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A Method for Analyzing Deformations of Plane Trusses

Méthode pour le calcul des déformations des treillis plans

Eine Methode zur Berechnung der Deformationen ebener Fachwerke

J. DUNDURS, Instructor in Civil Engineering, The Technological Institute of Northwestern University, Evanston, Illinois

Introduction

At least for the case of a statically determinate truss, it is possible to give the deflection problem a purely geometrical formulation: Given the axial deformations of the members, find the configuration in which the members will fit together. This statement of the problem leads to the simple graphical method of WILLIOT and MOHR. Attempts to put it on an algebraic basis, however, have resulted in manipulative difficulties.

In the analysis of plane mechanisms, complex numbers have been effectively used to deal with the geometrical problems[1]). The success of this approach is mainly due to the fact that a complex number, being an ordered pair of real numbers, permits one to operate with two components simultaneously and thus automatically takes into account the two-dimensional nature of the problem. This suggests the use of complex numbers in the analysis of the deformations of plane trusses.

In this discussion, it is convenient to begin with a special case and to generalize the results later. The basic relationships and a computational procedure will be developed for a simple truss consisting only of triangles. The results can be easily extended to cover other kinds of simple trusses as well as compound, complex and statically indeterminate types.

The Basic Relationships

Consider the triangle MNP in a plane truss formed by the bars k, $k+1$ and $k+2$ ($k=1, 2, \ldots$) as shown in fig. 1. Assume that a convenient coordinate

[1]) For the method and a bibliography on the subject see a paper by G. H. MARTIN and M. F. SPOTTS, Trans. ASME 79 (1957), 687.

system xy has been chosen, and let the joints M, N and P be located in the xy plane by the complex numbers z_M, z_N and z_P. Furthermore, assigning directions to the three bars allows them to be represented as the complex numbers z_k, z_{k+1} and z_{k+2}.

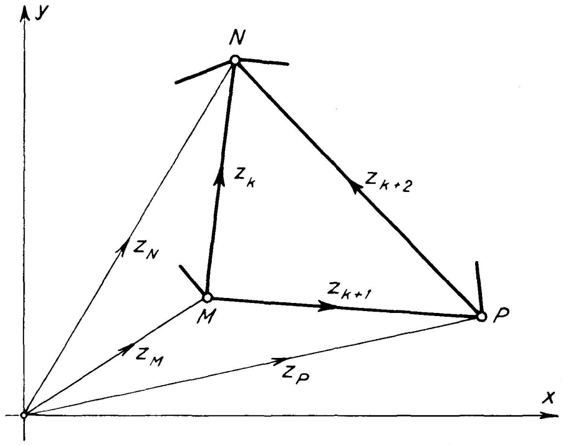

Fig. 1.

The starting relationships are established from the geometry of the truss in the undeformed state. Considering the complex numbers as vectors, it is seen from fig. 1 that

$$z_{k+1}+z_{k+2}=z_k, \tag{1a}$$

$$z_P=z_M+z_{k+1}. \tag{1b}$$

Eq. (1a) is simply a condition that the three bars form a closed polygon, in this case a triangle. Assuming that the position of the k-th bar (that is, the positions of joints M and N) has been established by some previous considerations, eq. (1b) locates the joint P in terms of z_M and z_{k+1}. Another relation that can be written in a similar manner is

$$z_P=z_N-z_{k+2}. \tag{2}$$

However, eq. (2) is not independent of the previous relationships, as is easily shown by substituting z_M+z_k for z_N in eq. (2),

$$z_P=z_M+z_k-z_{k+2}=z_M+z_{k+1}.$$

The deformation of a truss due to the changes in length of the bars involves rotations of the bars as well as displacements of the joints. Consequently, the complex numbers which describe the bars and locate the joints will experience certain changes as the truss is deformed. If the increment in the complex number z is denoted by Δz, the closing condition, which corresponds to eq. (1a) for the undeformed state, becomes

$$(z_{k+1}+\Delta z_{k+1})+(z_{k+2}+\Delta z_{k+2})=(z_k+\Delta z_k).$$

By the use of eq. (1a) this simplifies to

$$\Delta z_{k+1}+\Delta z_{k+2}=\Delta z_k. \tag{3a}$$

Similarly,

$$\Delta z_P = \Delta z_M + \Delta z_{k+1}. \tag{3b}$$

Thus the increments in the complex numbers satisfy relationships that are analogous to those satisfied by the original complex numbers, and, as a matter of fact, can be obtained from the latter if each z is replaced by the corresponding Δz.

Express the complex number representing bar k, i.e., z_k in the polar form

$$z_k = l_k e^{i\theta_k},$$

where l_k is the length of the k-th bar and θ_k is the argument of z_k or the angle measured from the x axis to the bar. At this point the usual assumption will be made that all deviations from the original configuration are very small. In other words, the elongations of the bars are small compared to their lengths, and the bars rotate through only small angles as the truss is deformed. Since both l_k and θ_k change,

$$\Delta z_k = \frac{\partial z_k}{\partial l_k}\Delta l_k + \frac{\partial z_k}{\partial \theta_k}\Delta \theta_k = e^{i\theta_k}\Delta l_k + l_k e^{i\theta_k} i \Delta\theta_k = \left(\frac{\Delta l_k}{l_k} + i\Delta\theta_k\right) l_k e^{i\theta_k},$$

where Δl_k and $\Delta\theta_k$ denote the changes in l_k and θ_k, respectively. Recognizing $\Delta l_k/l_k$ as the strain ϵ_k in the k-th bar,

$$\boxed{\Delta z_k = (\epsilon_k + i\Delta\theta_k) z_k.} \tag{4}$$

Eq. (4) has a very simple geometrical meaning as interpreted in fig. 2. Note that $\epsilon_k z_k$ for a positive ϵ_k (tension in the k-th bar) has the same direction as z_k, whereas $i\Delta\theta_k z_k$ for a positive $\Delta\theta_k$ is a vector turned through 90° counterclockwise from the direction of z_k.

