \beta) \sin \beta y \, d\beta ,$$

$$\bar{\tau}_{xy}\big|_{x=a} = \frac{KGa^2}{\pi h} \int_0^{\infty} \beta \eta_2 (\xi, \beta) \sin \beta y \, d\beta . \tag{2.8}$$

The source of heat at the point (ξ, 0) can be replaced by two sources symmetrical or anti-symmetrical in relation to the y'-axis (fig. 2).

For two symmetrical sources of intensity $W/2$ (fig. 2a) we have in the system of coordinates x', y':

$$\bar{\tau}^{(s)}_{x'y'}\big|_{x'=\frac{a}{2}} = \frac{KGa^2}{2\pi h} \int_0^{\infty} \beta \left[\eta_2 \left(\frac{a}{2} + \xi', \beta \right) + \eta_2 \left(\frac{a}{2} - \xi', \beta \right) \right] \sin \beta y' \, d\beta ,$$

or

$$\bar{\tau}^{(s)}_{x'y'}\Big|_{x'=\frac{a}{2}} = \frac{K G a^2}{8\pi h}\int\limits_0^\infty \beta\,\rho^{(s)}(\mu,\xi')\sin\beta y'\,d\beta, \tag{2.8'}$$

where

$$\rho^{(s)}(\mu,\xi') = \frac{\beta\xi'\cosh\mu\sinh\beta\xi' - \mu\sinh\mu\cosh\beta\xi'}{\mu^2\cosh^2\mu},$$

$$\mu = \frac{\beta a}{2}.$$

For anti-symmetrical sources of intensity $W/2$ (fig. 2b) we have

$$\bar{\tau}^{(a)}_{x'y'}\Big|_{x'=\frac{a}{2}} = \frac{K G a^2}{2\pi h}\int\limits_0^\infty \beta\left[\eta_2\left(\frac{a}{2}+\xi',\beta\right) - \eta_2\left(\frac{a}{2}-\xi',\beta\right)\right]\sin\beta y'\,d\beta,$$

or

$$\bar{\tau}^{(a)}_{x'y'}\Big|_{x'=\frac{a}{2}} = \frac{K G a^2}{8\pi h}\int\limits_0^\infty \beta\,\rho^{(a)}(\mu,\xi')\sin\beta y'\,d\beta, \tag{2.8''}$$

where

$$\rho^{(a)}(\mu,\xi') = \frac{\beta\xi'\sinh\mu\cosh\beta\xi' - \mu\cosh\mu\sinh\beta\xi'}{\mu^2\sinh^2\mu}.$$

Let us consider first the case of a symmetrical system of heat sources. In order to eliminate the shear stresses $\bar{\tau}^{(s)}_{x'y'}$ on the lines $x' = \pm a/2$ we should choose a state of stress $\bar{\bar{\sigma}}^{(s)}_{x'}$, $\bar{\bar{\sigma}}^{(s)}_{y'}$, $\bar{\bar{\tau}}^{(s)}_{x'y'}$ such that the following equation is satisfied

$$\nabla^2\nabla^2 F^{(s)} = 0 \tag{2.9}$$

together with the boundary conditions

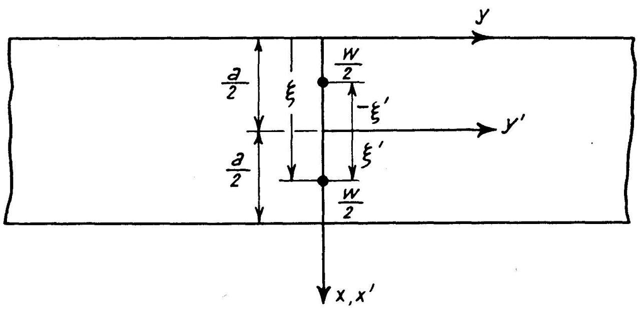

Fig. 2a

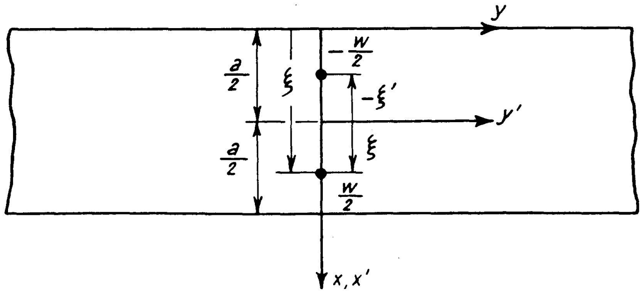

Fig. 2b

$$\bar{\bar{\sigma}}_{x'}^{(s)} = \frac{\partial^2 F^{(s)}}{\partial y'^2} = 0, \qquad \bar{\bar{\tau}}_{x'y'}^{(s)} = -\frac{\partial^2 F^{(s)}}{\partial x' \partial y'} = -\bar{\tau}_{x'y'}^{(s)} \quad \text{for} \quad x' = \pm\frac{a}{2}. \tag{2.10}$$

In view of the symmetry of the sources in relation to the y'-axis we can confine our considerations to the edge $x' = a/2$. The function $F^{(s)}$ will be assumed in the form

$$F^{(s)} = \frac{1}{h}\int_0^\infty \frac{1}{\beta^2}\left(A^{(s)}\cosh\beta x' + B^{(s)}\beta x'\sinh\beta x'\right)\cos\beta y'\, d\beta. \tag{2.11}$$

From the boundary conditions (2.10) we obtain

$$A^{(s)}\cosh\mu + B^{(s)}\mu\sinh\mu = 0,$$

$$(A^{(s)} + B^{(s)})\sinh\mu + B^{(s)}\mu\cosh\mu = \frac{K\,G\,a^2}{8\pi}\beta\,\rho^{(s)}(\mu, \xi').$$

Hence

$$A^{(s)} = -\frac{K\,G\,a}{4\pi}\,\frac{\mu^2\sinh\mu\,\rho^{(s)}(\mu,\xi')}{\cosh\mu\sinh\mu + \mu},$$

$$B^{(s)} = -\frac{\cosh\mu}{\mu\sinh\mu}A^{(s)}.$$

Let us determine the stresses from the eqs. (1.5)

$$\bar{\bar{\sigma}}_{x'}^{(s)} = \frac{K\,G\,a}{4\pi h}\int_0^\infty \mu\,\rho^{(s)}(\xi',\mu)\left[\frac{\mu\sinh\mu\cosh\beta x' - \beta x'\sinh\beta x'\cosh\mu}{\cosh\mu\sinh\mu + \mu}\right]\cdot \cos\beta y'\, d\beta,$$

$$\bar{\bar{\sigma}}_{y'}^{(s)} = \frac{K\,G\,a}{4\pi h}\int_0^\infty \mu\,\rho^{(s)}(\xi',\mu)\left[\frac{(\mu\sinh\mu - 2\cosh\mu)\cosh\beta x' - \beta x'\sinh\beta x'\cosh\mu}{\cosh\mu\sinh\mu + \mu}\right]\cdot \cos\beta y'\, d\beta, \tag{2.12}$$

$$\bar{\bar{\tau}}_{x'y'}^{(s)} = -\frac{K\,G\,a}{4\pi h}\int_0^\infty \mu\,\rho^{(s)}(\xi',\mu)\left[\frac{(\mu\sinh\mu - \cosh\mu)\sinh\beta x' - \beta x'\cosh\beta x'\cosh\mu}{\cosh\mu\sinh\mu + \mu}\right]\cdot \sin\beta y'\, d\beta.$$

Let us consider the case of sources of intensity $W/2$, which are anti-symmetrical in relation to the y'-axis.

In order to eliminate the shear stresses on the lines $x' = \pm a/2$ we choose the stresses $\bar{\bar{\sigma}}_{x'}^{(a)}$, $\bar{\bar{\sigma}}_{y'}^{(a)}$, $\bar{\bar{\tau}}_{x'y'}^{(a)}$ so that the differential equation

$$\nabla^2\nabla^2 F^{(a)} = 0 \tag{2.13}$$

is satisfied together with the boundary conditions

$$\bar{\bar{\sigma}}_{x'}^{(a)} = \frac{\partial^2 F^{(a)}}{\partial y'^2} = 0, \qquad \bar{\bar{\tau}}_{x'y'}^{(a)} = -\frac{\partial^2 F^{(a)}}{\partial x' \partial y'} = -\bar{\tau}_{x'y'}^{(a)}, \quad \text{for} \quad x = \pm\frac{a}{2}. \tag{2.14}$$

The function $F^{(a)}$ will be assumed in the form

$$F^{(a)} = \frac{1}{h}\int_0^\infty \frac{1}{\beta^2}(A^{(a)}\sinh\beta x' + B^{(a)}\beta x'\sinh\beta x')\cos\beta y'\,d\beta. \tag{2.15}$$

From the conditions (2.14) we obtain the system of two equations

$$A^{(a)}\sinh\mu + B^{(a)}\mu\cosh\mu = 0,$$
$$(A^{(a)} + B^{(a)})\cosh\mu + B^{(a)}\mu\sinh\mu = \frac{K G a^2}{8\pi}\beta\rho^{(a)}(\mu,\xi'),$$

from which we have

$$A^{(a)} = -\frac{K G a}{4\pi}\frac{\mu^2\rho^{(a)}(\mu,\xi')}{\cosh\mu\sinh\mu-\mu}, \qquad B^{(a)} = -A^{(a)}\frac{\sinh\mu}{\mu\cosh\mu}.$$

From the eqs. (1.5) we find the stresses

$$\bar{\bar{\sigma}}_{x'} = \frac{K G a}{4\pi h}\int_0^\infty \mu\rho^{(a)}(\mu,\xi')\left[\frac{\mu\cosh\mu\sinh\beta x' - \sinh\mu\,\beta x'\cosh\beta x'}{\cosh\mu\sinh\mu-\mu}\right]\cdot \cos\beta y'\,d\beta,$$

$$\bar{\bar{\sigma}}_{y'} = -\frac{K G a}{4\pi h}\int_0^\infty \mu\rho^{(a)}(\mu,\xi')\left[\frac{(\mu\cosh\mu - 2\sinh\mu)\sinh\beta x' - \beta x'\cosh\beta x'\sinh\mu}{\cosh\mu\sinh\mu-\mu}\right]\cdot \cos\beta y'\,d\beta, \tag{2.16}$$

$$\bar{\bar{\tau}}_{x'y'} = -\frac{K G a}{4\pi h}\int_0^\infty \mu\rho^{(a)}(\mu,\xi')\left[\frac{(\mu\cosh\mu - \sinh\mu)\cosh\beta x' - \beta x'\sinh\mu\sinh\beta x'}{\cosh\mu\sinh\mu-\mu}\right]\cdot \sin\beta y'\,d\beta.$$

It should be noted that for the two symmetric sources of intensity $W/2$ the stresses vanish on the lines $y'=0$ and $x'=0$. The distribution of normal stresses is symmetrical and that of shear stresses anti-symmetrical in relation to the x'- and y'-axes. On the contrary, for anti-symmetrical sources of intensity $W/2$ the distribution of normal stresses is anti-symmetrical and that of shear stresses is symmetrical.

