Analysis of thin doubly curved nearly cylindrical shells of rotation by the method of successive corrections

Analysis of Thin Doubly Curved Nearly Cylindrical Shells of Rotation by the Method of Successive Corrections

Calcul analytique des voiles de révolution à double courbure par une méthode d'approximations successives

Analytische Berechnung doppelt gekrümmter Zylinderrotationsschalen durch eine Methode sukzessiver Korrekturen

Dr. Phil. GUNHARD ORAVAS, Venezuela

Introduction

In engineering practice rotationally symmetrical thin cylindrical shells with a small positive or negative curvature of the generatrix find special application in various industrial structures subjected to surface pressures.

In this paper a method of successive form correction procedure is used for the analytical solution of such a shell. The basic ideas of this method are not novel and a recent use of it was made in the analysis of doubly curved plates [1][1]).

Basic Assumptions

In the following the usual simplifying assumptions for thin isotropic shells are made:

a) The thickness of the shell is very small in comparision with its other dimensions.
b) A normal to the middle surface of the undeformed shell remains a normal to the middle surface of the deformed shell.
c) Shear deformations are negligible.
d) Displacements are small in comparision with the thickness of the shell.
e) Shell obeys Hooke's law.

[1]) Numbers refer to Bibliography at the end of the paper.

Bending of Rotational Shells

General Differential Equations of Equilibrium

The following equations in general curvilinear coordinates have been given by other writers [2, 3, 4] (fig. 1).

The stress couple equation in radial plane is

$$\frac{d\,(r\,M_\xi)}{d\,\xi} - r\,\alpha\,Q_\xi - \frac{d\,r}{d\,\xi}\,M_\theta = 0. \tag{1}$$

Stress resultants in two orthogonal directions yield

$$\frac{d\,(r\,N_\xi)}{d\,\xi} - \frac{d\,r}{d\,\xi}\,N_\theta + \frac{\alpha\,r}{R_\xi}\,Q_\xi = 0 \tag{2}$$

and

$$\frac{d\,(r\,Q_\xi)}{d\,\xi} - \frac{\alpha\,r}{R_\theta}\,N_\theta - \frac{\alpha\,r}{R_\xi}\,N_\xi + \alpha\,r\,p_\zeta = 0, \tag{3}$$

where

p_ζ = normal pressure on a unit area of middle surface of the shell.

General Stress-Strain Relations

The following well known relations are [2, 3, 4]

$$N_\xi = C\,[\epsilon_\xi + \nu\,\epsilon_\theta], \tag{4}$$

$$N_\theta = C\,[\epsilon_\theta + \nu\,\epsilon_\xi], \tag{5}$$

$$M_\xi = D\,[\kappa_\xi + \nu\,\kappa_\theta], \tag{6}$$

$$M_\theta = D\,[\kappa_\theta + \nu\,\kappa_\xi], \tag{7}$$

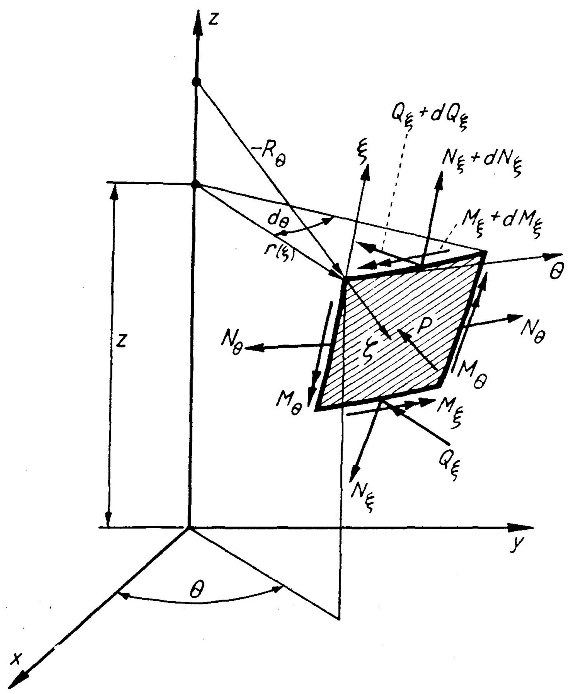

Fig. 1. An element of the shell with stress resultants, stress couples and surface loading.

where
$$\nu = \text{Poisson's ratio},$$
$$\epsilon_\theta = \frac{1}{r\alpha}\frac{dr}{d\xi}v + \frac{w}{R_\theta},$$
$$\epsilon_\xi = \frac{1}{\alpha}\frac{dv}{d\xi} + \frac{w}{R_\xi},$$
$$\kappa_\theta = \frac{1}{\alpha r}\frac{dr}{d\xi}\left[\frac{v}{R_\xi} - \frac{1}{\alpha}\frac{dw}{d\xi}\right],$$
$$\kappa_\xi = \frac{1}{\alpha}\frac{d}{d\xi}\left[\frac{v}{R_\xi} - \frac{1}{\alpha}\frac{dw}{d\xi}\right],$$
$$C = \frac{Eh}{1-\nu^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}.$$

An additional assumption is made at this point
$$\frac{v}{R_\xi} \ll \frac{1}{\alpha}\frac{dw}{d\xi}.$$

Retaining terms to the third power of "h" allows the following simplifications
$$\kappa_\theta = -\frac{1}{\alpha^2 r}\frac{dr}{d\xi}\cdot\frac{dw}{d\xi}$$
and
$$\kappa_\xi = -\frac{1}{\alpha}\frac{d}{d\xi}\left[\frac{1}{\alpha}\frac{dw}{d\xi}\right].$$