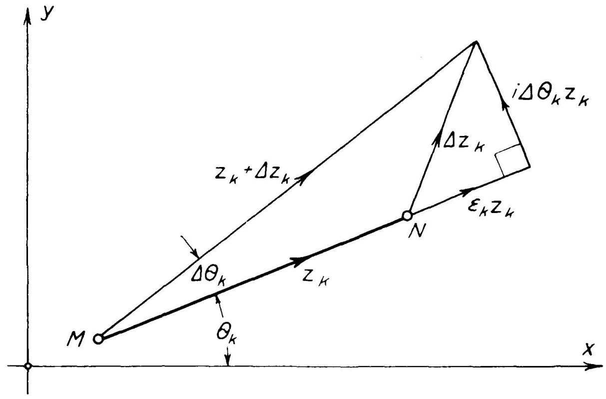

Fig. 2.

The substitution of Δz_k as given by eq. (4) and of Δz_{k+1} and Δz_{k+2} written by analogy into eq. (3) yields

$$(\epsilon_{k+1} + i\Delta\theta_{k+1}) z_{k+1} + (\epsilon_{k+2} + i\Delta\theta_{k+2}) z_{k+2} = (\epsilon_k + i\Delta\theta_k) z_k. \tag{5a}$$

$$\Delta z_P = \Delta z_M + (\epsilon_{k+1} + i\Delta\theta_{k+1}) z_{k+1}. \tag{5b}$$

Consider, first, eq. (5a): z_k, z_{k+1} and z_{k+2} are known from the initial geometry. After the forces in the members have been obtained in the usual way and the sizes of the members established, the strains can be computed from $\epsilon_k = P_k/E_k A_k$, where P_k is the force in the k-th member, E_k is the modulus of elasticity and A_k is the cross-sectional area of the k-th member[2]). Tensile forces and strains are to be considered as positive, compressive forces and strains as negative. This agrees with the sign convention as required by $\Delta l_k = \epsilon_k l_k$, positive Δl_k meaning an increase in length of the k-th bar (or modulus of z_k). Again assuming that the position of the k-th bar in the deformed state has been established by some previous considerations, the only unknowns in eq. (5a) are the rotations of bars $k+1$ and $k+2$, namely, $\Delta\theta_{k+1}$ and $\Delta\theta_{k+2}$. Also, note that eq. (5a) is linear in these quantities. Since every equation in complex numbers is equivalent to two equations in terms of real numbers, eq. (5a) can be solved for $\Delta\theta_{k+1}$ and $\Delta\theta_{k+2}$. There is a slight complication, however, which will be demonstrated in more detail in *example 1*. It will happen quite frequently that $\Delta\theta_k$ contains the unknown rotation of one bar dealt with previously. Consequently, $\Delta\theta_{k+1}$ and $\Delta\theta_{k+2}$ must be expressed in terms of this unknown rotation which, however, can be evaluated later in the calculation on the basis of conditions at one of the supports.

The interpretation of eq. (5b) is quite simple; it permits the finding of the displacement of joint P, provided the displacement of joint M is known since, presumably, $\Delta\theta_{k+1}$ has been evaluated from eq. (5a).

It is not worthwhile to develop additional relationships, as for example, to solve eq. (5a) for $\Delta\theta_{k+1}$ and $\Delta\theta_{k+2}$ in general terms, because the equations become rather unwieldy. Hence, with eq. (4) taken as the basic relationship, the application of complex numbers to the truss deformation problem is best demonstrated by a specific example.

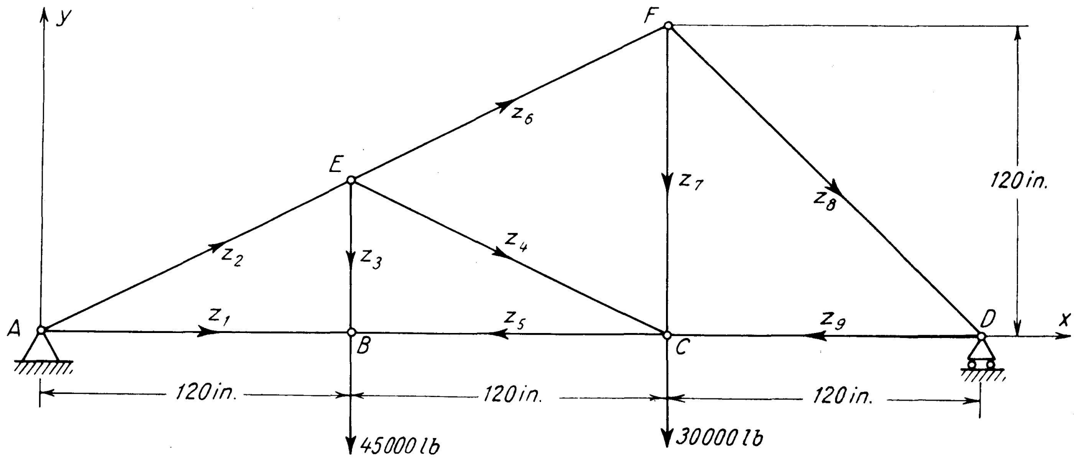

Fig. 3.

