Influence surfaces for support moments of continuous slabs

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Influence Surfaces for Support Moments of Continuous Slabs

Aires d'influence pour moments aux appuis dans les dalles continues

Einflußflächen für Stützmomente kontinuierlicher Platten

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1. Introduction

The first solutions to the problem of influence surfaces for bending moments of slabs were presented by WESTERGAARD [1][1]. Realizing the reciprocity between the bending moment M_x at point (u,v) produced by a unit load $P=1$ at point (x,y) and the bending moment M_x at point (x,y) caused by a unit load $P=1$ at point (u,v) of a simply supported plate strip (see fig. 1) he was able to determine the moment influence surface for a point (u,v) of such a plate. Subsequent investigators [2, 3] based their solutions on the same "reciprocity law". Unfortunately this law is not sufficiently general and holds only for a restricted number of cases such as, simply supported plate strip [1], simply supported rectangular plate [3], etc. Reference may be made to the problem of influence lines of beams. Only for the simply supported beam is the bending moment diagram for $P=1$ at (u) equal to the influence line of the bending moment at (u).

PUCHER [4, 5, 6] was the first to derive the general theory for influence

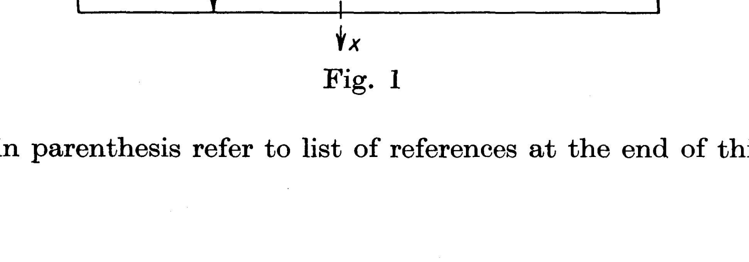

Fig. 1

[1]) Numbers in parenthesis refer to list of references at the end of this paper.

surfaces of slabs. Many new cases have since been solved. Reference [7] presents a collection of these solutions in graphical form readily applicable to design problems. However, to the author's knowledge, no solutions for the support moments of continuous slabs have previously been presented.

2. Fundamental Theorems

Before developing the solutions the fundamental theorems derived by PUCHER [4, 5, 8] are restated without proof:

1. The influence function $f(x,y)$ for any effect in a slab (such as bending moment, shearing force, etc.) at a given point (u,v) [2]) is a solution of the biharmonic equation $\Delta\Delta f(x,y) = 0$ [3]) with a singularity at the influence point (u,v).
2. The boundary conditions on $f(x,y)$ are identical with the boundary conditions for the deflection w of the given plate; e.g. a built-in boundary having $w = \frac{\partial w}{\partial n} = 0$ will have $f = \frac{\partial f}{\partial n} = 0$, n being the normal to the boundary line.

On the basis of these two theorems the following solutions will be derived.

3. Plate Strip Continuous Over Rigid Cross Beam

Consider an infinite plate strip with simply supported parallel edges (fig. 2). At $y = 0$ the plate is continuous over a rigid support beam. The problem is to derive the influence function for the bending moment M_y [4]) over this cross support, hence for M_y at $y = 0$. To distinguish the influence function from the bending moment M_y the influence function is labeled m_y. The function m_y is now taken in the form

$$m_y = m_{y0} + m_{y1} \tag{1}$$

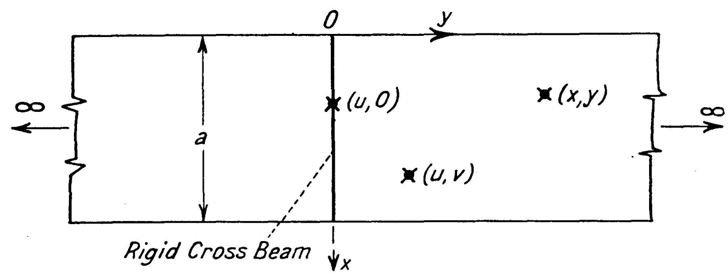

Fig. 2

[2]) Subsequently this point (u, v) will be referred to as the "influence point", being the point for which the influence function is determined.

[3]) Δ being the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

[4]) The notation is consistent with the usual engineering notation, especially with reference [9].

m_{y0} being the influence function for the simply supported plate strip without the cross beam and m_{y1} being a regular integral of $\Delta\Delta\, m_{y1}=0$, chosen in such a manner that the sum $(m_{y0}+m_{y1})$ fulfills all the boundary conditions. According to theorem (2) m_y is required to fulfill the same boundary conditions as the deflection w of the plate. The boundary conditions can then be formulated in the following manner:

$$\text{For } \left.\begin{array}{l} x=0 \\ x=a \end{array}\right\} \quad m_{y0}=m_{y1}=\frac{\partial^2 m_{y0}}{\partial y^2}=\frac{\partial^2 m_{y1}}{\partial y^2}=0. \tag{2}$$

$$\text{For } y=0 \quad m_{y0}+m_{y1}=0, \tag{3}$$

$$\frac{\partial m_{y1}}{\partial y}=0. \tag{4}$$

Of these conditions, (3) assures that the support provided by the cross beam is rigid. (4) expresses the continuity of the slab for the regular solution m_{y1}. No such condition on m_{y0} is required as it is a solution for the infinite plate strip and hence continuous.