Geometry of Doubly Curved Cylindrical Shells

The middle surface of the shell can be described by cylindrical coordinates (Fig. 2) with the radius
$$r(\xi) = r(z) = a + \mu f,$$

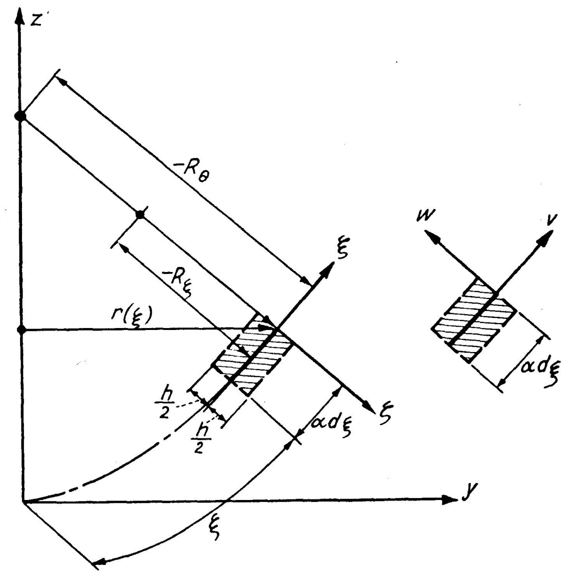

Fig. 2. Geometry of the middle surface.

where

$$f = f(z) \quad \text{and} \quad \mu = \text{dimensionless parameter.}$$

The elemental arc length in general coordinates is

$$d s^2 = \alpha^2 \left(1 + \frac{\zeta}{R_\xi}\right)^2 d\xi^2 + r^2 \left(1 + \frac{\zeta}{R_\theta}\right)^2 d\theta^2 + d\zeta^2.$$

When $\frac{\zeta}{R_\xi} \ll 1$ the arc length becomes

$$d s^2 = \alpha^2 d\xi^2 + r^2 d\theta^2 + d\zeta^2,$$

where the quantities α, r and 1 are the first fundamental quantities of the middle surface of the shell. For cylindrical coordinates the arc length becomes

$$d s^2 = d z^2 + r^2 d\theta^2 + d\zeta^2,$$

therefore $\alpha = 1$ and $\xi = z$ for this system.

The principal curvatures of the middle surface of the shell become in the cylindrical coordinates

$$\frac{1}{R_\xi} = \frac{1}{R_z} = \frac{\frac{d r^2}{d z^2}}{r\left[1 + \left(\frac{dr}{dz}\right)^2\right]^{3/2}},$$

$$\frac{1}{R_\theta} = -\frac{1}{r\left[1 + \left(\frac{dr}{dz}\right)^2\right]^{1/2}}.$$

Method of Analysis

In order to carry through the iterative solution of this shell a well known method in the solution of non-linear differential equations seems appropriate. This approach consists in expanding all dependent variables into power series in terms of an appropriately small dimensionless parameter and after substitution of these series into stress-strain relations and equilibrium equations a set of simultaneous equations results for each power of the parameter [1, 5, 6]. Therefore after solving these equations in succession, starting with the lowest powers of the parameter, a set of improving solutions is obtained.

In this case the parameter describes the fractional deviation of the middle surface of the shell from that of the perfect circular cylinder and therefore every higher parameter power solution corrects the fundamental circular cylinder solution to the end of describing the elastic behaviour of the cylinder with the deviatoric generator. Usually each successive solution indicates if the series of solutions is of a converging nature. For parameters $\mu = O\left(\frac{1}{10}\right)$ the second power approximation is considered satisfactory for most cases.

Power Series Expansions of Dependent Variables

The following expansions are obtained for the geometry of the middle surface of the shell by division and limiting the power series expansions to the exclusion of terms containing parameter μ to the third and higher powers.

By the use of binomial theorem

$$\alpha = \left[1+\left(\frac{dr}{dz}\right)^2\right]^{\frac{1}{2}} = [1+\mu^2 f_z^2]^{1/2} = 1+\mu^2\frac{f_z^2}{2}\cdots,$$

where

$$f_z = \frac{df}{dz}.$$

Further the following expressions can be derived by divisions:

$$\frac{1}{\alpha} = \frac{1}{1+\mu^2\frac{f_z^2}{2}} = 1-\mu^2\frac{f_z^2}{2}\cdots,$$

$$\frac{1}{r} = \frac{1}{a+\mu f} = \frac{1}{a}-\mu\frac{f}{a^2}+\mu^2\frac{f^2}{a^3}\cdots,$$

$$\frac{r_z}{r\alpha} = \frac{\frac{dr}{dz}}{r\alpha} = \frac{\mu f_z}{(a+\mu f)\left(1+\mu^2\frac{f_z^2}{2}\right)} = \mu\frac{f_z}{a}-\mu^2\frac{ff_z}{a^2}\cdots,$$

$$\frac{1}{R_z} = \frac{\frac{dr^2}{dz^2}}{\left[1+\left(\frac{dr}{dz}\right)^2\right]^{3/2}} = \frac{\mu f_{zz}}{1+\mu^2\frac{3}{2}f_z^2\cdots} = \mu f_{zz}\cdots,$$

$$\frac{1}{R_\theta} = -\frac{1}{r[1+r_z^2]^{1/2}} = -\frac{1}{(a+\mu f)(1+\mu^2 f_z^2)^{1/2}} = -\frac{1}{a}+\mu\frac{f}{a^2}-\mu^2\left(\frac{f^2}{a^3}-\frac{f_z^2}{2a}\right)\cdots.$$