For a source W at the origin of the system x', y' the stresses $\bar{\bar{\sigma}}^{(a)}_{x'}$, $\bar{\bar{\sigma}}^{(a)}_{y'}$, $\bar{\bar{\tau}}^{(a)}_{x'y'}$ vanish.

For a source W located in an asymmetrical manner the thermal stresses will be expressed by the equations

$$\sigma_x = \bar{\sigma}_x + \bar{\bar{\sigma}}^{(s)}_x + \bar{\bar{\sigma}}^{(a)}_x, \qquad \sigma_y = \bar{\sigma}_y + \bar{\bar{\sigma}}^{(s)}_y + \bar{\bar{\sigma}}^{(a)}_y, \qquad \tau_{xy} = \bar{\tau}_{xy} + \bar{\bar{\tau}}^{(s)}_{xy} + \bar{\bar{\tau}}^{(a)}_{xy}. \tag{2.17}$$

The solutions for a source at the point $(\xi, 0)$ can be used to determine the stresses due to a heat source distributed in an arbitrary manner along the segment $(\xi_2 - \xi_1)$ of the x-axis. If $\overline{W}(\xi)$ denotes the intensity of this source per unit length, the function expressing the thermal potential of displacement will take the form

$$\Phi = -\frac{2C}{a\pi h}\sum_{n=1}^{\infty} a_n \sin \alpha_n x \int_0^{\infty} \frac{\cos \beta y \, d\beta}{(\alpha_n^2+\beta^2)^2}, \tag{2.18}$$

where

$$C = \frac{(1+\nu)\,\alpha}{k}, \qquad a_n = \int_{\xi_1}^{\xi_2} \overline{W}(\xi) \sin \alpha_n \xi \, d\xi .$$

In the case of a source of intensity $W(\xi, \eta)$ per unit area of the plate, distributed over the area of the plate Ω the function Φ can be expressed in the form

$$\Phi = -\frac{2C}{a\pi h}\sum_{n=1}^{\infty} \sin \alpha_n x \int_0^{\infty} \frac{1}{(\alpha_n^2+\beta^2)^2}\left[\int_0^{\infty}\!\!\int W(\xi, \eta) \sin \alpha_n \xi \cos \beta (y-\eta) \, d\xi \, d\eta\right] d\beta . \tag{2.19}$$

If, in an analogous manner, $\sigma(x, y;\ \xi, \eta) = \overline{\sigma}(x, y;\ \xi, \eta) + \overline{\overline{\sigma}}(x, y;\ \xi, \eta)$ denotes the stress at the point (x, y), set up by a concentrated source $W = 1$, the stress $\sigma^*(x, y)$ due to a source of intensity $W(\xi, \eta)$ distributed over the area Ω, will be expressed by the integral

$$\sigma^*(x, y) = \iint_{(\Omega)} W(\xi, \eta)\, \sigma(x, y;\ \xi, \eta) \, d\xi \, d\eta . \tag{2.20}$$

Let us consider, in addition, an infinite plate in which sources of uniform intensity W are located in a periodic manner (fig. 3).

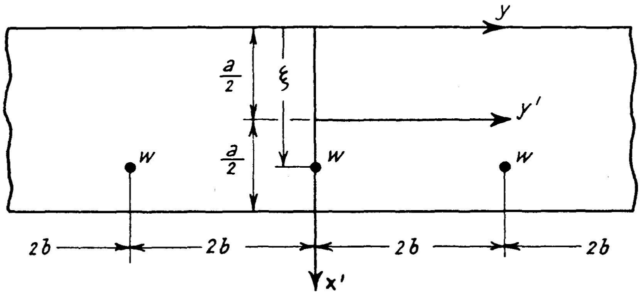

Fig. 3

It will be convenient to express the solution of eq. (1.3) in the form of a double trigonometrical series, the right-hand side of eq. (1.3) being expressed in the form

$$\frac{K}{h} = \frac{(1+\nu)\,\alpha W}{hK} = \frac{2K}{abh}\sum_{n=1}^{\infty} \sin \alpha_n \xi \sin \alpha_n x + \frac{4K}{abh}\sum_{n,m}^{\infty} \sin \alpha_n \xi \sin \alpha_n x \cos \beta_m y, \tag{2.21}$$

$$\alpha_n = \frac{n\pi}{a}, \qquad \beta_m = \frac{m\pi}{b},$$

and the function Φ by the series

$$\Phi = \sum_{n=1}^{\infty} A_n \sin \alpha_n x + \sum_{n,m}^{\infty} B_{n,m} \sin \alpha_n x \cos \beta_m y . \tag{2.22}$$

These series satisfy all the boundary conditions along the lines $x=0$, $x=a$ and $y=\pm b$. Substituting (2.21) and (2.22) in the eq. (1.3) we obtain

$$\Phi = -\frac{2K}{abh}\sum_{n=1}^{\infty}\frac{\sin\alpha_n\xi\sin\alpha_n x}{\alpha_n{}^4} - \frac{4K}{abh}\sum_{n,m}^{\infty}\frac{\sin\alpha_n\xi\sin\alpha_n x\cos\beta_m y}{(\alpha_n{}^2+\beta_m^2)^2}. \tag{2.23}$$

This function can also be represented in the form of a single trigonometrical series

$$\Phi = -\frac{K a^2}{2h\pi^3}\sum_{n=1}^{\infty}\frac{\sin\alpha_n\xi\sin\alpha x}{n^3}\left[e^{-\alpha_n y}(1+\alpha_n y)+R_n(y)\right], \tag{2.24}$$

where

$$R_n(y) = \frac{1}{e^{\delta_n}\sinh\delta_n}\left[(\cosh\alpha_n y-\alpha_n y\sinh\alpha_n y)+\frac{\delta_n e^{\delta_n}}{\sinh\delta_n}\cosh\alpha_n y\right],$$

$$\delta_n = \alpha_n b.$$

The stresses $\bar\sigma_x$, $\bar\sigma_y$, $\bar\tau_{xy}$ can be determined from the eqs. (1.4)

$$\begin{aligned}
\bar\sigma_y &= -2G\frac{\partial^2\Phi}{\partial x^2} = \frac{KG}{h}\left[\varphi - y\frac{\partial\varphi}{\partial y}+\theta-y\frac{\partial\theta}{\partial y}-\sum_{n=1}^{\infty}\delta_n\frac{\partial\theta_n}{\partial\delta_n}\right],\\
\bar\sigma_x &= -2G\frac{\partial^2\Phi}{\partial y^2} = \frac{KG}{h}\left[\varphi + y\frac{\partial\varphi}{\partial y}+\theta+y\frac{\partial\theta}{\partial y}+\sum_{n=1}^{\infty}\delta_n\frac{\partial\theta_n}{\partial\delta_n}\right],\\
\bar\tau_{xy} &= 2G\frac{\partial^2\Phi}{\partial x\,\partial y} = -\frac{KG}{h}\left[y\frac{\partial\varphi}{\partial x}+y\frac{\partial\theta}{\partial x}+\frac{b}{a}\sum_{n=1}^{\infty}\frac{\sin\alpha_n\xi\cos\alpha_n x\sinh\alpha_n y}{\sinh^2\delta_n}\right].
\end{aligned} \tag{2.25}$$

The function φ is identical with the function expressed by the eq. (2.5′) and the function θ is determined by the rapidly convergent series

$$\theta = \sum_{n=1}^{\infty}\theta_n = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin\alpha_n\xi\sin\alpha_n x\cosh\alpha_n y}{n\,e^{\delta_n}\sinh\delta_n}. \tag{2.26}$$

It is evident that the discontinuity of the logarithmic type is connected with the function φ, the function θ having no singularity. For $b\to\infty$ we have $\theta\to 0$ and the equations (2.25) become (2.5).

It will be convenient for the subsequent considerations to determine $\bar\tau_{xy}$ directly from eq. (2.23).

We have

$$\bar\tau_{xy} = 2G\frac{\partial^2\Phi}{\partial x\,\partial y} = \frac{8KG}{abh}\sum_{n,m}^{\infty}\frac{\alpha_n\beta_m\cos\alpha_n x\sin\alpha_n\xi\sin\beta_m y}{(\alpha_n{}^2+\beta_m^2)^2}. \tag{2.27}$$

Using the eqs. (2.7) we find

$$\bar\tau_{xy}\big|_{x=0} = \frac{2KGa^2}{bh}\sum_{m=1}^{\infty}\beta_m\,\eta_1(\xi',\beta_m)\sin\beta_m y,$$

$$\bar\tau_{xy}\big|_{x=a} = \frac{2KGa^2}{bh}\sum_{m=1}^{\infty}\beta_m\,\eta_2(\xi',\beta_m)\sin\beta_m y,$$

where the functions η_1, η_2 are taken from the eqs. (2.7′), introducing β_m and $\lambda_m = \beta_m a$ instead of β and $\lambda = \beta a$.