The influence function $m_{y0}\,(x,y)$ for the bending moment M_y at point (u,v) of an infinite plate strip (fig. 1) is given by [1]:

$$m_{y0}=\frac{1}{2\pi}\sum_1^{\infty}\frac{1}{n}\left[1+\nu\pm(1-\nu)\frac{n\pi(y-v)}{a}\right]e^{\pm\frac{n\pi(y-v)}{a}}\sin\frac{n\pi u}{a}\sin\frac{n\pi x}{a}.$$

In the case of the double sign the upper applies for $y\leqq v$, the lower for $y\geqq v$. The constant ν stands for Poisson's ratio. Using the following dimensionless coordinates:

$$\begin{array}{lll} \dfrac{\pi u}{a}=\alpha; & \dfrac{\pi v}{a}=\beta; & \dfrac{\pi x}{a}=\xi; \\[2ex] \dfrac{\pi y}{a}=\eta; & \dfrac{\pi c}{a}=\gamma; & \text{(used later on)} \end{array} \tag{5}$$

the expression becomes

$$m_{y0}=\frac{1}{2\pi}\sum_1^{\infty}\frac{1}{n}[1+\nu\pm(1-\nu)(n\eta-n\beta)]\,e^{\pm(n\eta-n\beta)}\sin n\alpha\cdot\sin n\xi. \tag{6}$$

Upper sign $\eta\leqq\beta$. Lower sign $\eta\geqq\beta$.

For $y=\frac{a}{\pi}\eta=0$ eq. (6) becomes:

$$m_{y0}\,(x,0)=\frac{1}{2\pi}\sum_1^{\infty}\frac{1}{n}[1+\nu-(1-\nu)\,n\beta]\,e^{-n\beta}\sin n\alpha\cdot\sin n\xi. \tag{7}$$

For m_{y1} a regular solution of $\Delta\Delta\, m_{y1}=0$ is required which fulfills the boundary conditions (2). Such a solution can readily be found in the form of an infinite series [9], p. 168.

$$m_{y1} = \sum_{1}^{\infty} (a_n + b_n n \eta) e^{-n\eta} \sin n\alpha \cdot \sin n\xi \tag{8}$$

for $\eta \geqq 0$. The coefficients a_n and b_n are determined from the boundary conditions (3) and (4) such that the sum $(m_{y0} + m_{y1})$ fulfils the condition of a rigid support at $\eta = 0$.

$$\frac{\partial m_{y1}}{\partial y} = \frac{\pi}{a} \frac{\partial m_{y1}}{\partial \eta} = 0 = -\sum_{1}^{\infty} n (a_n - b_n) \sin n\alpha \cdot \sin n\xi$$

or

$$a_n = b_n. \tag{9}$$

Making use of eqs. (7) and (9) in condition (3) gives:

$$a_n = -\frac{1}{2\pi} \frac{1}{n} [1 + \nu - (1-\nu) n\beta] e^{-n\beta}. \tag{10}$$

The general solution for the influence surface is the sum $(m_{y0} + m_{y1})$ or

$$\begin{aligned} m_y = \frac{1}{2\pi} \sum_{1}^{\infty} \frac{1}{n} \{ & [1 + \nu \pm (1-\nu)(n\eta - n\beta)] e^{\pm(n\eta - n\beta)} \\ & - [1 + \nu - (1-\nu) n\beta] e^{-n\beta} \cdot (1 + n\eta) e^{-n\eta} \} \sin n\alpha \cdot \sin n\xi \end{aligned} \tag{11}$$

for $\eta \geqq 0$. The supper sign is for $\eta \leqq \beta$, the lower for $\eta \geqq \beta$. To determine the influence function m_y for a point over the support $v = \frac{a}{\pi}\beta = 0$ in eq. (11) or:

For $\beta = 0$ (lower sign in (11) applies):

$$m_y = -\frac{\eta}{\pi} \sum_{1}^{\infty} e^{-n\eta} \sin n\alpha \cdot \sin n\xi. \tag{12}$$

An investigation of eq. (12) will show that in the neighborhood of the point $(\alpha, 0)$ the series converges very slowly. At $\eta = 0$, $\xi = \alpha$ the function exhibits singular behavior. It is therefore little suited for actual computation. Fortunately it is possible to sum this series such that m_y can be presented in finite form. As shown in the Appendix:

$$\sum_{1}^{\infty} e^{-n\eta} \sin n\alpha \cdot \sin n\xi = \frac{1}{4} \operatorname{Sinh} \eta \left(\frac{1}{\operatorname{Cosh} \eta - \cos(\xi - \alpha)} - \frac{1}{\operatorname{Cosh} \eta - \cos(\xi + \alpha)} \right) \tag{13}$$

which applied to (12) gives

$$m_y = -\frac{\eta}{4\pi} \operatorname{Sinh} \eta \left(\frac{1}{\operatorname{Cosh} \eta - \cos(\xi - \alpha)} - \frac{1}{\operatorname{Cosh} \eta - \cos(\xi + \alpha)} \right). \tag{14}$$