The strains can be expressed as follows:

$$\begin{aligned}\epsilon_z &= \epsilon_z^{(0)}+\mu\epsilon_z^{(1)}+\mu^2\epsilon_z^{(2)}\cdots = \left[1-\mu^2\frac{f_z^2}{2}\right]\left[\frac{dv^{(0)}}{dz}+\mu\frac{dv^{(1)}}{dz}+\mu^2\frac{dv^{(2)}}{dz}\cdots\right]+\\ &\quad+\mu f_{zz}[w^{(0)}+\mu w^{(1)}+\mu^2 w^{(2)}\cdots] = \\ &= \frac{dv^{(0)}}{dz}+\mu\left[\frac{dv^{(1)}}{dz}+f_{zz}w^{(0)}\right]+\mu^2\left[\frac{dv^{(2)}}{dz}+f_{zz}w^{(1)}-\frac{f_z^2}{2}\frac{dv_z^{(0)}}{dz}\right].\end{aligned} \tag{8}$$

Therefore

$$\epsilon_z^{(0)} = \frac{dv^{(0)}}{dz}; \qquad \epsilon_z^{(1)} = \frac{dv^{(1)}}{dz}+f_{zz}w^{(0)};$$

$$\epsilon_z^{(2)} = \frac{dv^{(2)}}{dz}+f_{zz}w^{(1)}-\frac{f_z^{(2)}}{2}\frac{dv^{(0)}}{dz}.$$

$$\begin{aligned}\epsilon_\theta &= \epsilon_\theta^{(0)}+\mu\epsilon_\theta^{(1)}+\mu^2\epsilon_\theta^{(2)}\cdots = \left[\frac{1}{a}-\mu\frac{f}{a^2}+\mu^2\left(\frac{f^2}{a^3}-\frac{f_z^2}{2a}\right)\right]\mu f_z[v^{(0)}+\mu v^{(1)}+\mu^2 v^{(2)}\cdots]-\\ &\quad-\left[\frac{1}{a}-\mu\frac{f}{a^2}+\mu^2\left(\frac{f^2}{a^3}-\frac{f_z^2}{2a}\right)\right][w^{(0)}+\mu w^{(1)}+\mu^2 w^{(2)}\cdots] = \\ &= -\frac{w^{(0)}}{a}+\mu\left[-\frac{w^{(1)}}{a}+\frac{f}{a^2}w^{(0)}+\frac{f_z}{a}v^{(0)}\right]+\mu^2\left[-\frac{w^{(2)}}{a}+\frac{f}{a^2}w^{(1)}-\left(\frac{f}{a^3}-\frac{f_z^2}{2a}\right)w^{(0)}+\right.\\ &\quad\left.+\frac{f_z}{a}v^{(1)}-\frac{ff_z}{a^2}v^{(0)}\right].\end{aligned} \tag{9}$$

Therefore

$$\epsilon_\theta^{(0)} = -\frac{w^{(0)}}{a},$$

$$\epsilon_\theta^{(1)} = -\frac{w^{(1)}}{a} + \frac{f}{a^2} w^{(0)} + \frac{f_z}{a} v^{(0)},$$

$$\epsilon_\theta^{(2)} = -\frac{w^{(2)}}{a} + \frac{f}{a^2} w^{(1)} - \left(\frac{f}{a^3} - \frac{f_z^2}{2a}\right) w^{(0)} + \frac{f_z}{a} v^{(1)} - \frac{f f_z}{a^2} v^{(0)}.$$

The changes of curvatures are as follows:

$$\begin{aligned} \kappa_z &= \kappa_z^{(0)} + \mu\,\kappa_z^{(1)} + \mu^2 \kappa_z^{(2)} \cdots = \\ &= -\left(1-\mu^2\frac{f_z^2}{2}\right)\frac{d}{dz}\left[\left(1-\mu^2\frac{f_z^2}{2}\right)\frac{d}{dz}(w^{(0)}+\mu\, w^{(1)}+\mu^2 w^{(2)})\right] = \\ &= -\frac{d^2 w^{(0)}}{dz^2} - \mu\frac{d^2 w^{(1)}}{dz^2} - \mu^2\left[\frac{d^2 w^{(2)}}{dz^2} - \frac{f_z^2}{2}\frac{d^2 w^{(0)}}{dz^2} - f_z f_{zz}\frac{d w^{(0)}}{dz} - \frac{f_z^2}{2}\frac{d^2 w^{(0)}}{dz^2}\right]. \end{aligned} \tag{10}$$

Therefore

$$\kappa_z^{(0)} = -\frac{d^2 w^{(0)}}{dz^2},$$

$$\kappa_z^{(1)} = -\frac{d^2 w^{(1)}}{dz^2},$$

$$\kappa_z^{(2)} = -\frac{d^2 w^{(2)}}{dz^2} + f_z^2\frac{d^2 w^{(0)}}{dz^2} + f_z f_{zz}\frac{d w^{(0)}}{dz}.$$

$$\begin{aligned} \kappa_\theta &= \kappa_\theta^{(0)} + \mu\,\kappa_\theta^{(1)} + \mu^2 \kappa_\theta^{(2)} \cdots = \left[-\mu\frac{f_z}{a} + \mu^2\frac{f f_z}{a^2}\right]\frac{d}{dz}[w^{(0)}+\mu\, w^{(1)}+\mu^2 w^{(2)}] = \\ &= -\mu\left[\frac{f_z}{a}\frac{d w^{(0)}}{dz}\right] - \mu^2\left[\frac{f f_z}{a^2}\frac{d w^{(0)}}{dz} + \frac{f_z}{a}\frac{d w^{(1)}}{dz}\right]. \end{aligned} \tag{11}$$