2) In principle, it makes no difference as to what causes the change in length of a member, hence, the deformation of a truss due to causes other than loading need not be excluded.

Example 1: Simple Truss

The truss to be analyzed is shown in fig. 3. Again, quantities with number subscripts are associated with the bars, while letters and quantities with letter subscripts refer to the joints. The cross-sectional areas of the members are given as $A_1=A_3=A_5=A_7=A_9=2.94\ \text{in}^2$, $A_2=A_4=A_6=A_8=19.22\ \text{in}^2$, and the modulus of elasticity for all members has the value $E=30\times 10^6\ \text{lb/in}^2$.

The numerical calculations will be carried to an accuracy that can be obtained from a slide rule. Computation of forces and strains in the members gives the values listed below:

$$
\begin{aligned}
P_1 &= +80000\ \text{lb} & \epsilon_1 &= +0.907\times 10^{-3}\\
P_2 &= -89400\ \text{lb} & \epsilon_2 &= -0.155\times 10^{-3}\\
P_3 &= +45000\ \text{lb} & \epsilon_3 &= +0.510\times 10^{-3}\\
P_4 &= -50300\ \text{lb} & \epsilon_4 &= -0.087\times 10^{-3}\\
P_5 &= +80000\ \text{lb} & \epsilon_5 &= +0.907\times 10^{-3}\\
P_6 &= -39100\ \text{lb} & \epsilon_6 &= -0.068\times 10^{-3}\\
P_7 &= +52500\ \text{lb} & \epsilon_7 &= +0.595\times 10^{-3}\\
P_8 &= -49500\ \text{lb} & \epsilon_8 &= -0.086\times 10^{-3}\\
P_9 &= +35000\ \text{lb} & \epsilon_9 &= +0.397\times 10^{-3}
\end{aligned}
$$

The next step is to assign directions to the bars so that they can be represented by complex numbers. Some foresight has been used here to avoid minus signs in the relationships corresponding to eq. (3a). The directions assumed and a convenient coordinate system are shown in fig. 3. Then

$$
\begin{aligned}
z_1 &= 120\ \text{in.} & z_6 &= 120+i\,60\ \text{in.}\\
z_2 &= 120+i\,60\ \text{in.} & z_7 &= -i\,120\ \text{in.}\\
z_3 &= -i\,60\ \text{in.} & z_8 &= 120-i\,120\ \text{in.}\\
z_4 &= 120-i\,60\ \text{in.} & z_9 &= -120\ \text{in.}\\
z_5 &= -120\ \text{in.}
\end{aligned}
$$

Using eq. (4), $\Delta z_k=(\epsilon_k+i\,\Delta\,\theta_k)\,z_k$, the changes in the complex numbers describing the bars can be computed:

$$
\begin{aligned}
\Delta z_1 &= (108.8\times 10^{-3})+i\,(120\,\Delta\,\theta_1),\\
\Delta z_2 &= (-18.6\times 10^{-3}-60\,\Delta\,\theta_2)+i\,(-9.3\times 10^{-3}+120\,\Delta\,\theta_2),\\
\Delta z_3 &= (60\,\Delta\,\theta_3)+i\,(-30.6\times 10^{-3}),\\
\Delta z_4 &= (-10.4\times 10^{-3}+60\,\Delta\,\theta_4)+i\,(5.2\times 10^{-3}+120\,\Delta\,\theta_4),\\
\Delta z_5 &= (-108.8\times 10^{-3})+i\,(-120\,\Delta\,\theta_5),\\
\Delta z_6 &= (-8.2\times 10^{-3}-60\,\Delta\,\theta_6)+i\,(-4.1\times 10^{-3}+120\,\Delta\,\theta_6),
\end{aligned}
$$

$$\Delta z_7 = (120\,\Delta\,\theta_7) + i\,(-71.4\times 10^{-3}),$$
$$\Delta z_8 = (-10.3\times 10^{-3} + 120\,\Delta\,\theta_8) + i\,(10.3\times 10^{-3} + 120\,\Delta\,\theta_8),$$
$$\Delta z_9 = (-47.6\times 10^{-3}) + i\,(-120\,\Delta\,\theta_9).$$

The condition that the triangle $A\,E\,B$ closes after the deformation of the truss is $\Delta z_2 + \Delta z_3 = \Delta z_1$, where this is most conveniently obtained by thinking about the relationship satisfied by the z's before the deformation (in this case $z_2 + z_3 = z_1$) and then replacing the z's with Δz's. This approach is to be preferred over an attempt to use eq. (3a) literally, since the algebraic signs of the Δz's depend upon the directions assigned to the bars (for example, if z_1 had been taken to the left, the condition would read $\Delta z_2 + \Delta z_3 = -\Delta z_1$). Substituting the previously computed values of Δz_1, Δz_2 and Δz_3 in the closing condition, separating the real and imaginary parts, and solving for $\Delta\,\theta_2$ and $\Delta\,\theta_3$ in terms of $\Delta\,\theta_1$ gives

$$\Delta\,\theta_2 = \Delta\,\theta_1 + 0.33\times 10^{-3},$$
$$\Delta\,\theta_3 = \Delta\,\theta_2 + 2.12\times 10^{-3} = \Delta\,\theta_1 + 2.45\times 10^{-3}.$$