Eq. (14) gives the influence function for the bending moment M_y at point $(\alpha, 0)$ over the support. Of special interest is the value m_y at $(\alpha, 0)$, hence $\eta = 0$ and $\xi = \alpha$. However, m_y assumes an indeterminate form $\frac{0}{0}$ such that de l'Hospital's rule must be applied to determine its value. It is readily found that for

$$\left.\begin{aligned}\eta &= 0\\ \xi &= \alpha\end{aligned}\right\} \quad m_y = -\frac{1}{2\pi}. \tag{15}$$

Actual computations of m_y do not offer any particular difficulties and can readily be made on a desk calculator. A typical example of an influence surface for $\left(u=\frac{a}{2}, v=0\right)$ or $\left(\alpha=\frac{\pi}{2}, \beta=0\right)$ respectively is shown in figure 5. It should be mentioned that following PUCHER [7] the plotted values are 8π times the influence values.

4. Plate Strip Continuous Over Two Rigid Cross Beams

As a second example the influence function for the support moment of a plate strip continuous over two rigid cross beams is considered (fig. 3). In the limiting case for $c=0$ — c being the distance between the two cross beams — this problem will give the influence function for a semi-infinite plate strip with built-in edge, already presented by PUCHER [7]. By varying the distance c different degrees of restraint can be obtained.

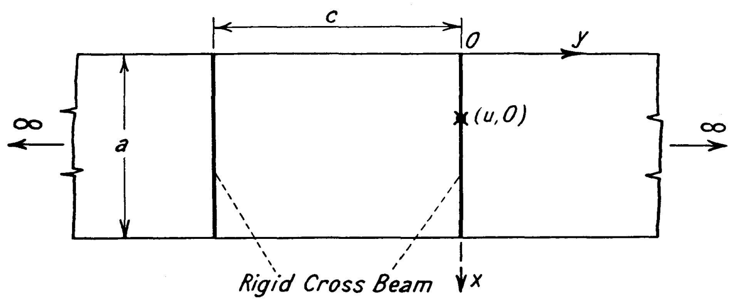

Fig. 3

The method of attack follows the pattern of the previous solution. The influence function for the bending moment M_y over the right support $y=0$ for point $(u, 0)$ is determined from the sum

$$m_y = m_{y0} + m_{y1} + m_{y2} \tag{16}$$

m_{y0} is the solution for the case of an infinite plate strip continuous over a rigid support at $y=0$, hence is given in series form by eq. (12) or in finite form by eq. (14). m_{y1} and m_{y2} are regular solutions for an unloaded plate strip except for reactive forces along $y=0$ and $y=-c$ respectively:

$$m_{y1} = \sum_{1}^{\infty}(a_{1n} \mp b_{1n}\, n\, \eta)\, e^{\mp n \eta} \sin n\alpha \cdot \sin n\xi. \tag{17}$$

Upper sign for $\eta \geqq 0$

Lower sign for $\eta \leqq 0$.

(See eq. (5) for dimensionless coordinates).

$$m_{y2} = \sum_{1}^{\infty}[a_{2n} \mp b_{2n}(n\gamma + n\eta)]\, e^{\mp(n\gamma+n\eta)} \sin n\alpha \cdot \sin n\xi. \tag{18}$$

Upper sign $\eta \geqq -\gamma$

Lower sign $\eta \leqq -\gamma$.

The three parts m_{y0}, m_{y1} and m_{y2} fulfil individually the boundary conditions for a simple support along $\xi = 0$ and $\xi = \pi$. The continuity of slope over the two cross beams requires that for

$$\eta = 0; \quad \frac{\partial m_{y1}}{\partial \eta} = 0 = -\sum_{1}^{\infty}(a_{1n} + b_{1n}),$$

$$\eta = -\gamma; \quad \frac{\partial m_{y2}}{\partial \eta} = 0 = -\sum_{1}^{\infty}(a_{2n} + b_{2n}),$$

$$\text{or} \quad \begin{aligned} b_{1n} &= -a_{1n}, \\ b_{2n} &= -a_{2n}. \end{aligned} \tag{19}$$

The final conditions imposed by the two rigid cross supports are given by the vanishing of m_y at $\eta = 0$ and $\eta = -\gamma$

$$\eta = 0; \quad m_{y0}(\xi, 0) + m_{y1}(\xi, 0) + m_{y2}(\xi, 0) = 0. \tag{20}$$

As m_{y0} $(\xi, 0) = 0$ it follows that

$$\sum_{1}^{\infty}[a_{1n} + a_{2n}(1 + n\gamma)\, e^{-n\gamma}] \sin n\alpha \cdot \sin n\xi = 0. \tag{21}$$

For the left cross beam the condition becomes

$$\eta = -\gamma; \quad m_{y0}(\xi, -\gamma) + m_{y1}(\xi, -\gamma) + m_{y2}(\xi, -\gamma) = 0 \tag{22}$$

or substituting from eqs. (12), (17), (18) and (19)

$$\sum_{1}^{\infty}\left[-\frac{\gamma}{\pi} e^{-n\gamma} + a_{1n}(1 + n\gamma)\, e^{-n\gamma} + a_{2n}\right] \sin n\alpha \cdot \sin n\xi = 0. \tag{23}$$