Proceeding as in the case of a strip of plate with one source we shall consider separately the case of two sources of intensity $W/2$ which are symmetrical in relation to the y'-axis and that of two anti-symmetrical sources of the same intensity.

For two sources of intensity $W/2$, which are symmetrical in relation to the y'-axis we have

$$\bar{\tau}^{(s)}_{x'y'}\Big|_{x'=\frac{a}{2}} = \frac{K G a^2}{4 b h} \sum_{m=1}^{\infty} \beta_m \rho^{(s)}(\mu_m, \xi') \sin \beta_m y', \qquad \mu_m = \frac{\beta_m a}{2} \tag{2.28}$$

and for anti-symmetrical sources

$$\bar{\tau}^{(a)}_{x'y'}\Big|_{x'=\frac{a}{2}} = \frac{K G a^2}{4 b h} \sum_{m=1}^{\infty} \beta_m \rho^{(a)}(\mu_m, \xi') \sin \beta_m y' \tag{2.29}$$

where $\rho^{(s)}$ and $\rho^{(a)}$ are taken from (2.8′) and (2.8″). For symmetrical sources we assume

$$F^{(s)} = \frac{1}{h} \sum_{n=1}^{\infty} \frac{1}{\beta_m^2} (A_m^{(s)} \cosh \beta_m x' + B_m^{(s)} \beta_m x' \sinh \beta_m x') \cos \beta_m y', \tag{2.30}$$

where

$$A_m^{(s)} = -\frac{K G a}{2 b} \frac{\mu_m^2 \sinh \mu_m \rho^{(s)}(\mu_m, \xi')}{\cosh \mu_m \sinh \mu_m + \mu_m},$$

$$B_m^{(s)} = -\frac{\operatorname{ctgh} \mu_m}{\mu_m} A_m^{(s)}.$$

The additional stresses can be calculated from

$$\begin{aligned}
\bar{\bar{\sigma}}^{(s)}_{x'} &= \frac{K G a}{2 b h} \sum_{m=1}^{\infty} \rho^{(s)}(\mu_m. \xi') \mu_m \cdot \\
&\quad \cdot \frac{\mu_m \sinh \mu_m \cosh \beta_m x' - \beta_m x' \sinh \beta_m x' \cosh \mu_m}{\cosh \mu_m \sinh \mu_m + \mu_m} \cos \beta_m y', \\
\bar{\bar{\sigma}}^{(s)}_{y'} &= -\frac{K G a}{2 b h} \sum_{m=1}^{\infty} \rho^{(s)}(\mu_m, \xi') \mu_m \cdot \\
&\quad \cdot \frac{(\mu_m \sinh \mu_m - 2 \cosh \mu_m) \cosh \beta_m x' - \beta_m x' \cosh \mu \sinh \beta_m x'}{\cosh \mu_m \sinh \mu_m + \mu_m} \cos \beta_m y', \\
\bar{\bar{\tau}}^{(s)}_{x'y'} &= -\frac{K G a}{2 b h} \sum_{m=1}^{\infty} \rho^{(s)}(\mu_m, \xi') \mu_m \cdot \\
&\quad \cdot \frac{(\mu_m \sinh \mu_m - \cosh \mu_m) \sinh \beta_m x' - \beta_m x' \cosh \mu_m \cosh \beta_m x'}{\cosh \mu_m \sinh \mu_m + \mu_m} \sin \beta_m y'.
\end{aligned} \tag{2.31}$$

In the case of two anti-symmetrical sources of intensity $W/2$ the following Airy function will be taken to determine the stresses $\bar{\bar{\sigma}}^{(a)}_{x'}$, $\bar{\bar{\sigma}}^{(a)}_{y'}$, $\bar{\bar{\tau}}^{(a)}_{x'y'}$

$$F^{(a)} = \frac{1}{h} \sum_{m=1}^{\infty} \frac{1}{\beta_m^2} (A_m^{(a)} \sinh \beta_m x' + B_m^{(a)} \beta_m x' \cosh \beta_m x') \cos \beta_m y', \tag{2.32}$$

where

$$A_m^{(a)} = -\frac{KG}{2b} \frac{\mu_m^2 \rho^{(a)}(\mu_m, \xi')}{\cosh \mu_m \sinh \mu_m - \mu_m},$$

$$B_m^{(a)} = -A_m^{(a)} \frac{\operatorname{tgh} \mu_m}{\mu_m}.$$

The stresses will be found from the eqs. (1.5)

$$\begin{aligned}
\bar{\bar{\sigma}}_{x'}^{(a)} &= \frac{KGa}{2bh} \sum_{m=1}^{\infty} \rho^{(a)}(\mu_m, \xi') \mu_m \cdot \\
&\cdot \frac{\mu_m \cosh \mu_m \sinh \beta_m x' - \sinh \mu_m \beta_m x' \cosh \beta_m x'}{\cosh \mu_m \sinh \mu_m - \mu_m} \cos \beta_m y', \\
\bar{\bar{\sigma}}_{y'}^{(a)} &= -\frac{KGa}{2bh} \sum_{m=1}^{\infty} \mu_m \rho^{(a)}(\mu_m, \xi') \cdot \\
&\cdot \frac{(\mu_m \cosh \mu_m - 2 \sinh \mu_m) \sinh \beta_m x' - \sinh \mu_m \beta_m x' \cosh \beta_m x'}{\cosh \mu_m \sinh \mu_m - \mu_m} \cos \beta_m y', \\
\bar{\bar{\tau}}_{x'y'}^{(a)} &= -\frac{KGa}{2bh} \sum_{m=1}^{\infty} \mu_m \rho^{(a)}(\mu_m, \xi') \cdot \\
&\cdot \frac{(\mu_m \cosh \mu_m - \sinh \mu_m) \cosh \beta_m x' - \sinh \mu_m \beta_m x' \sinh \beta_m x'}{\cosh \mu_m \sinh \mu_m - \mu_m} \sin \beta_m y'.
\end{aligned} \tag{2.33}$$

For sources of intensity W located in a periodic manner over a strip of plate (fig. 3) the thermal stress will be obtained by the superposition

$$\sigma_x = \bar{\sigma}_x + \bar{\bar{\sigma}}_x^{(s)} + \bar{\bar{\sigma}}_x^{(a)}, \quad \text{etc.}$$

It should be noted that for $b \to \infty$ the equations (2.31) and (2.33) become (2.12) and (2.16). The solution given here for the case of heat sources W uniformly spaced by $2b$, can be considered as Green's function. It can be used to determine the stresses due to linear or surface sources of heat located in a periodic manner over the area of the plate.

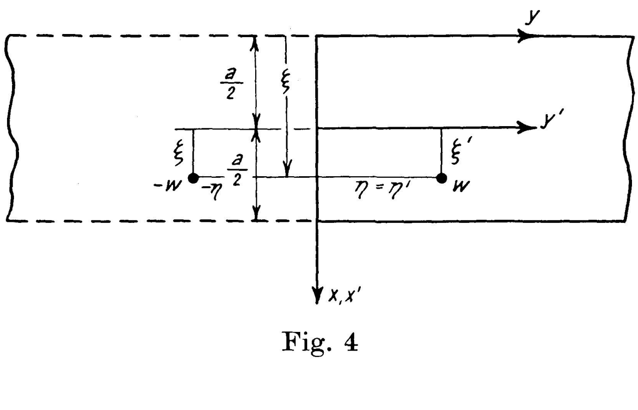

Fig. 4

3. A Semi-infinite Strip of Plate

The case of a semi-infinite strip of plate is equivalent to a strip of plate of infinite length with a source at the point (ξ, η) and a sink at the point $(\xi, -\eta)$. (Fig. 4). In this case we have $T=0$ at the edge $y=0$.

Using the eq. (2.2) we express the function Φ by the relation

$$\Phi = -\frac{2K}{a\pi h}\sum_{n=1}^{\infty}\sin\alpha_n\xi\sin\alpha_n x\int_0^{\infty}\frac{\cos\beta(y-\eta)-\cos\beta(y+\eta)}{(\alpha_n^2+\beta^2)^2}d\beta, \tag{3.1}$$

or

$$\Phi = -\frac{4K}{a\pi h}\sum_{n=1}^{\infty}\sin\alpha_n\xi\sin\alpha_n x\int_0^{\infty}\frac{\sin\beta\eta\sin\beta y}{(\alpha_n^2+\beta^2)^2}d\beta. \tag{3.2}$$

The function Φ can also be expressed by a simple trigonometrical series[3])

$$\Phi = -\frac{Ka^2}{\pi^3 h}\sum_{n=1}^{\infty}\frac{e^{-\alpha_n y}}{n^3}[(1+\alpha_n y)\sinh\alpha_n\eta-\alpha_n\eta\cosh\alpha_n\eta]\sin\alpha_n\xi\sin\alpha_n x. \tag{3.3}$$

This equation is valid for $\eta < y < \infty$. In the interval $0 < y < \eta$ y should be replaced by η in the eq. (3.3) and vice versa.