Then

$$\kappa_\theta^{(0)} = 0,$$

$$\kappa_\theta^{(1)} = -\frac{f_z}{a}\frac{d w^{(0)}}{dz},$$

$$\kappa_\theta^{(2)} = -\frac{f_z}{a}\frac{d w^{(1)}}{dz} + \frac{f f_z}{a^2}\frac{d w^{(0)}}{dz}.$$

The equilibrium equations (1), (2) and (3) are:

$$\frac{d}{dz}[(a+\mu f_z) M_z] - (a+\mu f)\left(1+\mu^2\frac{f_z^2}{2}\right) Q_z - \mu f_z M_\theta = 0,$$

$$\frac{d}{dz}[(a+\mu f) N_z] - \frac{d}{dz}[a+\mu f] N_\theta + (a+\mu f)\left(1+\mu^2\frac{f_z^2}{2}\right)(\mu f_{zz}) Q_z = 0,$$

$$\frac{d}{dz}[(a+\mu f)\,Q_z]+(a+\mu f)\left(1+\mu^2\frac{f_z^2}{2}\right)\left[\frac{1}{a}-\mu\frac{f}{a^2}+\mu^2\left(\frac{f^2}{a^3}-\frac{f_z^2}{2a}\right)\right]N_\theta-$$

$$-(a+\mu f)\left(1+\mu^2\frac{f_z^2}{2}\right)\mu f_{zz}N_z+(a+\mu f)\left(1+\mu^2\frac{f_z^2}{2}\right)p_\zeta=0.$$

Substituting the following expansions

$$\begin{aligned}
M_z &= M_z^{(0)}+\mu\,M_z^{(1)}+\mu^2\,M_z^{(2)}\cdots\\
M_\theta &= M_\theta^{(0)}+\mu\,M_\theta^{(1)}+\mu^2\,M_\theta^{(2)}\cdots\\
Q_z &= Q_z^{(0)}+\mu\,Q_z^{(1)}+\mu^2\,Q_z^{(2)}\cdots\\
N_z &= N_z^{(0)}+\mu\,N_z^{(1)}+\mu^2\,N_z^{(2)}\cdots\\
N_\theta &= N_\theta^{(0)}+\mu\,N_\theta^{(1)}+\mu^2\,N_\theta^{(2)}\cdots
\end{aligned}\tag{a}$$

into the equations above and equating the coefficients of successive powers of parameter μ equal to zero yields

$$\mu^0:\quad \frac{d}{dz}[a\,M_z^{(0)}]-a\,Q_z^{(0)}=0,\tag{1a}$$

$$\frac{d}{dz}[a\,N_z^{(0)}]=0,\tag{2a}$$

$$\frac{d}{dz}[a\,Q_z^{(0)}]+N_\theta^{(0)}+a\,p_\zeta=0.\tag{3a}$$

$$\mu:\quad \frac{d}{dz}[a\,M_z^{(1)}+f_z\,M_z^{(0)}]-a\,Q_z^{(1)}-f\,Q_z^{(0)}-f_z\,M_\theta^{(0)}=0,\tag{1b}$$

$$\frac{d}{dz}[a\,N_z^{(1)}]+\frac{d}{dz}[f_z\,N_z^{(0)}]-f_z N_\theta^{(0)}+a\,f_{zz}\,Q_z^{(0)}=0,\tag{2b}$$

$$\frac{d}{dz}[a\,Q_z^{(1)}]+N_\theta^{(1)}+\frac{d}{dz}[f\,Q_z^{(0)}]-a\,f_{zz}N_z^{(0)}+f\,p_\zeta=0.\tag{3b}$$

$$\mu^2:\quad \frac{d}{dz}[a\,M_z^{(2)}]-a\,Q^{(2)}+\frac{d}{dz}[f_z\,M_z^{(1)}]-f\,Q_z^{(1)}-f_z\,M_\theta^{(1)}-\frac{a f_z^2}{2}\,Q_z^{(0)}=0,\tag{1c}$$

$$\frac{d}{dz}[a\,N_z^{(2)}]-f_z\,N_\theta^{(1)}+a\,f_{zz}\,Q_z^{(1)}+f\,f_{zz}\,Q_z^{(0)}+\frac{d}{dz}[f_z\,N_z^{(1)}]=0,\tag{2c}$$

$$\frac{d}{dz}[a\,Q_z^{(2)}]+N_\theta^{(2)}+\frac{a f_z^2}{2}\,p_\zeta+\frac{d}{dz}[f\,Q_z^{(1)}]-a\,f_{zz}N_z^{(1)}-f\,f_{zz}\,N_z^{(0)}=0.\tag{3c}$$