Solving eqs. (21) and (23) for the constants gives

$$\begin{aligned} a_{1n}[e^{n\gamma} - (1 + n\gamma)^2 e^{-n\gamma}] &= -\frac{\gamma}{\pi}(1 + n\gamma)\, e^{-n\gamma}, \\ a_{2n}[e^{n\gamma} - (1 + n\gamma)^2 e^{-n\gamma}] &= \frac{\gamma}{\pi}. \end{aligned} \tag{24}$$

All the constants being determined the influence function m_y is found by proper substitution. For the region to the right of the cross beams (fig. 3) it is:

$$\text{For } \eta \geqq 0; \quad m_y = m_{y0} - \frac{\eta}{\pi} \sum_{1}^{\infty} \frac{n^2 \gamma^2}{e^{2n\gamma} - (1 + n\gamma)^2} e^{-n\eta} \sin n\alpha \cdot \sin n\xi \tag{25}$$

where m_{y0} is the solution of the plate strip continuous over one cross beam, hence given by eq. (14).

Of special interest is the case $\gamma = 0$, i. e. the edge $\eta = 0$ is fully restrained. As

$$\lim_{\gamma \to 0} \frac{n^2 \gamma^2}{e^{2n\gamma} - (1 + n\gamma)^2} = 1$$

it follows from eq. (12) that

$$\gamma = 0; \quad m_y = m_{y0} - \frac{\eta}{\pi} \sum_1^\infty e^{-n\eta} \sin n\alpha \cdot \sin n\xi = 2\, m_{y0}. \tag{26}$$

In words, the influence function for the bending moment at the built-in edge of a semi-infinite plate strip takes double the value of the influence function for the support moment of a plate strip continuous over a rigid cross beam. The value for $(\xi = \alpha, \eta = 0)$ is twice the value of eq. (15) or $m_y = -1/\pi$ which checks the result previously presented by PUCHER [7].

Table 1

Comparison between eq. (25) and approximate expression eq. (27)

Eq. (25)

$$m_y = m_{y0} - \frac{\eta}{\pi} \sum_1^\infty \frac{n^2\gamma^2}{e^{2n\gamma} - (1+n\gamma)^2} e^{-n\eta} \sin n\alpha \sin n\xi,$$

$$m_y = -\frac{\eta}{\pi} \sum_1^\infty e^{-n\eta} \sin n\alpha \sin n\xi \underline{\left(1 + \frac{n^2\gamma^2}{e^{2n\gamma} - (1+n\gamma)^2}\right)}.$$

Eq. (27)

$$m_y \cong m_{y0} - \frac{\eta}{\pi} \sum_1^\infty e^{-(n\eta + 1.25\, n\gamma)} \sin n\alpha \sin n\xi,$$

$$m_y \cong -\frac{\eta}{\pi} \sum_1^\infty e^{-n\eta} \sin n\alpha \sin n\xi \underline{\left(1 + e^{-1.25\, n\gamma}\right)}.$$

(1)	(2)	(3)	(4)
$n\gamma$	$\frac{n^2\gamma^2}{e^{2n\gamma} - (1+n\gamma)^2}$	$e^{-1.25\, n\gamma}$	$\frac{1+(2)}{1+(3)}$
0.00	1.0000	1.0000	1.000
0.10	0.8772	.8825	0.997
0.25	0.7251	.7316	0.996
0.50	0.5338	.5353	0.999
0.75	0.3964	.3916	1.003
1.00	0.2951	.2865	1.006
1.50	0.1626	.1534	1.008
2.00	0.0877	.0821	1.005
3.00	0.0232	.0235	1.000

Going back to the general case given by eq. (25), it is possible to closely approximate the second term of the right-hand side by a finite expression. The factor $\frac{n^2\gamma^2}{e^{2n\gamma}-(1+n\gamma)^2}$ can be approximated by $e^{-1.25n\gamma}$ as Table 1 shows for a number of values of $n\gamma$. Hence

$$m_y \cong m_{y0} - \frac{\eta}{\pi}\sum_1^\infty e^{-(n\eta+1.25\,n\gamma)}\sin n\alpha\cdot\sin n\xi\,. \tag{27}$$

Making use of the summation formula, eq. (13), it follows that

$$\begin{aligned} m_y &= m_{y0} - \frac{\eta}{4\pi}\,\mathrm{Sinh}\,(\eta+1.25\,\gamma)\cdot \\ &\cdot\left(\frac{1}{\mathrm{Cosh}\,(\eta+1.25\,\gamma)-\cos\,(\xi-\alpha)} - \frac{1}{\mathrm{Cosh}\,(\eta+1.25\,\gamma)-\cos\,(\xi+\alpha)}\right). \end{aligned} \tag{28}$$

It is interesting to note the behaviour of the influence function at the influence point, given by $\xi=\alpha,\ \eta=0$. As long as $\gamma>0$ the second term on the right-hand side disappears such that $m_y\,(\xi=\alpha,\ \eta=0)$ takes the value of $m_{y0}\,(\xi=\alpha,\ \eta=0)= -{}^1/_2\,\pi$. However, for $\gamma=0$ the second term becomes at first indeterminate $\frac{0}{0}$, taking the final value $-{}^1/_2\,\pi$, such that $m_y=-1/\pi$. It is therefore evident that a sudden jump occurs as γ approaches zero.