The closing conditions of the other triangles and the results obtained are listed below:

$$\Delta\,ECB:\quad \Delta z_4 + \Delta z_5 = \Delta z_3,$$
$$\Delta\,\theta_4 = \Delta\,\theta_1 + 4.44\times 10^{-3},$$
$$\Delta\,\theta_5 = \Delta\,\theta_1 + 4.74\times 10^{-3}.$$
$$\Delta\,EFC:\quad \Delta z_6 + \Delta z_7 = \Delta z_4,$$
$$\Delta\,\theta_6 = \Delta\,\theta_1 + 5.11\times 10^{-3},$$
$$\Delta\,\theta_7 = \Delta\,\theta_1 + 4.76\times 10^{-3}.$$
$$\Delta\,FDC:\quad \Delta z_8 + \Delta z_9 = \Delta z_7,$$
$$\Delta\,\theta_8 = \Delta\,\theta_1 + 5.24\times 10^{-3},$$
$$\Delta\,\theta_9 = \Delta\,\theta_1 + 5.92\times 10^{-3}.$$

The rotation $\Delta\,\theta_1$ can now be evaluated from the condition that the joint D is free to move in a horizontal direction only, i.e., the coefficient of the imaginary part of Δz_D must vanish. Since $z_D = z_1 - z_5 - z_9$,

$$\Im\,(\Delta z_D) = \Im\,(\Delta z_1 - \Delta z_5 - \Delta z_9) = 120\,\Delta\,\theta_1 + 120\,\Delta\,\theta_5 + 120\,\Delta\,\theta_9 = 0,$$

where $\Im$ denotes the coefficient of the imaginary part of a complex number. This value of $\Delta\,\theta_1$ is now used to find the rotations of all other members:

$$\Delta\,\theta_1 = -3.55\times 10^{-3}\text{ rad.} \qquad \Delta\,\theta_6 = +1.56\times 10^{-3}\text{ rad.}$$
$$\Delta\,\theta_2 = -3.22\times 10^{-3}\text{ rad.} \qquad \Delta\,\theta_7 = +1.21\times 10^{-3}\text{ rad.}$$
$$\Delta\,\theta_3 = -1.10\times 10^{-3}\text{ rad.} \qquad \Delta\,\theta_8 = +1.69\times 10^{-3}\text{ rad.}$$
$$\Delta\,\theta_4 = +0.89\times 10^{-3}\text{ rad.} \qquad \Delta\,\theta_9 = +2.37\times 10^{-3}\text{ rad.}$$
$$\Delta\,\theta_5 = +1.19\times 10^{-3}\text{ rad.}$$

where a minus sign indicates a clockwise rotation.

Again, thinking first about the relationship between the z's and then translating it into terms of Δz's and with the use of the $\Delta\theta$'s computed previously:

$$\begin{aligned}
\Delta z_A &= 0 \text{ in.}\\
\Delta z_B &= \Delta z_1 = 0.109 - i\,0.426 \text{ in.}\\
\Delta z_C &= \Delta z_B - \Delta z_5 = 0.218 - i\,0.283 \text{ in.}\\
\Delta z_D &= \Delta z_C - \Delta z_9 = 0.266 \text{ in.}\\
\Delta z_E &= \Delta z_B - \Delta z_3 = 0.175 - i\,0.395 \text{ in.}\\
\Delta z_F &= \Delta z_C - \Delta z_7 = 0.073 - i\,0.212 \text{ in.}
\end{aligned}$$

Since all real parts here are positive and all coefficients of the imaginary parts are negative, the displacements of all joints are to the right and down.

General Applicability of the Method

It remains to be shown that the approach described and illustrated with a specific example will lead to a solution for all types of plane trusses. Consider, first, only simple trusses.

To this end recall the rule for the generation of a simple truss: A simple truss will be obtained if, beginning with one bar, each additional joint is connected to the previous configuration by means of two bars which do not lie in the same straight line. For example, consider the truss of fig. 4, where

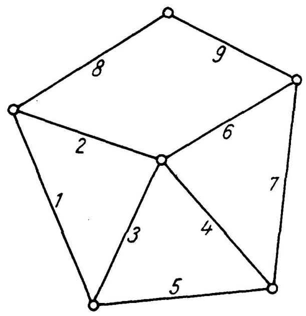

Fig. 4.

the members are numbered in the order of their inclusion into the structure. Obviously the first five bars must form triangles. Note that $\Delta\theta_2$, $\Delta\theta_3$, $\Delta\theta_4$ and $\Delta\theta_5$ can always be expressed in terms of $\Delta\theta_1$ as before. Furthermore, each additional pair of bars included (such as 6, 7 and 8, 9) gives an additional polygon whose closing condition is sufficient to allow evaluation of the rotations of the pair of bars added in terms of $\Delta\theta_1$. Thus the consideration of triangle 4-6-7 will give $\Delta\theta_6$ and $\Delta\theta_7$, whereas the quadrangle 2-6-8-9 can be used to find $\Delta\theta_8$ and $\Delta\theta_9$. It is seen that this process can be carried on indefinitely, or that for m members there will be $\frac{1}{2}(m-1)$ polygons whose closing conditions will give $m-1$ equations for the evaluation of $m-1$ rotations in terms of, say, $\Delta\theta_1$[3]). However, since the rotation of one bar is a part of the rigid body displacement of the truss in the deformed state, $\Delta\theta_1$ can always be found from the conditions at one of the supports. Also, it is clear from *example 1* that the position of one joint, which is known from the support conditions, and the rotations of the members are all that is needed to calculate the displacement of every joint.