On the basis of the eqs. (1.4) the stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$ can be calculated. We have

$$\begin{aligned}\bar{\sigma}_x &= \frac{KG}{h}\left[\varphi_1-\varphi_2+(y-\eta)\frac{\partial\varphi_1}{\partial y}-(y+\eta)\frac{\partial\varphi_2}{\partial y}\right],\\ \bar{\sigma}_y &= \frac{KG}{h}\left[\varphi_1-\varphi_2-(y-\eta)\frac{\partial\varphi_1}{\partial y}+(y+\eta)\frac{\partial\varphi_2}{\partial y}\right],\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\varphi_1 &= \frac{1}{4\pi}\ln\frac{\cosh\frac{\pi}{a}(y-\eta)-\cos\frac{\pi}{a}(x-\xi)}{\cosh\frac{\pi}{a}(y-\eta)-\cos\frac{\pi}{a}(x+\xi)},\\ \varphi_2 &= \frac{1}{4\pi}\ln\frac{\cosh\frac{\pi}{a}(y+\eta)-\cos\frac{\pi}{a}(x-\xi)}{\cosh\frac{\pi}{a}(y+\eta)-\cos\frac{\pi}{a}(x+\xi)}.\end{aligned} \tag{3.5}$$

It can easily be verified that at the edges $x=0$, $x=a$ we have $\bar{\sigma}_x=0$ and at the edge $y=0$ we have $\bar{\sigma}_y=0$. The only stresses different from zero at these edges are the shear stresses $\bar{\tau}_{xy}$.

The following formula for the shear stresses $\bar{\tau}_{xy}$ will be convenient for subsequent considerations.

$$\bar{\tau}_{xy} = 2G\frac{\partial^2\Phi}{\partial x\,\partial y} = -\frac{8KG}{a\pi h}\sum_{n=1}^{\infty}\alpha_n\sin\alpha_n\xi\cos\alpha_n x\int_0^{\infty}\frac{\beta\sin\beta\eta\cos\beta y}{(\alpha_n^2+\beta^2)^2}d\beta. \tag{3.6}$$

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[3]) A. Nadai: Elastische Platten, Berlin 1925, p. 160.

Bearing in mind that

$$\sum_{n=1}^{\infty} \frac{\alpha_n \sin \alpha_n \xi}{(\alpha_n^2+\beta^2)^2} = \frac{a^3}{4} \eta_1(\xi,\beta), \qquad \sum_{n=1}^{\infty} \frac{\alpha_n (-1) \sin \alpha_n \xi}{(\alpha_n^2+\beta^2)^2} = \frac{a^3}{4} \eta_2(\xi,\beta),$$

and

$$\int_0^{\infty} \frac{\beta \sin \beta \eta}{(\alpha_n^2+\beta^2)^2} d\beta = \frac{\pi}{4\alpha_n} \eta\, e^{-\alpha_n \eta}$$

the stresses $\bar{\tau}_{xy}$ on the edges of the semi-infinite strip of plate can be expressed by the equations

$$\bar{\tau}_{xy}\big|_{x=0} = -\frac{2KGa^2}{\pi h} \int_0^{\infty} \beta \sin \beta \eta\, \eta_1(\xi,\beta) \cos \beta y\, d\beta,$$

$$\bar{\tau}_{xy}\big|_{x=a} = -\frac{2KGa^2}{\pi h} \int_0^{\infty} \beta \sin \beta \eta\, \eta_2(\xi,\beta) \cos \beta y\, d\beta, \tag{3.7}$$

$$\bar{\tau}_{xy}\big|_{y=0} = -\frac{2KG}{\pi h} \sum_{n=1}^{\infty} \vartheta(\alpha_n, \eta) \sin \alpha_n \xi \cos \alpha_n x,$$

where

$$\vartheta(\alpha_n, \eta) = \frac{\pi}{a} \eta\, e^{-\alpha_n \eta}.$$

Here also it will be convenient for the determination of the additional stresses to replace the heat source of intensity W at the point (ξ, η) by two sources of intensity $W/2$ first symmetrical and then anti-symmetrical in relation to the y'-axis.

Let us consider first of all the two symmetrical sources. In the system of coordinates x', y' (see fig. 5a) we have

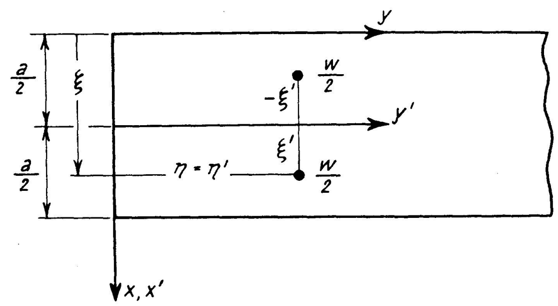

Fig. 5a

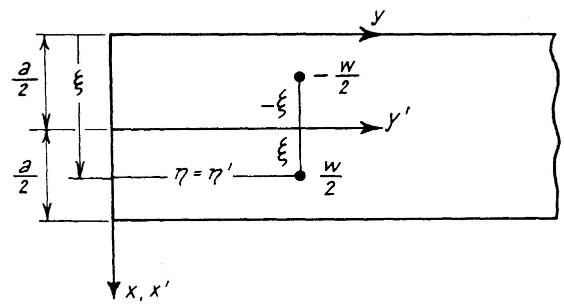

Fig. 5b

$$\left.\bar{\tau}^{(s)}_{x'y'}\right|_{x'=\frac{a}{2}} = -\frac{K\,G\,a^2}{4\,\pi\,h}\int\limits_0^\infty \beta\sin\beta\,\eta'\,\rho^{(s)}(\mu,\xi')\cos\beta\,y'\,d\beta,$$

$$\left.\bar{\tau}^{(s)}_{x'y'}\right|_{y'=0} = \frac{2\,K\,G}{\pi\,h}\sum_{n=1,2,\ldots}^{\infty}\vartheta(\alpha_n,\eta')\cos\alpha_n\,\xi'\sin\alpha_n\,x'. \tag{3.8}$$

The corresponding stresses $\bar{\bar{\sigma}}^{(s)}_x$, $\bar{\bar{\sigma}}^{(s)}_y$, $\bar{\bar{\tau}}^{(s)}_{xy}$ will be obtained by solving the Airy equation

$$\nabla^2\nabla^2 F^{(s)} = 0, \tag{3.9}$$

with the boundary conditions

$$\bar{\bar{\sigma}}^{(s)}_{x'} = \frac{\partial^2 F^{(s)}}{\partial y'^2} = 0, \qquad \bar{\bar{\tau}}^{(s)}_{x'y'} = -\frac{\partial^2 F^{(s)}}{\partial x'\,\partial y'} = -\bar{\tau}^{(s)}_{x'y'} \quad \text{for } x' = \frac{a}{2},$$

$$\bar{\bar{\sigma}}^{(s)}_{y'} = \frac{\partial^2 F^{(s)}}{\partial x'^2} = 0, \qquad \bar{\bar{\tau}}^{(s)}_{x'y'} = -\frac{\partial^2 F^{(s)}}{\partial x'\,\partial y'} = -\bar{\tau}^{(s)}_{x'y'} \quad \text{for } y' = 0. \tag{3.10}$$

These conditions will be satisfied assuming the function $F^{(s)}$ in the form

$$F^{(s)} = \frac{1}{h}\sum_{n=1,3,\ldots}^{\infty}\frac{1}{\alpha_n^2}\left(A_n^{(s)} + B_n^{(s)}\alpha_n\,y'\right)e^{-\alpha_n y'}\cos\alpha_n\,x' +$$

$$+\frac{1}{h}\int\limits_0^\infty\frac{1}{\beta}\left[A^{(s)}\cosh\beta\,x' + B^{(s)}\beta\,x'\sinh\beta\,x'\right]\sin\beta\,y'\,d\beta. \tag{3.11}$$

The boundary conditions (3.10) lead to the relations

$$A_n^{(s)} = 0, \tag{a}$$

$$A^{(s)}\cosh\mu + B^{(s)}\mu\sinh\mu = 0, \qquad \mu = \frac{\beta a}{2}, \tag{b}$$

$$\sum_{n=1,3,\ldots}^{\infty} B_n^{(s)}\sin\alpha_n\,x' - \int\limits_0^\infty\left[(A^{(s)}+B^{(s)})\sinh\beta\,x' + B^{(s)}\beta\,x'\cosh\beta\,x'\right]d\beta =$$

$$= -\frac{2\,K\,G}{\pi}\sum_{n=1,3,\ldots}^{\infty}\vartheta(\alpha_n,\eta')\cos\alpha_n\,\xi'\sin\alpha_n\,x', \tag{c} \quad (3.12)$$

$$\sum_{n=1,3,\ldots}^{\infty} B_n^{(s)}e^{-\alpha_n y'}(1-\alpha_n y')\sin\frac{n\pi}{2} - \int\limits_0^\infty\left[(A^{(s)}+B^{(s)})\sinh\mu + B^{(s)}\mu\cosh\mu\right]\cos\beta\,y'\,d\beta =$$

$$= \frac{K\,G\,a^2}{4\,\pi}\int\limits_0^\infty\beta\sin\beta\,\eta'\,\rho^{(s)}(\mu,\xi')\cos\beta\,y'\,d\beta. \tag{d}$$

Using the relations (3.12a, b) and substituting the relations

$$\sinh\beta\,x' = \sum_{n=1,2,\ldots}^{\infty} E_{n\beta}\sin\alpha_n\,x', \qquad \beta\,x'\cosh\beta\,x' = \sum_{n=1}^{\infty} F_{n\beta}\sin\alpha_n\,x',$$

$$e^{-\alpha_n y'}(1-\alpha_n y') = \int\limits_0^\infty C_{n\beta}\cos\beta\,y'\,d\beta$$

in (3.12c, d) we obtain the system of two equations

$$B_n^{(s)} - \int_0^\infty A^{(s)} [r(\mu) E_{n\beta} - g(\mu) F_{n\beta}] \, d\beta = -\frac{2KG}{\pi} \vartheta(\alpha_n, \eta') \cos \alpha_n \xi',$$
$$\sum_{k=1}^\infty B_k^{(s)} C_{K\beta} \sin \frac{k\pi}{2} + A^{(s)} t(\mu) = \frac{KGa^2}{4\pi} \beta \sin \beta \eta' \rho^{(s)}(\mu, \xi'), \tag{3.13}$$

where
$$r(\mu) = \frac{\mu \sinh \mu - \cosh \mu}{\mu \sinh \mu}, \qquad g(\mu) = \frac{\cosh \mu}{\mu \sinh \mu},$$
$$t(\mu) = \frac{\mu + \sinh \mu \cosh \mu}{\mu \sinh \mu}.$$