To complete the solution the expression for $\eta<0$ is needed:

$$\begin{aligned} m_y &= m_{y0} + \sum_1^\infty a_{1n}\,(1-n\,\eta)\,e^{+n\eta}\sin n\alpha\cdot\sin n\xi\; + \\ &\quad + \sum_1^\infty a_{2n}\,[1\pm(n\gamma+n\,\eta)]\,e^{\mp(n\gamma+n\eta)}\sin n\alpha\cdot\sin n\xi\,. \end{aligned} \tag{29}$$

In the case of double sign the upper and lower signs hold for $\eta\geqq-\gamma$ and $\eta\leqq-\gamma$ respectively. In the preceding equation m_{y0} is given by eq. (14) and the constants a_{1m} and a_{2n} by eq. (24). Unfortunately no summation nor simple approximation of the second and third terms in eq. (29) could be developed; hence their computation in series form becomes necessary. In figure 6 the results for computations taking $c=a$ (or $\gamma=\alpha$) are presented. Again the plotted values are 8π times the influence ordinates. In this particular instance the series were found to converge very rapidly giving sufficiently accurate results using only one term.

5. Two-Span Continuous Plate

The influence function for the support moment at $(u, 0)$ of the two-span continuous plate shown in figure 4 is taken as the sum

$$m_y = m_{y0} + m_{y1} \tag{30}$$

m_{y0} is given by eq. (12) or eq. (14) fulfilling the boundary conditions along $\xi=0$, $\xi=\pi$ and $\eta=0$. Due to symmetry, only the portion $\eta\geqq0$ must be considered.

The solution of m_{y1} — m_{y1} being an integral of $\Delta\Delta\, m_{y1}=0$ — is given by ([9], p. 199)

$$m_{y1} = \sum_1^\infty (a_n \operatorname{Sinh} n\,\eta + b_n \operatorname{Cosh} n\,\eta + c_n n\,\eta \operatorname{Sinh} n\,\eta + + d_n n\,\eta \operatorname{Cosh} n\,\eta) \sin n\,\alpha \cdot \sin n\,\xi\,. \tag{31}$$

The constants a_n to d_n are determined from the following boundary conditions:

For $\eta = 0$;
$$m_{y1} = 0\,, \tag{32}$$
$$\frac{\partial\, m_{y1}}{\partial\, y} = 0\,. \tag{33}$$

As m_{y0} already assures continuity over the support the two equations above express this condition for m_{y1}.

For $\eta = \gamma$;
$$m_y = m_{y0} + m_{y1} = 0\,, \tag{34}$$
$$\frac{\partial^2\, m_y}{\partial\, y^2} = \frac{\partial^2\, m_{y0}}{\partial\, y^2} + \frac{\partial^2\, m_{y1}}{\partial\, y^2} = 0\,. \tag{35}$$

It may be once more remembered that the influence function should fulfil the same boundary conditions as the deflections which explains the form of eq. (34) and (35).

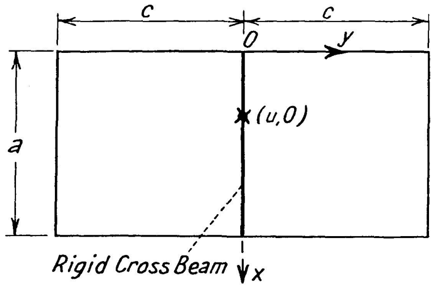

Fig. 4

Making use of these conditions (32) to (35) the constants are as follows:

$$\begin{aligned} a_n &= -d_n = \frac{e^{-n\gamma}}{\operatorname{Sinh} n\gamma \operatorname{Cosh} n\gamma - n\gamma}\, \frac{\gamma}{\pi} [\operatorname{Sinh} n\gamma + \operatorname{Cosh} n\gamma]\,, \\ b_n &= 0\,, \\ c_n &= \frac{e^{-n\gamma}}{\operatorname{Sinh} n\gamma \operatorname{Cosh} n\gamma - n\gamma}\, \frac{\gamma}{\pi} \left[\operatorname{Sinh} n\gamma + \operatorname{Cosh} n\gamma - \frac{1}{n\gamma} \operatorname{Sinh} n\gamma\right]. \end{aligned} \tag{36}$$

The solution takes the final form:

$$m_y = m_{y0} + \sum_1^\infty A_n [B_n (\operatorname{Sinh} \eta - n\,\eta \operatorname{Cosh} n\,\eta) + C_n n\,\eta \operatorname{Sinh} n\,\eta] \sin n\,\alpha \cdot \sin n\,\xi \tag{37}$$

in which

$$\begin{aligned} A_n &= \frac{e^{-n\gamma}}{\operatorname{Sinh} n\gamma \operatorname{Cosh} n\gamma - n\gamma}, \\ B_n &= \frac{\gamma}{\pi}(\operatorname{Sinh} n\gamma + \operatorname{Cosh} n\gamma), \\ C_n &= \frac{\gamma}{\pi}\left(\operatorname{Sinh} n\gamma + \operatorname{Cosh} n\gamma - \frac{1}{n\gamma}\operatorname{Sinh} n\gamma\right). \end{aligned} \tag{38}$$