It is interesting to note that a systematic evaluation of the rotations must proceed in the same order as that followed when the truss was generated, whereas the analysis of forces by the method of joints is done in the opposite order or by "breaking down" the truss. Thus for the K-truss shown in fig. 5, the calculation of rotations should be started with triangle 1-2-3 and then continued with triangle 3-4-5, etc. Of course, it is possible to start with triangle 14-16-17, but then in the next step the closing condition of quadrangle 11-13-14-15 will contain three unknown rotations, and the analysis must be carried further in terms of two $\Delta\theta$'s (say $\Delta\theta_{16}$ and $\Delta\theta_{15}$) until triangle 1-2-3 is reached, where one of them can be eliminated.

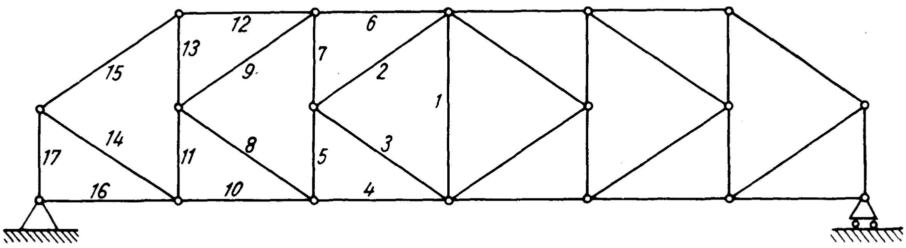

Fig. 5.

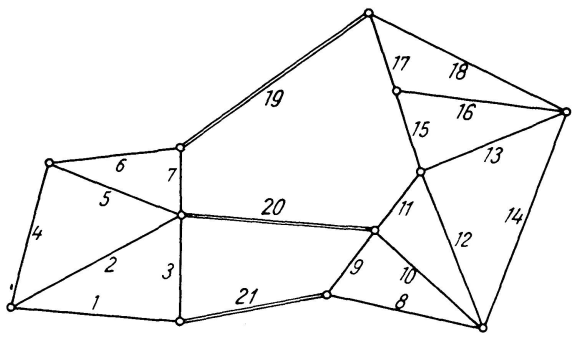

Fig. 6.

In order to discuss compound trusses, recall that a compound truss can be formed by interconnecting two simple trusses with three bars whose axes do not intersect at a point; a common joint between the two simple trusses may be used to replace two of the interconnecting bars. An example of this is shown in fig. 6, where the interconnecting bars are 19, 20 and 21. In each

[3]) Only those polygons are to be counted which give independent closing relationships.

of the simple trusses, rotations of all bars can be expressed in terms of the rotation of one bar as was done previously. The three interconnecting bars introduce three unknown rotations, but there always will be two additional polygons (such as 7-19-17-15-11-20 and 3-20-9-21 in fig. 6), so that the four additional relationships between the rotations can be used to reduce the number of unknown rotations to one. Or, if m_1 and m_2 denote the numbers of bars in the simple trusses, the total number of bars in the compound truss is $m = m_1 + m_2 + 3$, whereas the total number of polygons in the truss is $\frac{1}{2}(m_1 - 1) + \frac{1}{2}(m_2 - 1) + 2 = \frac{1}{2}(m_1 + m_2 + 3 - 1) = \frac{1}{2}(m - 1)$, thus giving us $m - 1$ relationships for the rotations. The last unknown rotation can again be evaluated on the basis of a condition at one of the supports. The situation is similar when two of the interconnecting bars are replaced by a common joint between the simple trusses. Then there is available only one additional polygon giving two relationships, but since one interconnecting bar introduces only one unknown rotation, again the rotations in the two simple trusses can be related. The argument is easily extended to cases where the compound truss is formed by interconnecting three or more simple trusses.

The last type of statically determinate trusses to be considered, the so-called complex truss, is formed by removing one or more bars from a simple or a compound truss and adding the same number of new bars between existing joints. If a bar is removed from a statically determinate truss, one part of the truss is able to move with respect to the other part with one degree of freedom. For example, if bar 5 is removed from the simple truss shown in fig. 7a, a four-bar linkage is obtained consisting of I, 4, II and III interconnected with 12 and IV as indicated in fig. 7b. Then there are several pairs of joints, such as A and B, which could move with respect to each other. In case the bar to be added is used to connect a pair of such joints, the stiffness of the truss will be restored, provided that the distance between the two joints did not have a maximum or minimum value in the relative motion between the joints. Thus, substituting bar 16 for 5, the complex truss shown in fig. 7c is obtained.