Bearing in mind that

$$E_{n\beta} = \frac{4\beta}{a} \frac{\cosh \mu \sin \frac{n\pi}{2}}{\alpha_n^2 + \beta^2},$$
$$F_{n\beta} = \frac{4\beta}{a} \frac{\sin \frac{n\pi}{2}}{\alpha_n^2 + \beta^2} \left[\mu \sinh \mu + \frac{\alpha_n^2 - \beta^2}{\alpha_n^2 + \beta^2} \cosh \mu \right],$$
$$C_{n\beta} = \frac{4}{\pi} \frac{\alpha_n \beta^2}{(\alpha_n^2 + \beta^2)^2}.$$

we can represent the eqs. (3.15) in the form

$$B_n^{(s)} + \frac{32 n^2}{a \pi^2} \sin \frac{n\pi}{2} \int_0^\infty \frac{A^{(s)} \cosh^2 \mu \, d\mu}{\sinh \mu \left(n^2 + \frac{4\mu^2}{\pi}\right)^2} = -\frac{2KG}{a} \eta' e^{-\frac{n\pi\eta'}{a}} \cos \frac{n\pi}{a} \xi',$$
$$\frac{16\mu^2 a}{\pi^4} \sum_{k=1,3,\ldots}^\infty B_k^{(s)} \frac{k \sin \frac{k\pi}{2}}{\left(k^2 + \frac{4\mu^2}{\pi^2}\right)^2} + A^{(s)} t(\mu) = \frac{KGa}{2\pi} \mu \sin \frac{2\mu\eta'}{a} \rho^{(s)}(\mu, \xi'), \tag{3.14}$$
$$\mu = \frac{\beta a}{2}.$$

Let us eliminate from this system the function $A^{(s)}(\mu)$. We obtain the infinite system of equations

$$B_n^{(s)} - \frac{512}{\pi^6} n^2 \sin \frac{n\pi}{2} \sum_{k=2} B_K^{(s)} k \sin \frac{k\pi}{2} \int_0^\infty \frac{\mu^3 \cosh^2 \mu \, d\mu}{\left(n^2 + \frac{4\mu^2}{\pi^2}\right)\left(k^2 + \frac{4\mu^2}{\pi^2}\right)^2 (\sinh \mu \cosh \mu + \mu)} =$$
$$= -\left[\frac{16 n^2}{\pi^3} \sin \frac{n\pi}{2} k G \int_0^\infty \frac{\mu^2 \cosh^2 \mu \, g^{(s)}(\mu, \xi') \sin \frac{2\mu\eta'}{a} \, d\mu}{\left(n^2 + \frac{4\mu}{\pi^2}\right)(\mu + \sinh \mu \cosh \mu)} + \right. \tag{3.15}$$
$$\left. + \frac{KG}{\pi} \left(\frac{\pi\eta'}{a}\right) e^{-\frac{n\pi\eta'}{a}} \cos \frac{n\pi\xi'}{a} \right],$$
$$n = 1, 3, 5, \ldots$$

After the determination of the integrals in a numerical manner we obtain a system of equations containing the unknown coefficients B_n. From the second of the eqs. (3.14) we determine the parameter $A(\mu)$. Thus all the quantities appearing in the function $F^{(s)}$ are determined. The stresses will be determined from the equations

$$\overline{\overline{\sigma}}{}^{(s)}_{x'} = \frac{\partial^2 F^{(s)}}{\partial y'^2}, \qquad \overline{\overline{\sigma}}{}^{(s)}_{y'} = \frac{\partial^2 F^{(s)}}{\partial x'^2}, \qquad \overline{\overline{\tau}}{}^{(s)}_{x'y'} = -\frac{\partial^2 F^{(s)}}{\partial x' \partial y'}. \tag{3.16}$$

Consider now the case of two sources of intensity $W/2$ anti-symmetrical in relation to the y'-axis (fig. 5b). In the system of coordinates x', y' we obtain

$$\begin{aligned} \overline{\tau}{}^{(a)}_{x'y'}\Big|_{x'=\frac{a}{2}} &= -\frac{KGa}{4\pi h}\int_0^\infty \beta \sin\beta\eta'\, \rho^{(a)}\mu, \xi')\cos\beta y'\, d\beta, \\ \overline{\tau}{}^{(a)}_{x'y'}\Big|_{y'=0} &= -\frac{2KG}{\pi h}\sum_{n=2,4,\ldots}^{\infty} \vartheta(\alpha_n, \eta')\sin\alpha_n\xi'\cos\alpha_n x'. \end{aligned} \tag{3.17}$$

The stresses $\overline{\overline{\sigma}}{}^{(a)}_{x'}$, $\overline{\overline{\sigma}}{}^{(a)}_{y'}$, $\overline{\overline{\tau}}{}^{(a)}_{x'y'}$ corresponding to this state will be found by solving the differential equation

$$\nabla^2\nabla^2 F^{(a)} = 0 \tag{3.18}$$

with the boundary conditions

$$\overline{\overline{\sigma}}{}^{(a)}_{x'} = \frac{\partial^2 F^{(s)}}{\partial y'^2} = 0, \qquad \overline{\overline{\tau}}{}^{(a)}_{x'y'} = -\frac{\partial^2 F^{(a)}}{\partial x'\partial y'} = -\overline{\tau}{}^{(a)}_{x'y'} \quad \text{for } x' = \frac{a}{2}, \tag{3.19}$$

$$\text{and } \overline{\overline{\sigma}}{}^{(a)}_{y'} = \frac{\partial^2 F^{(a)}}{\partial x'^2} = 0, \qquad \overline{\overline{\tau}}_{x'y'} = -\frac{\partial^2 F^{(a)}}{\partial x'\partial y'} = -\overline{\tau}{}^{(a)}_{x'y'} \quad \text{for } y' = 0.$$

The Airy function will be taken in the form

$$\begin{aligned} F^{(a)} = {} & \frac{1}{h}\sum_{n=2,4,\ldots}^{\infty}\frac{1}{\alpha_n^2}(A_n^{(a)} + B_n^{(a)}\alpha_n y')e^{-\alpha_n y'}\sin\alpha_n x' + \\ & + \frac{1}{h}\int_0^\infty \frac{1}{\beta^2}(A^{(a)}\sinh\beta x' + B^{(a)}\beta x'\cosh\beta x')\sin\beta y'\, d\beta. \end{aligned} \tag{3.20}$$

The boundary conditions (3.10) lead to the system of equations

$$A_n^{(a)} = 0, \tag{a}$$

$$A^{(a)}\sinh\mu + B^{(a)}\mu\cosh\mu = 0, \tag{b}$$

$$\begin{aligned} & \sum_{n=2,4,\ldots}^{\infty} B_n^{(a)}\cos\alpha_n x' + \int_0^\infty [(A^{(a)} + B^{(a)})\cosh\beta x' + B^{(a)}\beta x'\sinh\beta x']\, d\beta = \\ & = \frac{2KG}{\pi}\sum_{n=2,4,\ldots}^{\infty}\vartheta(\alpha_n, \eta')\sin\alpha_n\xi'\cos\alpha_n x', \end{aligned} \tag{c) (3.21}$$

$$\begin{aligned} & \sum_{n=2,4,\ldots}^{\infty} B_n^{(a)} e^{-\alpha_n y'}(1-\alpha_n y')\cos\frac{n\pi}{2} + \int_0^\infty [(A^{(a)}+B^{(a)})\cosh\mu + B^{(a)}\mu\sinh\mu]\cos\beta y'\, d\beta = \\ & = \frac{KGa^2}{4\pi}\int_0^\infty \beta\sin\beta\eta'\,\rho^{(a)}(\mu, \xi')\cos\beta y'\, d\beta. \end{aligned} \tag{d}$$

Using the eq. (3.21 b) and introducing the series

$$\cos \beta x' = \sum_{n=1}^{\infty} G_{n\beta} \cos \alpha_n x', \qquad \beta x' \sinh \beta x' = \sum_{n=1}^{\infty} H_{n\beta} \cos \alpha_n x',$$

where

$$G_{n\beta} = \frac{4}{a} \cosh \mu \frac{\alpha_n \sin \frac{n\pi}{2}}{\alpha_n^2 + \beta^2},$$

$$H_{n\beta} = \frac{2\beta \alpha_n \sin \frac{n\pi}{2}}{\alpha_n^2 + \beta_n^2} \left[\sinh \mu - \frac{4\beta}{a} \frac{\cosh \mu}{(\alpha_n^2 + \beta^2)^2} \right],$$

we obtain the system of two equations

$$\begin{aligned} B_n^{(a)} + \int_0^{\infty} A^{(a)} [c(\mu) G_{n\beta} - d(\mu) H_{n\beta}] \, d\beta &= \frac{2KG}{\pi} \vartheta(\alpha_n, \eta') \sin \alpha_n \xi', \\ \sum_{k=1}^{\infty} B_k^{(a)} \cos \frac{k\pi}{2} C_{k\beta} - A^{(a)} f(\mu) &= \frac{KGa^2}{4\pi} \beta \sin \beta \eta' \rho^{(a)}(\mu, \xi'), \end{aligned} \tag{3.22}$$

where

$$c(\mu) = \frac{\mu \cosh \mu - \sinh \mu}{\mu \cosh \mu}, \qquad d(\mu) = \frac{\sinh \mu}{\mu \cosh \mu}, \qquad f(\mu) = \frac{\sinh \mu \cosh \mu - \mu}{\mu \cosh \mu}.$$