A numerical solution of eq. (37) for the case of two continuous square plates — $c=a$ or $\gamma=\pi$ respectively — was computed. The influence point was fixed at mid span over the cross beam, hence $u=\frac{a}{2}$, $v=0$ or $\alpha=\frac{\pi}{2}$, $\beta=0$. Excellent convergence of the series was observed requiring the computation of the first term only. The results are presented in form of an influence surface in figure 7 (again times 8π).

6. Application of Influence Surfaces

In case of point loads the ordinate of the influence surface below the load multiplied by the load represents directly 8π times the moment at the influence point produced by this load. For uniformly distributed loads the volume of the influence surface below the loaded area times the load intensity yields 8π times the bending moment at the influence point. As such numerical computations are beyond the objective of this paper reference is made to the pertinent literature [4, 7].

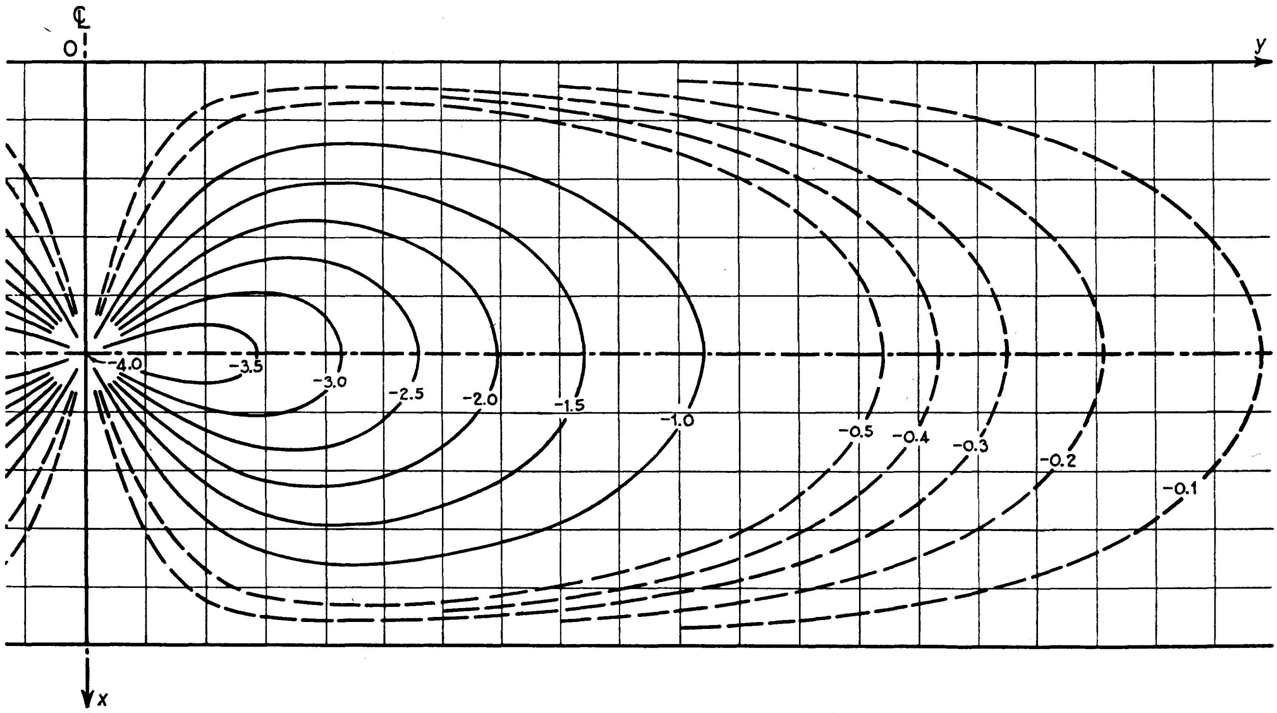

Fig. 5. m_y-influence surface for moment over cross beam (8π times).