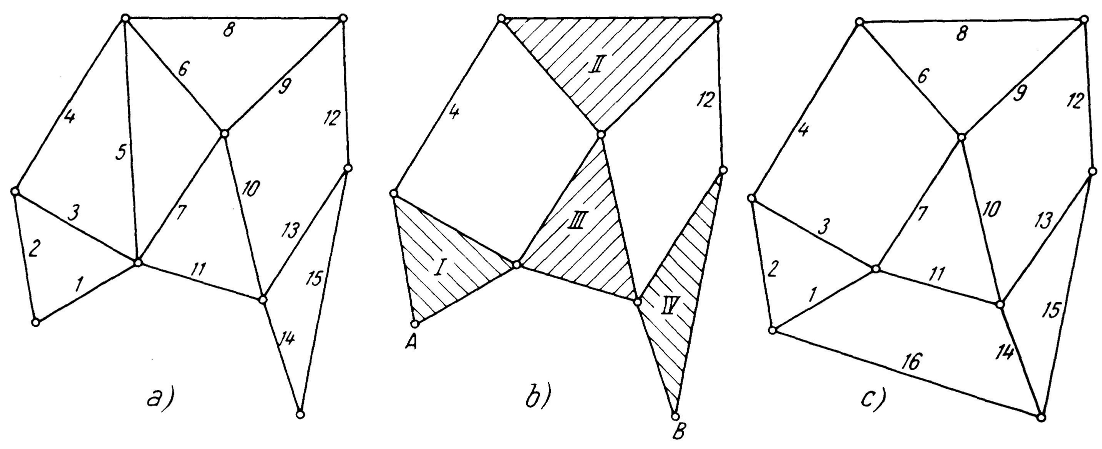

Fig. 7.

It is clear from the process of generating a complex truss that the number of bars is not changed, and that the number of polygons giving independent closing conditions remains the same. In the truss of fig. 7c, the closing conditions of 3-4-6-7 and 1-11-14-16 would replace the closing conditions of 3-4-5 and 5-6-7. The only complication that arises in this type of truss is that the rotations of certain bars cannot be expressed in terms of the rotation of only one bar, as in the case of a simple truss. Thus for the truss shown in fig. 7c, triangle 1-2-3 permits, say, $\Delta\,\theta_2$ and $\Delta\,\theta_3$ to be expressed in terms of $\Delta\,\theta_1$, but then the quadrangle 3-4-6-7 introduces three bars not dealt with previously and, say, $\Delta\,\theta_6$ and $\Delta\,\theta_7$ must be expressed in terms of $\Delta\,\theta_1$ and $\Delta\,\theta_4$. The same applies for $\Delta\,\theta_8, \ldots, \Delta\,\theta_{15}$. The one missing relationship can be recovered later, however, from the polygon 1-11-14-16.

It may be remarked in passing that the application of complex numbers to the deformation problem permits one to take advantage of special simplifying circumstances, *such as symmetry*. For example, in the truss of fig. 8, half of the work in analyzing the rotations can be saved by noticing that $\Delta\,\theta_{10}=0$ (this permits one to solve for $\Delta\,\theta_1$ regardless of the condition at the right support), and that $\Delta\,\theta_{12}=-\Delta\,\theta_9, \Delta\,\theta_{13}=-\Delta\,\theta_{11}$, etc.

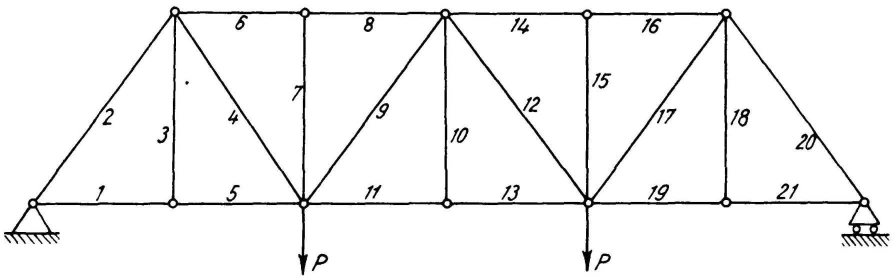

Fig. 8.

Example 2: Statically Indeterminate Truss

Before proceeding with a general discussion of statically indeterminate trusses, a specific example for illustrative purposes will be considered.

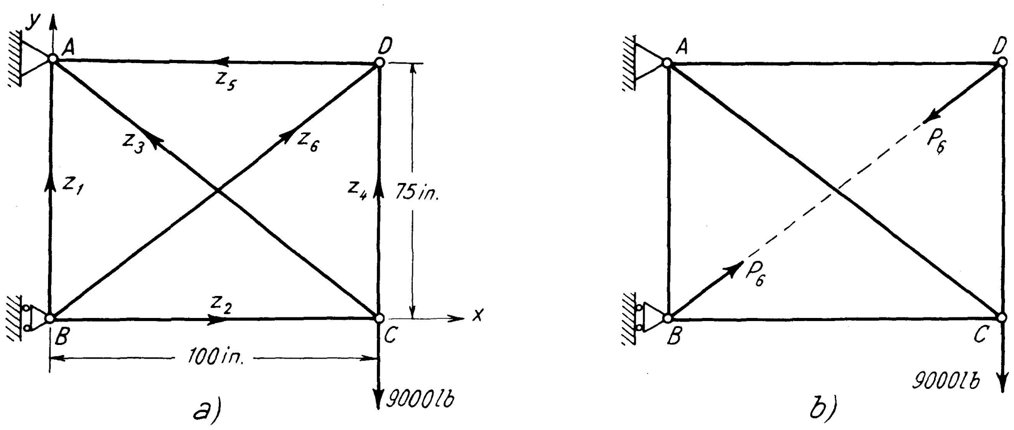

Fig. 9.

The hypothetical truss to be analysed for forces in the members and deflections is shown in fig. 9a. The cross-sectional area of every member is $1.00\,\text{in}^2$ and the modulus of elasticity has the value of $30\times 10^6\,\text{lb/in}^2$.