The quantities $G_{n\beta}$, $H_{n\beta}$ being equal to zero for $n = 2, 4, \ldots$ the integral in the first equation of the group (3.22) vanishes and the system of eqs. (3.22) can be represented in the form

$$\begin{aligned} B_n^{(a)} &= \frac{2KG}{\pi} \vartheta(\alpha_n, \eta') \sin \alpha_n \xi', \\ \frac{16\mu^2}{\pi^3} \sum_{k=2,4,\ldots}^{\infty} B_k^{(a)} \frac{k^2 \cos \frac{K\pi}{2}}{\left(k^2 + \frac{4\mu^2}{\pi^2}\right)^2} + A^{(a)} \frac{\sinh \mu \cosh \mu - \mu}{\mu \cosh \mu} &= \\ = \frac{KGa}{2\pi} \mu \sin \frac{2\mu\eta'}{a} \rho^{(a)}(\mu, \xi'), \\ n = 2, 4, \ldots, \infty. \end{aligned} \tag{3.23}$$

The additional stresses will be obtained from the equations

$$\bar{\bar{\sigma}}_{x'}^{(a)} = \frac{\partial^2 F^{(a)}}{\partial y'^2}, \qquad \bar{\bar{\sigma}}_{y'}^{(a)} = \frac{\partial^2 F^{(a)}}{\partial x'^2}, \qquad \bar{\bar{\tau}}_{x\,y'}^{(a)} = -\frac{\partial^2 F^{(a)}}{\partial x' \partial y'}. \tag{3.24}$$

The stresses provoked by the action of a source of heat W located at the point (ξ, η) will be obtained as the sum of the stresses obtained from the eqs. (3.4), (3.16) and (3.24).

4. A Rectangular Plate

Let a source of heat of intensity W be located at the point (ξ, η) (fig. 6). Using the known solution for the deflection of a rectangular plate subjected to the action of a concentrated force at the point (ξ, η)[4]), the function Φ can be represented in the form

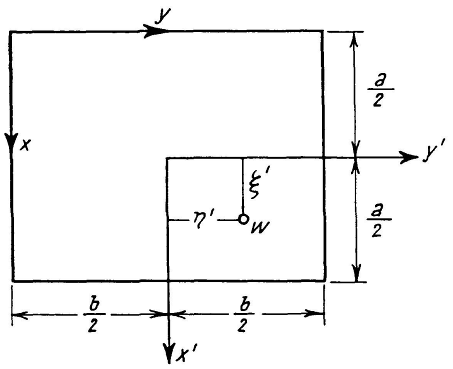

Fig. 6

$$\Phi = -\frac{4K}{abh} \sum_{n,m}^{\infty} \frac{\sin \alpha_n \xi \sin \beta_m \eta}{(\alpha_n{}^2 + \beta_m^2)^2} \sin \alpha_n x \sin \beta_m y, \qquad \alpha_n = \frac{n\pi}{a}, \quad \beta_m = \frac{m\pi}{b}. \tag{4.1}$$

This function can also be represented by a simple trigonometrical series

$$\Phi = -\frac{K a^2}{\pi^3 h} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \alpha_n \xi \sin \alpha_n x \frac{\sinh \alpha_n (b-\eta)}{\sinh \alpha_n b} \cdot$$
$$\cdot \{[1 + \alpha_n b \operatorname{ctgh} \alpha_n b - \alpha_n (b-\eta) \operatorname{ctgh} \alpha_n (b-\eta)] \sinh \alpha_n y - \alpha_n y \cosh \alpha_n y\}, \tag{4.2}$$
$$0 \leqq y \leqq \eta.$$

Using the eqs. (1.4), we find the stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$ from eq. (4.2). It will be convenient for subsequent considerations to represent the stress $\bar{\tau}_{xy}$ in the form

$$\bar{\tau}_{xy} = 2G \frac{\partial^2 \Phi}{\partial x \, \partial y} = -\frac{8KG}{abh} \sum_{n,m}^{\infty} \frac{\alpha_n \beta_m \sin \alpha_n \xi \sin \beta_m \eta}{(\alpha_n{}^2 + \beta_m^2)^2} \cos \alpha_n x \cos \beta_m y \tag{4.3}$$

following directly from eq. (4.1).

It can easily be found that the shear stresses do not vanish at the edge of the plate. In order to find the additional stresses $\bar{\bar{\sigma}}_x$, $\bar{\bar{\sigma}}_y$, $\bar{\bar{\tau}}_{xy}$ the single source of intensity W will be replaced by four sources of intensity $W/2$ symmetric or anti-symmetric in relation to the x' and y' axes (fig. 7a–d).

[4]) K. Girkmann: Flächentragwerke, Wien 1954, p. 195.

Let us first consider the case of symmetry in relation to the x' and y' axes.

For a single source of intensity W, at the point (ξ, η), we obtain from eq. (4.3)

$$\bar{\tau}_{xy}\Big|_{x=a} = -\frac{8\,K\,G}{a\,b\,h}\sum_{m=1}^{\infty}\beta_m\cos\beta_m y\sin\beta_m\eta\sum_{n=1}^{\infty}\frac{\alpha_n(-1)^n\sin\alpha_n\xi}{(\alpha_n^2+\beta_m^2)^2}$$
$$= -\frac{2\,K\,G\,a^2}{b\,h}\sum_{m=1}^{\infty}\beta_m\,\eta_2(\beta_m,\xi)\cos\beta_m y\sin\beta_m\eta, \tag{4.4}$$
$$\bar{\tau}_{xy}\Big|_{y=b} = -\frac{2\,K\,G\,b^2}{a\,h}\sum_{n=1}^{\infty}\alpha_n\,\eta_2(\alpha_n,\eta)\sin\alpha_n\xi\cos\alpha_n x,$$

where η_2 should be taken from the eq. (2.7′).

For four sources $W/4$ located as in fig. 7a we have in the coordinates x', y'.

$$\bar{\tau}_{x'y'}\Big|_{x'=\frac{a}{2}} = -\frac{K\,G\,a^2}{2\,b\,h}\sum_{m=1}^{\infty}\beta_m\left[\eta_2\left(\frac{a}{2}+\xi',\beta_m\right)+\eta_2\left(\frac{a}{2}-\xi',\beta_m\right)\right]\cdot$$
$$\cdot\left[\sin\beta_m\left(\frac{b}{2}+\eta'\right)+\sin\beta_m\left(\frac{b}{2}-\eta'\right)\right]\cos\beta_m\left(\frac{b}{2}+y'\right),$$
$$\bar{\tau}_{x'y'}\Big|_{y'=\frac{b}{2}} = -\frac{K\,G\,b^2}{2\,b\,h}\sum_{n=1}^{\infty}\alpha_n\left[\eta_2\left(\frac{b}{2}+\eta',\alpha_n\right)+\eta_2\left(\frac{b}{2}-\eta',\alpha_n\right)\right]\cdot \tag{4.5}$$
$$\cdot\left[\sin\alpha_n\left(\frac{a}{2}+\xi'\right)+\sin\alpha_n\left(\frac{a}{2}-\xi'\right)\right]\cos\alpha_n\left(\frac{a}{2}+x'\right).$$

After some simple transformations we have

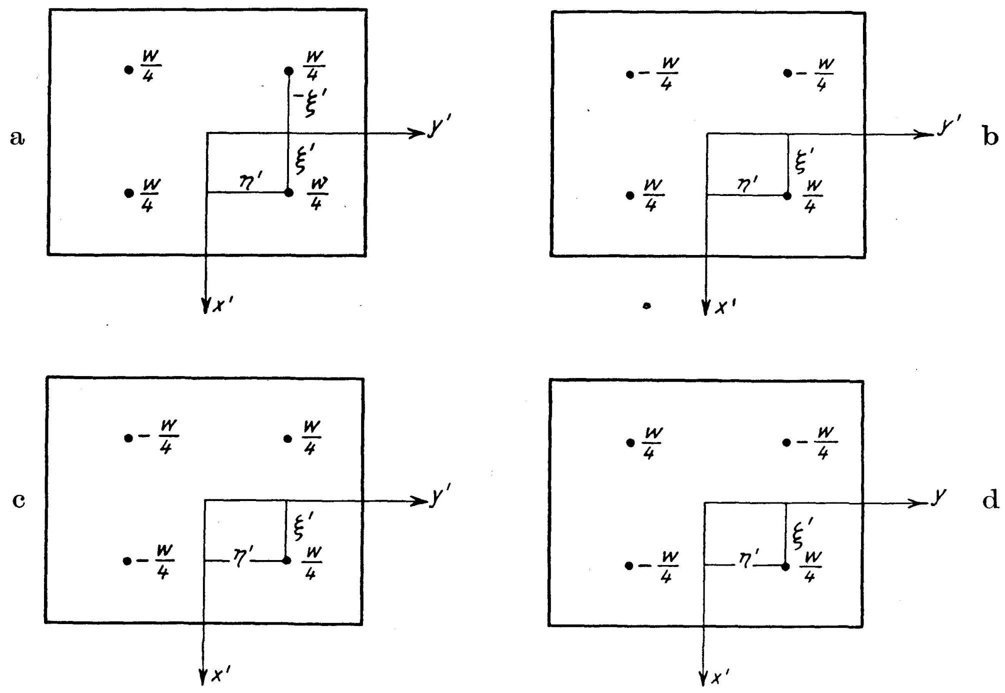

Fig. 7a—d

$$\bar{\tau}_{x'y'}\Big|_{x'=\frac{a}{2}} = -\frac{KGa^2}{4bh}\sum_{m=1}^{\infty}\beta_m\,\rho^{(s)}(\mu_m,\xi')\cos\beta_m\eta'\sin\beta_m y',$$
$$\bar{\tau}_{x'y}\Big|_{y'=\frac{b}{2}} = -\frac{KGb^2}{4ah}\sum_{n=1}^{\infty}\alpha_n\,\rho^{(s)}(\delta_n,\eta')\cos\alpha_n\xi'\sin\alpha_n x',\tag{4.6}$$
$$n,m = 1,3,5,\dots$$

where

$$\rho^{(s)}(\mu_m,\xi') = \frac{\beta_m\xi'\cosh\mu_m\sinh\beta_m\xi' - \mu_m\sinh\mu_m\cosh\beta_m\xi'}{\mu_m^2\cosh^2\mu_m},$$
$$\rho^{(s)}(\delta_n,\eta') = \frac{\alpha_n\eta'\cosh\delta_n\sinh\alpha_n\eta' - \delta_n\sinh\delta_n\cosh\alpha_n\eta'}{\delta_n{}^2\cosh^2\delta_n},$$
$$\mu_m = \frac{\beta_m a}{2},\qquad \delta_n = \frac{\alpha_n b}{2}.$$

