General instability of low framed buildings

General Instability of Low Framed Buildings

Instabilité générale des constructions formées de portiques à un étage

Allgemeine Instabilität einstöckiger Rahmentragwerke

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Introduction

In the design and stability analysis of one-story framed structures such as industrial buildings, it is usual at most to consider the individual plane frames or bents as acting independently. Each plane frame is thus usually required to be stable under its own loadings without dependence upon the possible support which may be provided by other elements of the structure. In many cases, however, such support could be, and in fact is, provided by the horizontal bracing system, the end walls, and perhaps those frames which are not themselves loaded to their critical limit.

Suppose, for example, that the building has reasonably stiff end walls and, in the roof or ceiling plane, an adequate horizontal bracing system supported by the end walls. This system can provide the individual frames with a definite amount of elastic support against transverse buckling in their respective planes. Including this additional support in the stability analysis will, of course, increase the calculated collapse loads for the sidesway mode of collapse.

As a second example, suppose that the critical column loading condition for the individual frames of a one-story industrial building is due primarily to a heavy overhead traveling crane. Since the crane can be at only one location at any time, only one frame will be severely loaded at a given time. The remaining frames, being less than critically loaded, will have at that time what may be considered to be an excess of stiffness. Hence if an adequate horizontal bracing system exists, this excess stiffness together with the stiffness of the end-walls will be transmitted to the critically loaded frame in the form of elastic support against lateral buckling. If an analysis is made in which this elastic support is considered, it may be possible either to reduce the sizes of columns in a structure yet to be designed, or, in a structure already designed, either to justify a materially greater load or to show an increased factor of safety.

The problems of general instability of complete framed structures usually are complicated to such a degree that their exact solution is extremely difficult and laborious. While it is true that the formulation of such problems is not in itself difficult, although usually lengthy, the high degree of elastic coupling and the fact that the axial loads enter the equations in a transcendental form complicate the problem to such an extent that the rigorous solution or even a dependable rational solution is not attempted.

In the case of the one-story building, however, a rigorous solution is not only possible but also practicable. It is the purpose of this paper to present the theory and demonstrate the procedure for an exact stability analysis of one-story buildings. This approach is applicable to buildings in which the upper parts of the frames are either trusses or girders. For simplicity, the following discussion will be centered about the case of plane frames consisting of columns and girders. The modifications which become necessary in using this approach for frames having roof trusses will be obvious.

If the individual plane frames are of the rectilinear, rigidly connected portal type, the solution proceeds in the following way:

1. Assume each bent is disconnected from the bracing system, and apply the design axial loads to the columns of the bents.

2. Taking into account the effect of the axial loads on the various members, find the shear stiffness or horizontal spring rate of each frame, i. e., the force required at the level of the bracing system to produce a unit transverse displacement. This may be done easily by a generalized slope deflection method or by a generalized moment distribution method, using coefficients which have been previously[1]) presented. For frames which are less than critically loaded, it will be found that the spring rates are positive; for frames which are more than critically loaded, the spring rates will be negative.

3. Treating the bracing system and its supports as an elastic structure disconnected from the frames, determine the influence coefficients for displacements at the locations of the critically loaded frames due to unit forces at those locations. The frames which are not critically loaded are to be considered as a part of the elastic support for the bracing system. When only two non-critically loaded frames exist, or only the two end-walls are available for supporting the bracing system, and the bracing system is itself statically determinate, then the computation of the influence coefficients is also simply the problem of deflections of a statically determinate structure. On the other hand, if the number of walls plus non-critically loaded frames exceeds two,

[1]) See "Stiffness Charts for Gusseted Members Under Axial Load", by JOHN E. GOLDBERG, *Transactions*, American Society of Civil Engineers, Vol. 119 (1954), p. 43.

the computation of influence coefficients becomes the straightforward problem of determining the deflections in a statically indeterminate structure.

4. Apply to the bracing system a set of loads proportional to the product of the shear stiffnesses of the critically loaded frames as determined in (2) times the corresponding deflections and compute the resulting lateral deflections of the bracing system. This calculation is made by use of the influence coefficients determined in (3). It will be seen that the pertinent equations form a linear, homogeneous, algebraic set and can therefore be treated as an eigenvalue problem, the solution of which can be obtained by an iterative procedure as well as by other and more formal methods.

5. For any one critically loaded frame location on the bracing system, compute from the results of (4) the ratio of applied force to deflection, i. e., the spring rate of the bracing system and its supports.

6. Compare the spring rate of the bracing system at this location to the negative spring rate of the critically loaded frame. If the spring rate of the bracing system exceeds the negative spring rate of the frame, the entire structure is stable. Conversely, if the negative spring rate of the frame exceeds the spring rate of the bracing system, the entire structure is unstable, i. e., a condition of general instability exists. Note that this comparison can be made for any frame location and the conclusion regarding stability or instability is not dependent upon the choice.

7. Alternatively, if the iterative procedure is used at step (4), the stability or instability of the structure is deduced from the comparison of the initial values and resulting values of a cycle.

Lateral Stiffness of Frames

The lateral stiffness of each of the individual frames is easily obtained by any one of several techniques, including a generalized slope deflection method or a generalized moment distribution method. The slope deflection approach is here used inasmuch as charts which have previously been prepared by the author are directly applicable to this technique. However, these charts may also be used to provide the necessary factors in the event the moment distribution technique is to be used. The reader is referred to a previous publication [2]) which presents a complete set of charts giving constants for various axial load ratios and for various gusset lengths. Only the first of the charts, for gussets of zero length, are reproduced at this time as fig. 4. The numerical values of the coefficients are also listed in table 1.