Formulation and Solution of Differential Equations

First Approximation ($\mu^0 = 1$):
Equation (1 a) gives

$$\frac{d^2}{dz^2}[a\,M_z^{(0)}] - \frac{d}{dz}[a\,Q_z^{(0)}] = 0.$$

Substituting $\frac{d}{dz}[a\,Q_z^{(0)}]$ from (3b)

$$\frac{d^2}{dz^2}[a\,M_z^{(0)}] + N_\theta^{(1)} + a\,p_\zeta = 0$$

or

$$\frac{d^2 M_z^{(0)}}{dz^2} + \frac{N_\theta^{(0)}}{a} = -p_\zeta. \tag{12}$$

Equation (6) gives the meridional moment

$$M_z^{(0)} = D[\kappa_z^{(0)} + \nu\,\kappa_\theta^{(0)}] = D\left[-\frac{d^2 w^{(0)}}{dz^2}\right] \tag{6'}$$

and the parallel force from (5)

$$N_\theta^{(0)} = C[\epsilon_\theta^{(0)} + \nu\,\epsilon_z^{(0)}] = C\left[-\frac{w^{(0)}}{a} + \nu\frac{dv^{(0)}}{dz}\right]. \tag{5'}$$

From

$$\frac{d}{dz}[a\,N_z^{(0)}] = \frac{d}{dz}\left\{C\left[\frac{dv^{(0)}}{dz} - \nu\frac{w^{(0)}}{a}\right]\right\} = 0$$

or

$$\frac{dv^{(0)}}{dz} - \nu\frac{w^{(0)}}{a} = K.$$

Assuming the axial strain to be very small

$$\frac{dv^{(0)}}{dz} = \nu\frac{w^{(0)}}{a}.$$

Substituting this relationship in (5′) gives

$$N_\theta^{(0)} = C\left[-\frac{w^{(0)}}{a} + \nu^2\frac{w^{(0)}}{a}\right] = -E\,h\frac{w^{(0)}}{a}.$$

Elimination of $M_z^{(0)}$ and $N_\theta^{(0)}$ from (12) yields

$$\frac{d^4 w^{(0)}}{dz^4} + 4\beta^4 w^{(0)} = \frac{p_\zeta}{D} = \frac{12(1-\nu^2)}{E\,h^3}p_\zeta, \tag{13}$$

where

$$\beta^4 = \frac{3(1-\nu^2)}{h^2 a^2}.$$

The solution of (13) is

$$w^{(0)} = A_0 e^{(1+i)\beta z} + \bar{A}_0 e^{(1-i)\beta z} + B_0 e^{-(1+i)\beta z} + \bar{B}_0 e^{-(1-i)\beta z} + \frac{a^2}{E h} p_\zeta,$$

where $\bar{A}_0$ is the conjugate complex of A_0 and $i = \sqrt{-1}$.

For the first approximation it was assumed that

$$N_z^{(0)} \approx 0,$$
$$M_\theta^{(0)} \approx \nu M_z^{(0)}.$$

Second Approximation (μ):

Equation (1b) gives after substitution of $\frac{d}{dz}(a\, Q_z^{(1)}]$ from (3b)

$$\frac{d^2 M_z^{(1)}}{d z^2} + \frac{N_\theta^{(1)}}{a} = -\frac{f}{a} p_\zeta + \frac{\nu}{a}\frac{d}{dz}[f_z M_z^{(0)}] - \frac{1}{a}\frac{d^2}{dz^2}(f_z M_z^{(0)}).$$

Substituting

$$M_z^{(1)} = -D\left[\frac{d^2 w^{(1)}}{d z^2} + \nu \frac{f_z}{a}\frac{d w^{(0)}}{d z}\right]$$

and

$$N_\theta^{(1)} = C\left[-\frac{w^{(1)}}{a} + \frac{f}{a^2} w^{(0)} + \nu \frac{f_z}{a^2}\int^z w^{(0)}\, d\bar{z} + \nu \frac{d v^{(1)}}{d z} + \nu f_{zz} w^{(0)}\right]$$

into the above equation yields

$$\frac{d^4 w^{(1)}}{d z^4} + 4\beta^4 w^{(1)} = \Theta^{(1)},$$

where

$$\Theta^{(1)} = \frac{C}{D a}\left\{\left[1 - \nu + (1-\nu^2)\frac{f}{a^2}\right] w^{(0)} + \frac{\nu}{a}(1-\nu^2)\int^z w^{(0)}\, d\bar{z} + \right.$$
$$\left. + \frac{\nu}{C a}\int^z f_{\bar{z}} N_\theta^{(0)}\, d\bar{z} - \frac{\nu}{C}\int^z f_{\bar{z}\bar{z}} Q_z^{(0)}\, d\bar{z}\right\} - \frac{\nu}{a}\frac{d}{dz}\left[\frac{d}{dz}\left(f_z \frac{d w^{(0)}}{d z}\right) + \frac{f_z}{D} M_z^{(0)}\right] +$$
$$+ \frac{1}{a D}\frac{d^2}{d z^2}(f_z M_z^{(0)}) + \frac{f}{D a} p_\zeta .$$

The substitution

$$\frac{d v^{(1)}}{d z} = \frac{\nu}{a} w^{(1)} - \left(f_{zz} + \nu \frac{f}{a^2}\right) w^{(0)} - \nu^2 \frac{f_z}{a^2}\int^z w^{(0)}\, d\bar{z} + \frac{1}{C a}\int^z f_{\bar{z}} N_\theta^{(0)}\, d\bar{z} -$$
$$- \frac{1}{C}\int^z f_{\bar{z}\bar{z}} Q_z^{(0)}\, d\bar{z}$$

was made from (2b) in order to eliminate $\frac{d v^{(1)}}{dz}$ from $N_\theta^{(1)}$.