In order to determine the stresses $\bar{\bar{\sigma}}_{x'}$ etc. the following Airy equation should be solved

$$\nabla^2\nabla^2 F = 0,\tag{4.7}$$

the boundary conditions being

$$\bar{\bar{\sigma}}_{x'} = \frac{\partial^2 F}{\partial y'^2} = 0,\qquad \bar{\bar{\tau}}_{x'y'} = -\frac{\partial^2 F}{\partial x'\,\partial y'} = -\bar{\tau}_{x'y'}\quad \text{for } x' = \frac{a}{2},$$
$$\bar{\bar{\sigma}}_{y'} = \frac{\partial^2 F}{\partial x'^2} = 0,\qquad \bar{\bar{\tau}}_{x'y'} = -\frac{\partial^2 F}{\partial x'\,\partial y'} = -\bar{\tau}_{x'y'}\quad \text{for } y' = \frac{b}{2}.\tag{4.8}$$

The function F will be chosen in the form of the series

$$F = \frac{1}{h}\sum_{m=1,3,\dots}^{\infty}\frac{1}{\beta_m^2}\left[A_m\cosh\beta_m x' + B_m\beta_m x'\sinh\beta_m x'\right]\cos\beta_m y +$$
$$+\frac{1}{h}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{\alpha_n{}^2}\left[C_n\cosh\alpha_n y' + D_n\alpha_n y'\sinh\alpha_n y'\right]\cos\alpha_n x.\tag{4.9}$$

The boundary conditions (4.8) lead to the system of equations

$$A_m\cosh\mu_m + B_m\mu_m\sinh\mu_m = 0,\tag{a}$$
$$C_n\cosh\delta_n + D_n\delta_n\sinh\delta_n = 0,\tag{b}$$
$$\sum_{m=1}^{\infty}\left[(A_m+B_m)\sinh\mu_m + B_m\mu_m\cosh\mu_m\right]\sin\beta_m y' +$$
$$+\sum_{n=1}^{\infty}\left[(C_n+D_n)\sinh\alpha_n y' + D_n a_n y'\cosh\alpha_n y'\right]\sin\frac{n\pi}{2} = \qquad \text{(c)}\quad (4.10)$$
$$= \frac{KGa^2}{4b}\sum_{m=1}^{\infty}\beta_m\,\rho^{(s)}(\mu_m,\xi')\cos\beta_m\eta'\sin\beta_m y',$$
$$\sum_{m=1}^{\infty}\left[(A_m+B_m)\sinh\beta_m x' + B_m\beta_m x'\cosh\beta_m x'\right]\sin\frac{m\pi}{2} + \sum_{n=1}^{\infty}\left[(C_n+D_n)\sinh\delta_n +\right.$$
$$\left. + D_n\delta_n\cosh\delta_n\right]\sin\alpha_n x' = \frac{KGb^2}{4a}\sum_{n=1}^{\infty}\alpha_n\,\rho^{(s)}(\delta_n,\eta')\cos\alpha_n\xi'\sin\alpha_n x'.\tag{d}$$

Expressing the following functions by trigonometrical series

$$\sinh\alpha_n y' = \sum_{m=1}^{\infty} E_{n,m}\sin\beta_m y', \qquad \alpha_n y'\cosh\alpha_n y' = \sum_{m=1}^{\infty} F_{n,m}\sin\beta_m y',$$

$$\sin\beta_m x' = \sum_{n=1}^{\infty} G_{n,m}\sin\alpha_n x', \qquad \beta_m x'\cosh\beta_m x' = \sum_{n=1}^{\infty} H_{n,m}\sin\alpha_n x',$$

where

$$E_{n,m} = \frac{4\alpha_n}{b}\,\frac{\sin\frac{m\pi}{2}}{\alpha_n{}^2+\beta_m^2}\cosh\delta_n,$$

$$F_{n,m} = \frac{4\alpha_n}{b}\,\frac{\sin\frac{m\pi}{2}}{\alpha_n{}^2+\beta_m^2}\left[\delta_n\sinh\delta_n - \frac{\alpha_n{}^2-\beta_m^2}{\alpha_n{}^2+\beta_m^2}\cosh\delta_n\right],$$

we reduce the system of equations (4.10) to the system of two equations

$$\begin{aligned}
&A_m t(\mu_m) + \frac{16}{b^2}\beta_m^2\sin\frac{m\pi}{2}\sum_{n=1,3,\ldots}^{\infty} C_n \frac{\sin\frac{n\pi}{2}\cosh^2\delta_n}{(\alpha_n{}^2+\beta_m^2)^2\sinh\delta_n} = \\
&\qquad = -\frac{KGa^2}{4b}\beta_m\rho^{(s)}(\mu_m,\xi')\cos\beta_m\eta', \\
&C_n t(\delta_n) + \frac{16}{a^2}\alpha_n{}^2\sin\frac{n\pi}{2}\sum_{m=1,3,\ldots}^{\infty} A_m \frac{\sin\frac{m\pi}{2}\cosh^2\mu_m}{(\alpha_n{}^2+\beta_m^2)^2\sinh\mu_m} = \\
&\qquad = -\frac{KGb^2}{4a}\alpha_n\rho^{(s)}(\delta_n,\eta')\cos\alpha_n\xi', \\
&\qquad n, m = 1, 3, 5, \ldots,
\end{aligned} \tag{4.11}$$

where

$$t(\mu_m) = \frac{\mu_m+\sinh\mu_m\cosh\mu_m}{\mu_m\sinh\mu_m},$$

$$t(\delta_n) = \frac{\delta_n+\sinh\delta_n\cosh\delta_n}{\delta_n\sinh\delta_n}.$$

We have obtained an infinite system of equations. Confining our attention to r terms of the series (4.6) we obtain $2r$ equations (4.11). By solving these equations we obtain the coefficients $A_m, \ldots, D_n$, which, when substituted in the function (4.9), will enable us to find approximate expressions for the additional surfaces $\bar{\bar{\sigma}}_x$ etc.

In the particular case of a square plate and a single source W at the origin we obtain for $A_m = C_n$ the following system of equations

$$
\begin{aligned}
A_m \frac{m\pi + \sinh m\pi}{m\pi \sinh \frac{m\pi}{2}} + \frac{16}{\pi^2} m^2 \sin\frac{m\pi}{2} \sum_{n=1,3,\ldots}^{\infty} A_n \frac{\sin\frac{n\pi}{2}\cosh^2\frac{n\pi}{2}}{(n^2+m^2)^2 \sinh\frac{m\pi}{2}} = \\
= \frac{KG}{2} \frac{\operatorname{tgh}\frac{m\pi}{2}}{\cosh\frac{m\pi}{2}}, \qquad m = 1, 3, 5, \ldots
\end{aligned} \tag{4.12}
$$

For four sources of intensity $W/4$ located as in fig. 7 b the eq. (4.7) with the boundary conditions (4.8) should be solved. We have

$$
\begin{aligned}
\bar{\tau}_{x'y'}\big|_{x'=\frac{a}{2}} &= \frac{KGa^2}{4bh} \sum_{m=1,3,\ldots}^{\infty} \beta_m \rho^{(a)}(\mu_m, \xi') \cos\beta_m \eta' \sin\beta_m y', \\
\bar{\tau}_{x'y'}\big|_{y'=\frac{b}{2}} &= -\frac{KGb^2}{4ah} \sum_{n=2,4,\ldots}^{\infty} \alpha_n \rho^{(s)}(\delta_n, \eta') \sin\alpha_n \xi' \cos\alpha_n x',
\end{aligned} \tag{4.13}
$$

where

$$
\rho^{(a)}(\mu_m, \xi') = \frac{\beta_m \xi' \sinh\mu_m \cosh\beta_m \xi' - \mu_m \cosh\mu_m \sinh\beta_m \xi'}{\mu_m^2 \sinh^2\mu_m}.
$$

We assume the Airy function in the form

$$
\begin{aligned}
F = & \frac{1}{h} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{\beta_m^2} [A_m \sinh\beta_m x' + B_m \beta_m x' \cosh\beta_m x'] \cos\beta_m y' + \\
& + \frac{1}{h} \sum_{n=2,4,\ldots}^{\infty} \frac{1}{\alpha_n^2} [C_n \cosh\alpha_n y' + D_n \alpha_n y' \sinh\alpha_n y'] \sin\alpha_n x'.
\end{aligned} \tag{4.15}
$$