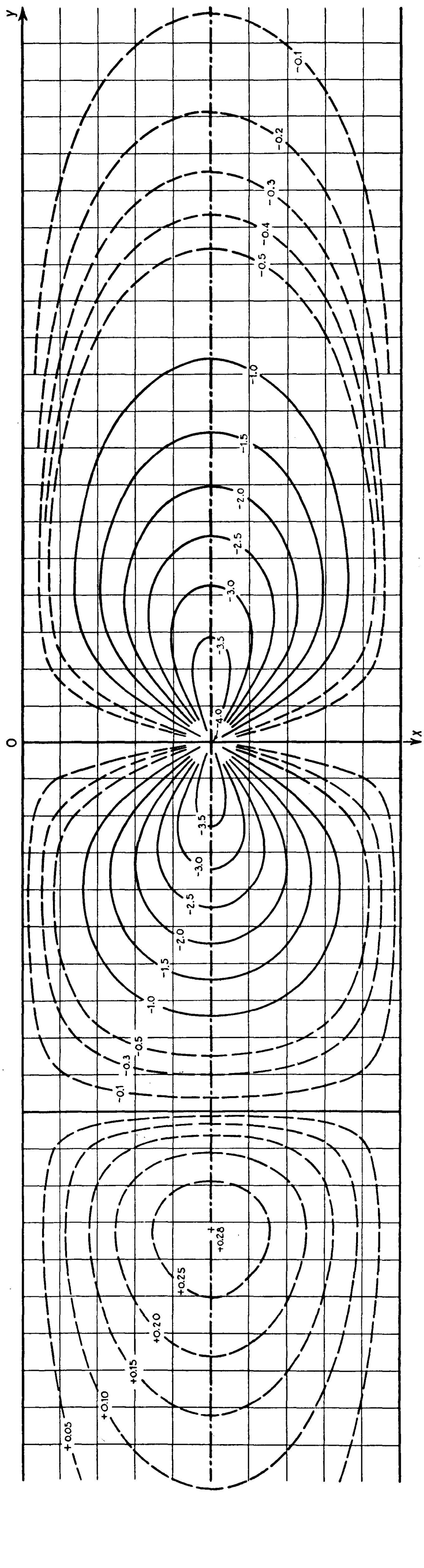

Fig. 6. m_y-influence surface for moment over right cross beam (8π times).

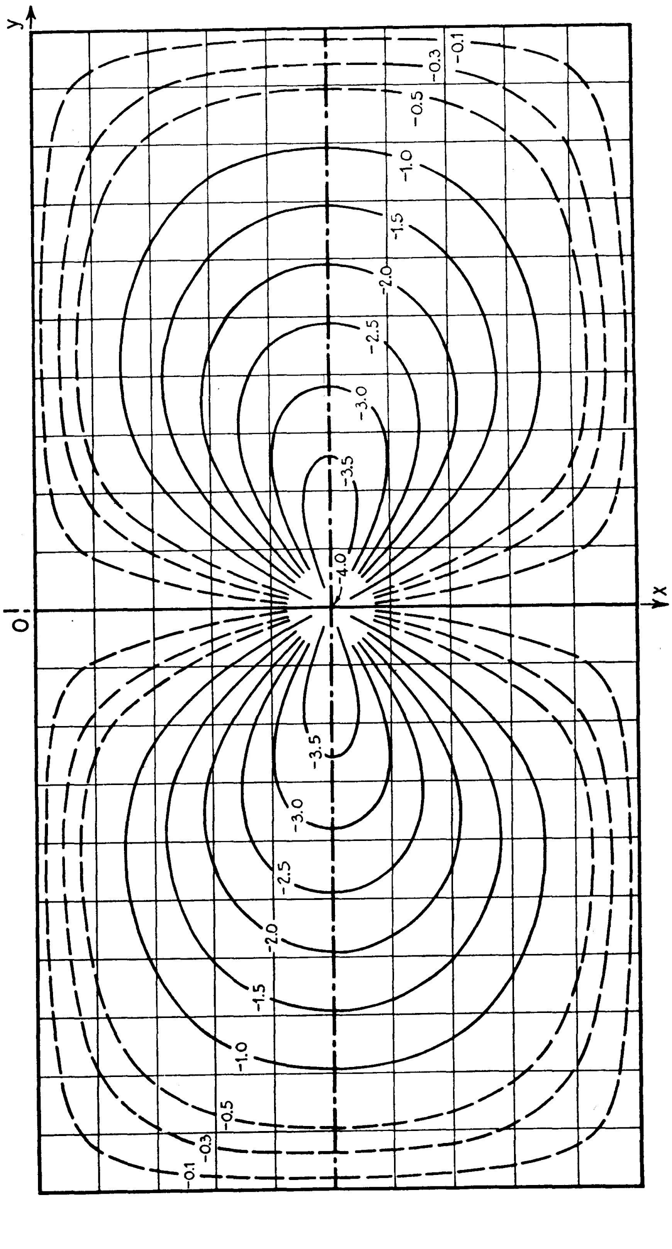

Fig. 7. m_y-influence surface for moment over cross beam (8π times).

7. Acknowledgement

The author would like to express his appreciation for the help he received in the preparation of this paper. Mr. Tadahiko Kawai, Research Assistant at Fritz Engineering Laboratory, indicated to him the possibility of summing up the series in eq. (12) resulting in a finite expression as given in eq. (14). The checking of the theoretical derivations, the numerical computations and the drawing of the influence surfaces presented in figures 5 to 7 were done by Mr. Arthur J. Romanelli, Instructor in the Department of Civil Engineering.