The solution of this equation is

$$w^{(1)} = A_1 e^{(1+i)\beta z} + \bar{A}_1 e^{-(1+i)\beta z} + B_1 e^{(1-i)\beta z} + \bar{B}_1 e^{-(1-i)\beta z} -$$

$$- \frac{1+i}{16\beta^3}\left[\int^z \Theta^{(1)} e^{-(1+i)\beta \bar{z}} d\bar{z}\right] e^{(1+i)\beta z} + \frac{1+i}{16\beta^3}\left[\int^z \Theta^{(1)} e^{(1+i)\beta \bar{z}} d\bar{z}\right] e^{-(1+i)\beta z} -$$

$$- \frac{1-i}{16\beta^3}\left[\int^z \Theta^{(1)} e^{-(1-i)\beta \bar{z}} d\bar{z}\right] e^{(1-i)\beta z} + \frac{1-i}{16\beta^3}\left[\int^z \Theta^{(1)} e^{(1-i)\beta \bar{z}} d\bar{z}\right] e^{-(1-i)\beta z}.$$

Stress couples and stress resultants are given by

$$M_z^{(1)} = -D\left[\frac{d^2 w^{(1)}}{dz^2} + \nu \frac{f_z}{a}\frac{dw^{(0)}}{dz}\right],$$

$$M_\theta^{(1)} = -D\left[\frac{f_z}{a}\frac{dw^{(1)}}{dz} + \nu \frac{d^2 w^{(1)}}{dz^2}\right],$$

$$Q_z^{(1)} = \frac{dM_z^{(1)}}{dz} + \frac{d}{dz}\left(\frac{f_z}{a} M_z^{(0)}\right) - \nu \frac{f_z}{a} M_z^{(0)} - \frac{f}{a} Q_z^{(0)},$$

$$N_\theta^{(1)} = -a\frac{dQ_z^{(1)}}{dz} - \frac{d}{dz}(f\, Q_z^{(0)}) + a f_{zz} N_z^{(0)} - f p_\zeta.$$

Third Approximation (μ^2):

From (1b) and (3b) the following relationship is obtained

$$\frac{d^2 M_z^{(2)}}{dz^2} + \frac{N_\theta^{(2)}}{a} = -\frac{f_z^2}{2} p_\zeta + f_{zz} N_z^{(1)} - \frac{1}{a}\frac{d^2}{dz^2}(f_z M_z^{(1)}) + \frac{1}{a}\frac{d}{dz}(f_z M_\theta^{(1)}) +$$
$$+ \frac{1}{2}\frac{d}{dz}(f_z^2 Q_z^{(0)}).$$

Substitution of

$$M_z^{(2)} = -D\left[\frac{d^2 w^{(2)}}{dz} + \nu \frac{f_z}{a}\frac{dw^{(1)}}{dz} - f_z^2 \frac{d^2 w^{(0)}}{dz^2} + f_z\left(f_{zz} + \nu \frac{f}{a^2}\right)\frac{dw^{(0)}}{dz}\right]$$

and

$$N_\theta^{(2)} = C\left[-\frac{w^{(2)}}{a} + \frac{f}{a^2} w^{(1)} - \right.$$
$$\left. - \left(\frac{f}{a^2} - \frac{f_z^2}{2}\right)\frac{w^{(0)}}{a} - \nu \frac{f f_z}{a^3}\int^z w^{(0)} d\bar{z} + \frac{f_z}{a} v^{(1)} + \nu\left(\frac{dv^{(2)}}{dz} + f_{zz} w^{(1)} - \frac{f_z^2}{2}\frac{dv^{(0)}}{dz}\right)\right]$$

into the equation above with additional condition

$$\frac{dv^{(2)}}{dz} = \nu \frac{w^{(2)}}{a} - \left[\left(f_{zz} + \nu \frac{f}{a^2}\right) w^{(1)} - \nu \frac{f}{a^3} w^{(0)} + \nu \frac{f_z}{a} v^{(1)} - \nu^2 \frac{f f_z}{a^3}\int^z w^{(0)} d\bar{z}\right] +$$
$$+ \frac{1}{C}\int^z \left[\frac{f_z}{a} N_\theta^{(1)} - f_{zz} Q_z^{(1)} - \frac{f f_{zz}}{a} Q_z^{(0)}\right] d\bar{z} - \frac{1}{aC} f_z N_z^{(1)}$$

yields
$$\frac{d^4 w^{(2)}}{dz^4} + 4\beta w^{(2)} = \Theta^{(2)}$$

with

$$\Theta^{(2)} = -\frac{C\nu}{Da}\left\{\left[f_{zz}+\nu\frac{f}{a^2}\right]w^{(1)}+\nu\frac{f_z}{a}v^{(1)}-\nu^2\frac{ff_z}{a^3}\int^z w^{(0)}d\bar{z}\right\}+$$
$$+\frac{\nu}{Da}\int^z\left[\frac{f_z}{a}N_\theta^{(1)}-f_{zz}Q_z^{(1)}-\frac{ff_{zz}}{a}Q_z^{(0)}\right]d\bar{z}-\frac{\nu}{Da^2}f_zN_z^{(1)}+$$
$$+\frac{d^2}{dz^2}\left\{f_z^2\frac{d^2w^{(0)}}{dz^2}+\left[f_zf_{zz}+\nu\frac{ff_z}{a^2}\right]\frac{dw^{(0)}}{dz}-\nu\frac{f_z}{a}\frac{dw^{(1)}}{dz}\right\}+$$
$$+\frac{C}{Da^2}\left\{\left(\frac{f}{a^2}+\nu af_{zz}\right)w^{(1)}+f_zv^{(1)}-\left(\frac{f}{a^2}-\frac{1-\nu}{2}f_z^2\right)w^{(0)}-\nu\frac{ff_z}{a^2}\int^z w^{(0)}d\bar{z}\right\}-$$
$$-\frac{f_{zz}}{D}N_z^{(1)}+\frac{1}{aD}\frac{d^2}{dz^2}(f_zM_z^{(1)})-\frac{1}{aD}\frac{d}{dz}(f_zM_\theta^{(1)})-\frac{1}{2D}\frac{d}{dz}[f_z^2Q_z^{(0)}]+\frac{f_z^2}{2D}.$$