From the boundary conditions (4.8) we obtain the system of equations

$$
\begin{aligned}
A_m s(\mu_m) + \frac{16}{b^2} \beta_m^2 \sin\frac{m\pi}{2} \sum_{n=2,4,\ldots}^{\infty} \frac{C_n \cosh^2\delta_n \cos\frac{n\pi}{2}}{(\alpha_n^2 + \beta_m^2)^2 \sinh\delta_n} = \\
= \frac{KGa^2}{4b} \beta_m \rho^{(a)}(\mu_m, \xi') \cos\beta_m \eta',
\end{aligned} \tag{4.16}
$$

$$
C_n t(\delta_n) = \frac{KGb^2}{4a} \alpha_n \rho^{(s)}(\delta_n, \eta') \sin\alpha_n \xi,
$$

$$
n = 2, 4, 6, \ldots, \qquad m = 1, 3, 5, \ldots
$$

where

$$
s(\mu_m) = \frac{\sinh\mu_m \cosh\mu_m - \mu_m}{\mu_m \cosh\mu_m}.
$$

From this system we find the values of A_m, C_n, the remaining coefficients, B_m and D_n, being determined from the relations

$$
B_m = -A_n \frac{\sinh\mu_m}{\mu_m \cosh\mu_m}, \qquad D_n = -C_n \frac{\cosh\delta_n}{\delta_n \sinh\delta_n}.
$$

For four sources of intensity $W/4$ located as in fig. 7c the Airy function will be taken in the form

$$F = \frac{1}{h} \sum_{m=2,4,\dots}^{\infty} \frac{1}{\beta_m^2} [A_m \cosh \beta_m x' + B_m \beta_m x' \sinh \beta_m x'] \sinh \beta_m y' + \\ + \frac{1}{h} \sum_{n=1,3,\dots}^{\infty} \frac{1}{\alpha_n^2} [C_n \sinh \alpha_n y' + D_n \alpha_n y' \cosh \alpha_n y'] \cos \alpha_n x'. \tag{4.17}$$

The boundary conditions (4.8) lead to the system of equations

$$A_m t(\mu_m) = \frac{K G a^2}{4b} \beta_m \rho^{(s)}(\mu_m, \xi') \sin \beta_m \eta',$$

$$C_n s(\delta_n) + \frac{16}{a^2} \alpha_n^2 \sin \frac{n\pi}{2} \sum_{m=2,4,\dots}^{\infty} \frac{A_m \cosh^2 \mu_m \cos \frac{m\pi}{2}}{(\alpha_n^2 + \beta_m^2)^2 \sinh \mu_m} = \\ = \frac{K G b^2}{4a} \alpha_n \rho^{(a)}(\delta_n, \eta') \cos \alpha_n \xi'. \tag{4.18}$$

The constants B_m and D_n are found from the equations

$$B_m = -A_m \frac{\cosh \mu_m}{\mu_m \sinh \mu_m}, \qquad D_n = -C_n \frac{\sinh \delta_n}{\delta_n \cosh \delta_n}. \tag{4.19}$$

Finally, for four sources of heat, of intensity $W/4$, located in an anti-symmetrical manner in relation to the x'- and y'-axes (fig. 7d) the Airy function should be assumed in the form

$$F = \frac{1}{h} \sum_{m=2,4,\dots}^{\infty} \frac{1}{\beta_m^2} [A_m \sinh \beta_m x' + B_m \beta_m x' \cosh \beta_m x'] \sin \beta_m y' + \\ + \frac{1}{h} \sum_{n=2,4,\dots}^{\infty} \frac{1}{\alpha_n^2} [C_n \sinh \alpha_n y' + D_n \alpha_n y' \cosh \alpha_n y'] \sin \alpha_n x'. \tag{4.20}$$

The boundary conditions are as follows

$$\bar{\bar{\tau}}_{x'y'} = -\frac{\partial^2 F}{\partial x \, \partial y} = -\bar{\tau}_{x'y'} \quad \text{for} \quad x' = \frac{a}{2}, \quad y' = \frac{b}{2}, \\ \bar{\bar{\sigma}}_{x'} = 0 \quad \text{for} \quad x' = \frac{a}{2}, \qquad \bar{\bar{\sigma}}_{y'} = 0 \quad \text{for} \quad y' = \frac{b}{2}, \tag{4.21}$$

where

$$\bar{\tau}_{x'y'}\Big|_{x'=\frac{a}{2}} = -\frac{K G a^2}{4bh} \sum_{m=2,4,\dots}^{\infty} \beta_m \rho^{(a)}(\mu_m, \xi') \sin \beta_m \eta' \cos \beta_m y', \\ \bar{\tau}_{x'y'}\Big|_{y'=\frac{b}{2}} = -\frac{K G b^2}{4ah} \sum_{n=2,4,\dots}^{\infty} \alpha_n \rho^{(a)}(\delta_n, \eta') \sin \alpha_n \xi' \cos \alpha_n x. \tag{4.22}$$

The coefficients $A_m, \ldots, D_n$, can be determined directly from the conditions (4.21).

We have

$$\begin{aligned} A_m s(\mu_m) &= \frac{K G a^2}{4b} \beta_m \rho^{(a)}(\mu_m, \xi') \sin \beta_m \eta', \\ C_n s(\delta_n) &= \frac{K G b^2}{4a} \alpha_n \rho^{(a)}(\delta_n, \eta') \sin \alpha_n \xi', \\ B_m &= -A_m \frac{\sinh \mu_m}{\mu_m \cosh \mu_m}, \qquad D_n = -C_n \frac{\sinh \delta_n}{\delta_n \cosh \delta_n}. \end{aligned} \tag{4.23}$$

Summing up the stresses due to the states represented in the figs. 7a–d we obtain the additional stresses $\bar{\bar{\sigma}}_x \ldots$, which, together with the stresses $\bar{\sigma}_x, \ldots$ determine the state of stress in the plate due to the action of a source of heat of intensity W, located at the point (ξ, η).

The solutions described in this paper can be used in the case of plane strain. The heat sources at the level h will be replaced by linear sources parallel to the z axis.

The differential equation of the thermal potential of displacement take the form

$$\nabla^2 \nabla^2 \Phi = -\frac{1+\nu}{1-\nu} \frac{\alpha W}{k}.$$

The stresses $\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}$ are found from the relations (1.4). For a plane state of deformation where the stresses are independent of the variable z we have $\bar{\tau}_{xz} = 0$, $\bar{\tau}_{yz} = 0$, $\bar{\sigma}_z = -2G\nabla^2\Phi$.

The additional state of stress $(\bar{\bar{\sigma}}, \bar{\bar{\tau}})$ is obtained by solving the Airy function, using the relations

$$\begin{aligned} \bar{\bar{\sigma}}_x &= \frac{\partial^2 F}{\partial y^2}, \qquad & \bar{\bar{\sigma}}_y &= \frac{\partial^2 F}{\partial x^2}, \qquad & \bar{\bar{\tau}}_{xy} &= -\frac{\partial^2 F}{\partial x \, \partial y}, \\ \bar{\bar{\tau}}_{xz} &= 0, & \bar{\bar{\tau}}_{yz} &= 0, & \bar{\bar{\sigma}}_z &= \nu \nabla^2 F. \end{aligned}$$

Summary

A formal analogy can be observed between the differential equation of the thermal potential of displacement for a steady temperature field due to a source of heat and that of the deflection of a thin plate due to a concentrated force. This analogy is used in the present paper to determine the thermal potential of displacement and the stresses $(\bar{\sigma}, \bar{\tau})$.

The normal stresses at the edges are found to be zero, whereas the shear stresses are other than zero. In order to eliminate the shear stresses at the edges, on additional state of stress $(\bar{\bar{\sigma}}, \bar{\bar{\tau}})$ (obtained by solving the boundary problem for a plane state of stress by means of the Airy function) must be superposed on the state $(\bar{\sigma}, \bar{\tau})$.

The state of stress due to the action of heat sources is treated in detail for the following cases:

a) a strip of plate of infinite length

b) a semi-infinite strip of plate

c) a rectangular plate.

Résumé

L'équation différentielle du potentiel thermique de déplacement est formellement analogue à celle exprimant la déflection d'une plaque mince causée par l'action d'une force concentrée. Dans le présent mémoire on emploie cette analogie pour déterminer le potentiel thermique de déplacement et les contraintes $(\bar{\sigma}, \bar{\tau})$ dans une plaque. On trouve que les tensions normales sur le contour de la plaque disparaissent tandis que les tensions tangentielles sont différentes de zéro.

Pour annuler les tensions tangentielles au contour de la plaque il faut superposer à l'état de tension $(\bar{\sigma}, \bar{\tau})$ l'état $(\bar{\bar{\sigma}}, \bar{\bar{\tau}})$ qu'on obtient par la solution du problème aux limites pour l'état plan de tension en employant la fonction d'Airy.

L'état de tension provoqué par l'action des sources de chaleur a été étudié pour les plaques suivantes:

a) une bande indéfinie,

b) une bande semi-indéfinie,

c) une plaque rectangulaire.

Zusammenfassung

Zwischen der Differentialgleichung des thermischen Verschiebungspotentials für stationäre Temperaturfelder, hervorgerufen durch Wärmequellen, und der Differentialgleichung für die Durchbiegung einer dünnen Platte, hervorgerufen durch die Wirkung einer Einzellast, besteht eine formale Analogie. In der vorliegenden Arbeit wurde diese Analogie zur Bestimmung des thermischen Verschiebungspotentials und der Spannungen $(\bar{\sigma}, \bar{\tau})$ benützt; es wurden hierbei an den Rändern der Scheibe Nullwerte für Normalspannungen, jedoch von Null verschiedene Werte für die Schubspannungen gefunden. Zwecks Beseitigung dieser Schubspannungen auf den Scheibenrändern muß

auf den Spannungszustand $(\bar{\sigma}, \bar{\tau})$ ein zusätzlicher Spannungszustand $(\bar{\bar{\sigma}}, \bar{\bar{\tau}})$ überlagert werden, welcher aus der Lösung des ebenen Randwertproblems durch Benutzung der Airyschen Spannungsfunktion hervorgeht.

In der Arbeit wurden eingehend die Spannungszustände behandelt, die durch das Auftreten von Wärmequellen verursacht werden:

a) im unendlich langen Scheibenstreifen,

b) im scheibenartigen Halbstreifen,

c) in einer rechteckigen Scheibe.