Appendix

Derivation of eq. (13)

If z is a complex variable and $|z| < 1$, the following expansion holds

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_0^\infty z^n. \tag{A}$$

Expressing z in polar form

$$z = r e^{i\phi} = r(\cos\phi + i\sin\phi)$$

and its conjugate

$$\bar{z} = r e^{-i\phi} = r(\cos\phi - i\sin\phi)$$

yields

$$\frac{1}{1-z} = \frac{1-\bar{z}}{(1-z)(1-\bar{z})} = \frac{1 - r\cos\phi + i\,r\sin\phi}{1 - 2r\cos\phi + r^2} \tag{B}$$

and

$$\sum_0^\infty z^n = \sum_0^\infty r^n \cos n\phi + i\sum_0^\infty r^n \sin n\phi. \tag{C}$$

Comparing eqs. (A), (B) and (C) the following expressions can be derived, provided $0 \leqq r < 1$.

$$\sum_0^\infty r^n \cos n\phi = \frac{1 - r\cos\phi}{1 - 2r\cos\phi + r^2}$$

and

$$\sum_1^\infty r^n \cos n\phi = \frac{1 - r\cos\phi}{1 - 2r\cos\phi + r^2} - 1 = \frac{1}{2}\left(\frac{1 - r^2}{1 - 2r\cos\phi + r^2} - 1\right). \tag{D}$$

Eq. (D) can be applied to sum the following series:

$$\begin{aligned}\sum_1^\infty e^{-n\eta}\sin n\alpha \sin n\xi &= \tfrac{1}{2}\sum_1^\infty e^{-n\eta}\cos n(\xi-\alpha) - \tfrac{1}{2}\sum_1^\infty e^{-n\eta}\cos n(\xi+\alpha),\\ &= \frac{1}{4}\left(\frac{1 - e^{-2\eta}}{1 - 2e^{-\eta}\cos(\xi-\alpha) + e^{-2\eta}} - 1\right) -\\ &\quad - \frac{1}{4}\left(\frac{1 - e^{-2\eta}}{1 - 2e^{-\eta}\cos(\xi+\alpha) + e^{-2\eta}} - 1\right),\\ &= \frac{1}{4}\frac{\operatorname{Sinh}\eta}{\operatorname{Cosh}\eta - \cos(\xi-\alpha)} - \frac{1}{4}\frac{\operatorname{Sinh}\eta}{\operatorname{Cosh}\eta - \cos(\xi+\alpha)}\end{aligned}$$

provided $\eta > 0$. Therefore

$$\sum_{1}^{\infty} e^{-n\eta} \sin n\alpha \sin n\xi = \tfrac{1}{4} \operatorname{Sinh} \eta \left(\frac{1}{\operatorname{Cosh} \eta - \cos(\xi - \alpha)} - \frac{1}{\operatorname{Cosh} \eta - \cos(\xi + \alpha)}\right) \quad \text{(E)}$$

which corresponds to eq. (13).

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Summary

In the present paper the theory of influence functions for bending moments in slabs is extended to the case of continuous plates. An expression for the bending moment over a rigid cross beam across a simply supported plate strip, is derived. By summing the series solution into a finite expression it becomes possible to accurately compute the values even in the neighborhood of the influence point (point for which influence function is derived). A series solution would not allow such a computation as this influence point exhibits a singularity. On the basis of this solution two other cases are treated, namely the infinite plate strip with two rigid cross beams and the two span continuous plate. Expressions for the influence function of the support moments are derived. Graphical representations of influence surfaces are given for three specific examples which are directly applicable to actual design problems.

Résumé

La théorie des fonctions d'influence pour les moments fléchissants dans les planchers est étendue au cas des dalles continues. L'auteur établit une expression pour le moment fléchissant sur une poutre transversale rigide, appartenant à un platelage reposant librement sur ses appuis. Par sommation de la solution de la série sous forme d'une expression finie, il est possible de calculer exactement les valeurs correspondantes, même au voisinage du point d'appui (pour lequel la fonction d'influence est calculée). Une solution de série ne permettrait pas un tel calcul, car ce point représente une singularité. Sur la base de cette solution, l'auteur traite deux autres cas, celui d'un platelage infini comportant deux poutres transversales rigides et celui d'une dalle continue à deux travées. Il établit des expressions pour la fonction d'influence des moments aux appuis. Pour trois exemples déterminés, il donne des représentations graphiques, qui peuvent être appliquées directement aux calculs.

Zusammenfassung

In diesem Aufsatz wird die Theorie der Einflußfunktionen für Biegemomente in Decken erweitert für den Fall der durchlaufenden Platten. Ein Ausdruck für das Biegemoment über einem steifen Querträger eines freiaufliegenden Plattenstreifens wird abgeleitet. Durch Summieren der Reihenlösung zu einem endlichen Ausdruck wird es möglich, die Werte sogar in der Umgebung des Aufpunktes (für den die Einflußfunktion abgeleitet wird) genau zu berechnen. Eine Reihenlösung würde eine solche Berechnung nicht erlauben, da dieser Aufpunkt eine Singularität darstellt. Auf der Basis dieser Lösung werden zwei andere Fälle behandelt, der unendliche Plattenstreifen mit zwei steifen Querträgern und die zweifeldrige, durchlaufende Platte. Es werden Ausdrücke für die Einflußfunktion der Stützmomente abgeleitet. Für drei bestimmte Beispiele werden graphische Darstellungen der Einflußflächen angegeben, die direkt für die Berechnungen anwendbar sind.