The solution of this differential equation is

$$w^{(2)} = A_2e^{(1+i)\beta z}+\bar{A}_2e^{-(1+i)\beta z}+B_2e^{(1-i)\beta z}+\bar{B}_2e^{-(1-i)\beta z}-$$
$$-\frac{1+i}{16\beta^3}\left[\int^z\Theta^{(2)}e^{-(1+i)\beta\bar{z}}d\bar{z}\right]e^{(1+i)\beta z}+\frac{1+i}{16\beta^3}\left[\int^z\Theta^{(2)}e^{(1+i)\beta\bar{z}}d\bar{z}\right]e^{-(1+i)\beta z}-$$
$$-\frac{1-i}{16\beta^3}\left[\int^z\Theta^{(2)}e^{-(1-i)\beta\bar{z}}d\bar{z}\right]e^{(1-i)\beta z}+\frac{1-i}{16\beta^3}\left[\int^z\Theta^{(2)}e^{(1-i)\beta\bar{z}}d\bar{z}\right]e^{-(1-i)\beta z}.$$

The stress couples and stress resultants are given by

$$M_\theta^{(2)} = D\left\{-\nu\frac{d^2w^{(2)}}{dz^2}-\frac{f_z}{a}\frac{dw^{(1)}}{dz}+\nu f_z^2\frac{d^2w^{(0)}}{dz^2}+\left(\frac{ff_z}{a^2}+\nu f_zf_{zz}\right)\frac{dw^{(0)}}{dz}\right\},$$

$$M_z^{(2)} = D\left\{-\frac{d^2w^{(2)}}{dz^2}-\nu\frac{f_z}{a}\frac{dw^{(1)}}{dz}+f_z\frac{d^2w^{(0)}}{dz^2}+\left(\nu\frac{ff_z}{a^2}+f_zf_{zz}\right)\frac{dw^{(0)}}{dz}\right\},$$

$$N_\theta^{(2)} = -\frac{d}{dz}[aQ_z^{(2)}]-\frac{d}{dz}[fQ_z^{(1)}]+af_{zz}N_z^{(1)}-\frac{af_z^2}{2}p_\zeta,$$

$$N_z^{(2)} = -\frac{f_z}{a}N_z^{(1)}+\frac{1}{a}\int^z f_{\bar{z}}N_\theta^{(1)}d\bar{z}-\int^z f_{\bar{z}\bar{z}}Q_z^{(1)}d\bar{z}-\frac{1}{a}\int^z ff_{\bar{z}\bar{z}}Q_z^{(0)}d\bar{z}.$$

The complete solution can now be expressed by the power series

$$w = w^{(0)}+\mu w^{(1)}+\mu^2w^{(2)}.$$

The stress couples and stress resultants are given by power series (a).

Bibliography

1. "Bending of Thin Plates with Compound Curvature" by H. G. Lew, *N.A.C.A. TN 2782*, 1951.
2. "A New Derivation of the Equations for the Deformation of Elastic Shells" by Eric Reissner, *American Journal of Mathematics*, v. 63, No. 1, Jan. 1941, pp. 177—184.

3. "Notes on the Foundations of the Theory of Small Displacements of Orthotropic Shells" by F. B. Hildebrand, E. Reissner and G. Thomas, *N.A.C.A. TN 1833*, 1949.
4. "Applied Elasticity" by Chi-Teh Wang, *McGraw-Hill*, New York 1953.
5. "Large Deflection of a Circular Clamped Plate under Uniform Pressure" by Wei-Zang Chien, *Chinese Journal of Physics*, v. 7, No. 2, Dec. 1947, pp. 102—113.
6. "A Nonlinear Theory of Bending and Buckling of Thin Elastic Shallow Spherical Shells" by A. Kaplan and Y. C. Fung, *N.A.C.A. TN 3212*, 1954.

Summary

A method of successive corrections is applied to the solution of rotationally symmetrical cylindrical shells with small double curvatures subjected to applied

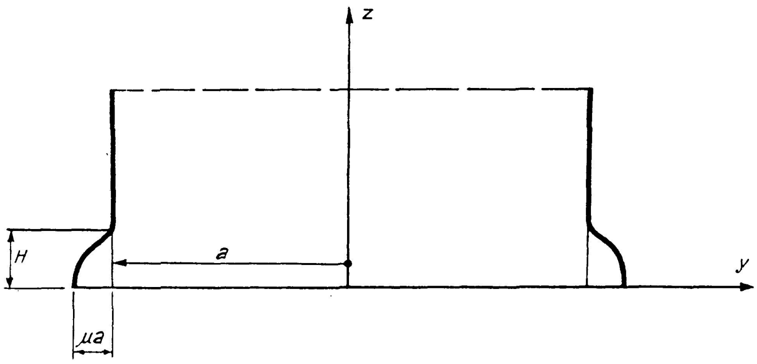

Fig. 3. The shell with the bell end.