By taking bar 6 (BD) as the redundant member and assuming it to be in tension, the given truss can be thought of as a statically determinate structure loaded, besides the 9000-lb external load, by the two forces P_6 whose magnitude must be determined on the basis of the compatibility of the deformations (fig. 9b). The forces and strains in members 1 to 5 then can be easily evaluated in terms of P_6 or ϵ_6. Thus,

$$
\begin{aligned}
P_1 &= -0.6\,P_6 \text{ lb} & \epsilon_1 &= -0.6\,\epsilon_6\\
P_2 &= -12000-0.8\,P_6 \text{ lb} & \epsilon_2 &= -0.400\times 10^{-3}-0.8\,\epsilon_6\\
P_3 &= +15000+P_6 \text{ lb} & \epsilon_3 &= +0.500\times 10^{-3}+\epsilon_6\\
P_4 &= -0.6\,P_6 \text{ lb} & \epsilon_4 &= -0.6\,\epsilon_6\\
P_5 &= -0.8\,P_6 \text{ lb} & \epsilon_5 &= -0.8\,\epsilon_6
\end{aligned}
$$

After assigning directions to the bars and with a coordinate system such as that shown in fig. 9a, the bars are described by the following complex numbers:

$$
\begin{aligned}
z_1 &= i\,75 \text{ in.} & z_4 &= i\,75 \text{ in.}\\
z_2 &= 100 \text{ in.} & z_5 &= -100 \text{ in.}\\
z_3 &= -100+i\,75 \text{ in.} & z_6 &= 100+i\,75 \text{ in.}
\end{aligned}
$$

The changes in the complex numbers describing the bars can be computed from eq. (4). In this case, however, bar 1 cannot rotate ($\varDelta\,\theta_1=0$):

$$
\begin{aligned}
\varDelta\,z_1 &= i\,(-45\,\epsilon_6),\\
\varDelta\,z_2 &= (-40.0\times 10^{-3}-80\,\epsilon_6)+i\,(100\,\varDelta\,\theta_2),\\
\varDelta\,z_3 &= (-50.0\times 10^{-3}-100\,\epsilon_6-75\,\varDelta\,\theta_3)+i\,(37.5\times 10^{-3}+75\,\epsilon_6-100\,\varDelta\,\theta_3),\\
\varDelta\,z_4 &= (-75\,\varDelta\,\theta_4)+i\,(-45\,\epsilon_6),\\
\varDelta\,z_5 &= (80\,\epsilon_6)+i\,(-100\,\varDelta\,\theta_5),\\
\varDelta\,z_6 &= (100\,\epsilon_6-75\,\varDelta\,\theta_6)+i\,(75\,\epsilon_6+100\,\varDelta\,\theta_6).
\end{aligned}
$$

The solution is conveniently started by considering the primary truss of fig. 9b. The closing condition of triangle BCA (which is $\varDelta\,z_2+\varDelta\,z_3=\varDelta\,z_1$) gives upon separation of real and imaginary parts and solving for $\varDelta\,\theta_2$ and $\varDelta\,\theta_3$,

$$
\begin{aligned}
\varDelta\,\theta_2 &= -3.60\,\epsilon_6-1.575\times 10^{-3},\\
\varDelta\,\theta_3 &= -2.40\,\epsilon_6-1.200\times 10^{-3}.
\end{aligned}
$$

Similarly, the closing condition of triangle CDA ($\varDelta\,z_4+\varDelta\,z_5=\varDelta\,z_3$) gives

$$
\begin{aligned}
\varDelta\,\theta_4 &= -0.533\times 10^{-3},\\
\varDelta\,\theta_5 &= -3.60\,\epsilon_6-1.575\times 10^{-3}.
\end{aligned}
$$

Finally, because of the presence of the redundant bar 6, an additional condition is imposed which can be obtained either from the triangle BCD or the triangle BDA. Using the triangle BDA (which gives $\Delta z_6 + \Delta z_5 = \Delta z_1$), two relations are obtained upon separation of the real and imaginary parts:

$$100\,\epsilon_6 - 75\,\Delta\theta_6 + 80\,\epsilon_6 = 0,$$
$$75\,\epsilon_6 + 100\,\Delta\theta_6 - 100\,\Delta\theta_5 = -45\,\epsilon_6.$$

The first of these equations gives

$$\Delta\theta_6 = 2.40\,\epsilon_6,$$

whereas the second can be used to find ϵ_6, since $\Delta\theta_5$ had been expressed in terms of $\Delta\theta_6$ before:

$$\epsilon_6 = -0.219 \times 10^{-3}.$$

The known strain in bar 6 permits the force in this bar and, consequently, the forces in all other bars to be calculated. Furthermore, the rotations of bars can be evaluated. The results follow:

$$\begin{aligned} P_1 &= +3940 \text{ lb} & \qquad \Delta\theta_1 &= 0 \text{ rad.} \\ P_2 &= -6750 \text{ lb} & \Delta\theta_2 &= -0.788 \times 10^{-3} \text{ rad.} \\ P_3 &= +8440 \text{ lb} & \Delta\theta_3 &= -0.675 \times 10^{-3} \text{ rad.} \\ P_4 &= +3940 \text{ lb} & \Delta\theta_4 &= -0.533 \times 10^{-3} \text{ rad.} \\ P_5 &= +5250 \text{ lb} & \Delta\theta_5 &= -0.788 \times 10^{-3} \text{ rad.} \\ P_6 &= -6560 \text{ lb} & \Delta\theta_6 &= -0.525 \times 10^{-3} \text{ rad.} \end{aligned}$$

The minus sign on a force indicates compression, and a negative $\Delta\theta$ represents a clockwise rotation.