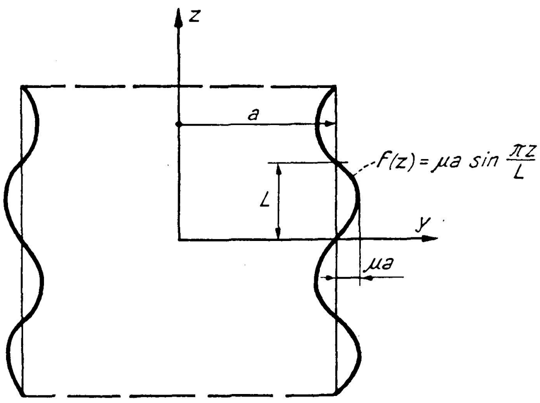

Fig. 4. The shell with surface corrugations.

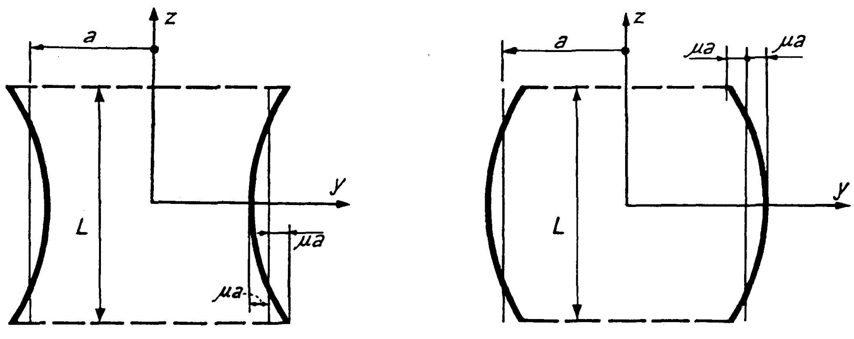

Fig. 5. The shells with positive and negative surface curvature.

normal pressure. The axial loads are assumed to be of small order of magnitude and therefore have been neglected in this analysis.

The computations in the second and third approximations are time consuming and tedious. There are a number of simplifications which can relieve the burden of computation work for the shell designer. Many shells have such proportions that they can be analyzed as semi-infinite shells. Also terms with coefficients of small order of magnitude can be neglected. If the shell is made of material of negligibly small Poisson's ratio, then the terms containing "ν" can be rejected in the analysis. Concrete for instance admits such a simplification.

The presented method can be used for cylindrical shells with flared ends. The function $f(z) = \mu a e^{-\frac{n}{H}z}\left(1 + \cos\frac{\pi z}{H}\right)$, $\left[\frac{\mu a}{H} \ll 1\right]$, can describe satisfactorily the bell end of a pipe (Fig. 3). Corrugated pipes can be analyzed by taking $f(z) = \mu a \sin\frac{\pi z}{L}$, $\left[\frac{\mu a}{L} \ll 1\right]$, (Fig. 4). Shells with small positive and negative curvature can also be solved approximately by the method of successive corrections (Fig. 5).

Résumé

L'auteur applique une méthode d'approximations successives à la résolution des voiles de révolution admettant de petits rayons de courbure doubles, sous l'influence des efforts normaux. Les efforts axiaux sont supposés faibles et par suite négligés. Les calculs dans les deuxième et troisième approximations sont pénibles et fastidieux. Il existe toutefois une série de simplifications qui facilitent les opérations de calcul pour l'étude des projets. De nombreux voiles admettent des proportions qui permettent le calcul sous forme de voiles semi-infinis. De même, il est possible de négliger des expressions comportant des coefficients d'un faible ordre de grandeur. Lorsque le voile est constitué par un matériau admettant un coefficient de Poisson faible et négligeable, les expressions en ν peuvent être laissées de côté. Cette simplification est admissible par exemple pour le béton.

La méthode décrite peut être appliquée aux voiles cylindriques dont les extrémités sont irrégulières. L'extrémité en cloche d'un tube peut être représentée par une fonction; il en est de même des tubes ondulés. Les voiles de faible courbure, positive ou négative, peuvent également être calculés par cette méthode d'approximations successives.

Zusammenfassung

In diesem Aufsatz wird eine Methode sukzessiver Korrekturen angewendet auf die Lösung rotationssymmetrischer Zylinderschalen mit kleinen, doppelten Krümmungsradien und unter Einfluß von Normalkräften. Die Axialkräfte

werden als klein vorausgesetzt und deshalb vernachlässigt. Die Berechnungen in den 2. und 3. Approximationen sind zeitraubend und ermüdend. Es gibt eine Reihe von Vereinfachungen, die die Rechnungsarbeiten für den Entwerfenden erleichtern. Viele Schalen haben Proportionen, die die Berechnung als halb-infinite Schalen gestatten. Ebenso können Ausdrücke mit Koeffizienten von kleiner Größenordnung vernachlässigt werden. Wenn die Schale aus einem Material mit vernachlässigbar kleiner Poisson-Zahl besteht, können die Ausdrücke mit ν wegfallen. Für Beton ist diese Vereinfachung beispielsweise zulässig.

Die beschriebene Methode kann für zylindrische Schalen mit unregelmäßigen Enden verwendet werden. Das Glockenende eines Rohres kann durch eine Funktion dargestellt werden, ebenso gewellte Rohre. Auch Schalen mit kleiner positiver oder negativer Krümmung können mit der Methode sukzessiver Approximationen berechnet werden.