Finally, by use of the $\Delta\theta$'s computed above,

$$\begin{aligned} \Delta z_A &= 0 \text{ in.} \\ \Delta z_B &= -\Delta z_1 = -i\,0.0098 \text{ in.} \\ \Delta z_C &= \Delta z_B + \Delta z_2 = -0.0225 - i\,0.0886 \text{ in.} \\ \Delta z_D &= -\Delta z_5 = 0.0175 - i\,0.0788 \text{ in.} \end{aligned}$$

In the coordinate system used, negative components of a displacement indicate motion to the left and down, respectively.

Comments on Statically Indeterminate Trusses

The previous example indicates that in analysing a statically indeterminate truss by means of complex numbers the conventional approach used in other methods can be employed successfully. That is, the force in a redundant member or from a redundant support is carried through the calculation as an unknown until it can be evaluated from the conditions of geometric compatibility of deformations.

A truss with one or more internal indeterminacies may be thought of as being obtained from a determinate truss by including one or more bars between existing joints. An example is shown in fig. 10, where bar 10 may be considered to be the redundant member. In the analysis by complex numbers, each redundant bar introduces two unknowns, namely, the force or strain in the bar and the rotation of the bar. At the same time, however, an additional polygon is obtained whose closing condition is always sufficient for finding the two unknowns. It is interesting to note that, although each redundant member generates several additional polygons, only one of these will yield an independent closing condition. Thus for the truss of fig. 10, the primary system gives

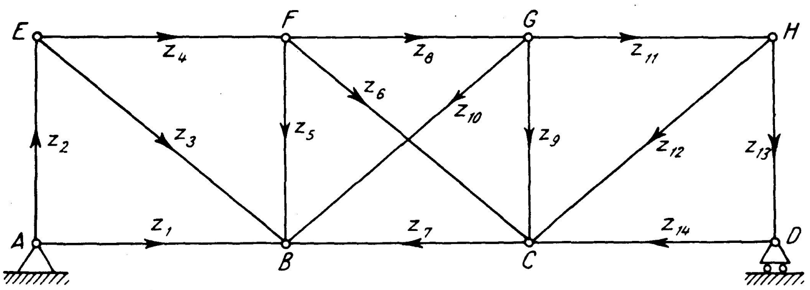

Fig. 10.

$\Delta z_6+\Delta z_7=\Delta z_5$ and $\Delta z_8+\Delta z_9=\Delta z_6$. In addition, the redundant member dictates the condition $\Delta z_8+\Delta z_{10}=\Delta z_5$ if the triangle FGB is used. On the other hand, using triangle GCB, $\Delta z_9+\Delta z_7=\Delta z_{10}$. It is easy to show, however, that the last relationship can be obtained from the previous three, and hence is not independent.

A truss becomes externally indeterminate if it is constrained in more ways than are required for support in a statically determinate manner. In case the redundant support is a roller or its equivalent, one unknown force is introduced, but at the same time an additional condition is obtained pertaining to the deflections. A redundant hinge gives two unknown components of a force and also two conditions on deflections. In either case, the number of additional unknowns is equal to the number of additional conditions. As a matter of fact, it makes little difference in the application of the method of what type the indeterminacy is, except that external indeterminacies tend to affect the forces in a larger number of bars than do most internal indeterminacies.

The analysis of assembly and thermal stresses as well as the effect of the settlement of supports requires no special extension of this method, and such problems are easily formulated in terms of complex numbers.

Summary

Assigning directions to the members of any plane truss allows the members to be described by complex numbers. The increments in these complex numbers upon the deformation of the truss may be expressed in terms of the strains and the rotations of the members by eq. (4). The conditions of geometric compatibility that must be satisfied by the increments are readily obtained from the original geometry of the truss, and are of a sufficient number to permit the solution for the unknown rotations of the members and, in the case of a statically indeterminate truss, also for the strains in the redundant members. The displacements of the joints may be computed by using the rotations of the members previously evaluated.

Résumé

Lorsqu'une direction est assignée à chacune des barres d'un treillis plan, il est possible de représenter ces barres par des nombres complexes. Les variations de ces nombres complexes par suite de la déformation du treillis sont exprimées suivant l'équation (4) en fonction des allongements et des rotations des barres. Le nombre des conditions géométriques de compatibilité qui résultent de la forme initiale du système porteur et qui doivent être satisfaites par ces variations des nombres complexes est suffisant pour la détermination des rotations inconnues des barres, ainsi que des rotations qui se produisent dans les barres surabondantes dans le cas d'un ouvrage statiquement indéterminé. Le déplacement des nœuds peut être calculé à l'aide des rotations ainsi déterminées des barres.

Zusammenfassung

Wenn allen Stäben eines ebenen Fachwerkes je eine Richtung zugewiesen wird, können sie mit komplexen Zahlen beschrieben werden. Die Veränderungen dieser komplexen Zahlen infolge der Deformation des Fachwerkes werden nach Gl. (4) in Funktion der Dehnungen und Drehungen der Stäbe ausgedrückt. Dabei genügt die Anzahl der sich aus der ursprünglichen Tragwerksform ergebenden geometrischen Verträglichkeitsbedingungen, die durch diese Veränderungen der komplexen Zahlen erfüllt werden müssen, für die Bestimmung der unbekannten Stabdrehungen sowie auch der Drehungen in den überzähligen Stäben im Falle eines statisch unbestimmten Fachwerkes. Die Verschiebung der Knotenpunkte kann mit Hilfe der oben ermittelten Drehungen der Stäbe berechnet werden.