It may be shown that the bending moment at the end a of a uniform member ab (see fig. 2) which is subjected to an axial compressive load P, can be written

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[2]) *Loc. cit.*

$$M_{ab} = K(A\,\theta_a + B\,\theta_b), \tag{1}$$

where $K = \frac{EI}{L}$. θ_a, θ_b = slopes at ends a and b, respectively, taken positive in the clockwise direction;

$$A = \frac{\sin pL - pL\cos pL}{\frac{2}{pL}(1-\cos pL) - \sin pL} \quad (P \text{ positive, compression}),$$

$$B = \frac{pL - \sin pL}{\frac{2}{pL}(1-\cos pL) - \sin pL} \quad (P \text{ positive, compression}),$$

$$A = \frac{pL\cosh pL - \sinh pL}{\frac{2}{pL}(1-\cosh pL) + \sinh pL} \quad (P \text{ negative, tension}),$$

$$B = \frac{\sinh pL - pL}{\frac{2}{pL}(1-\cosh pL) + \sinh pL} \quad (P \text{ negative, tension}),$$

P = axial compressive load,

$$p = \sqrt{\left|\frac{P}{EI}\right|}.$$

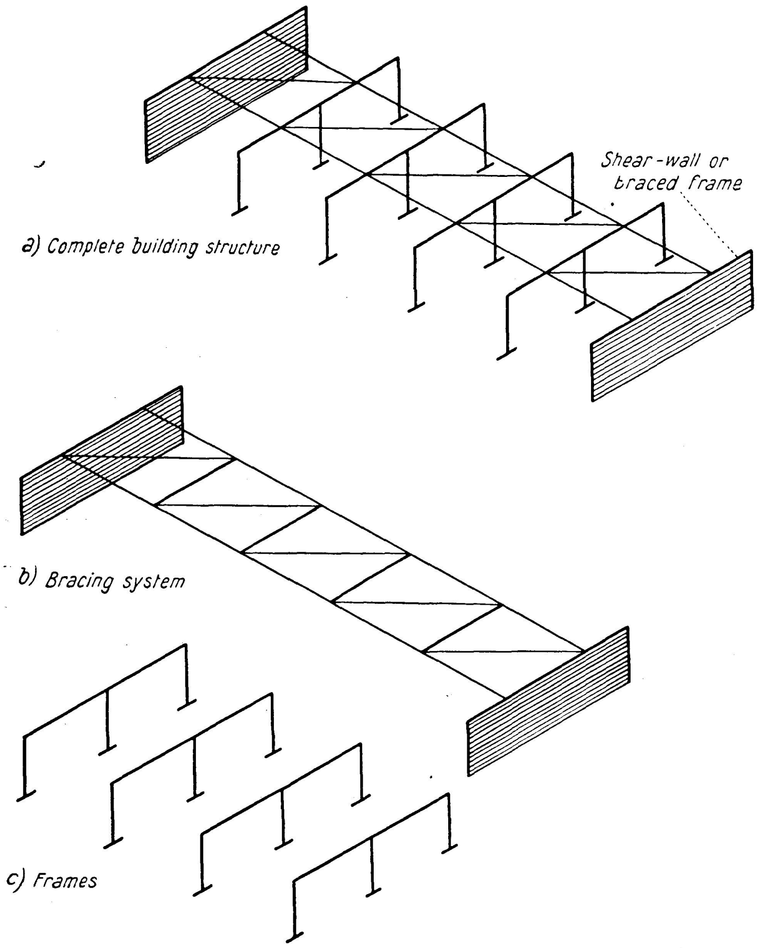

Fig. 1. Complete Structure and its Components.

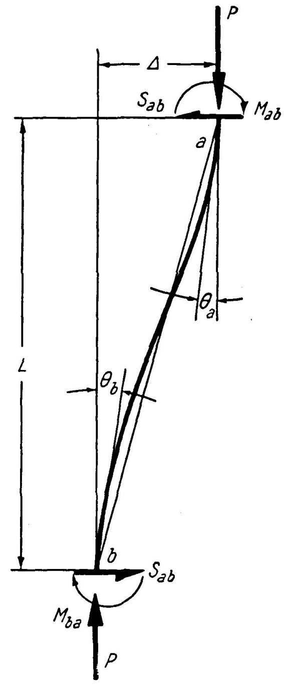

Fig. 2. Applied Forces and Distortion of a Column.

Table 1. Slope Deflection Coefficients A and B for Various Values of Load Ratio ρ

ρ	A	B
3.9	-78.34	78.58
3.8	-39.05	39.54
3.7	-24.69	25.39
3.6	-17.87	18.79
3.5	-13.73	14.86
3.4	-10.91	12.24
3.3	- 8.86	10.40
3.2	- 7.30	9.02
3.1	- 6.05	7.96
3.0	- 5.03	7.12
2.8	- 3.449	5.884
2.6	- 2.252	5.019
2.5	- 1.749	4.678
2.4	- 1.300	4.383
2.2	- 0.519	3.901
2.0	0.143	3.521
1.8	0.717	3.224
1.6	1.224	2.980
1.5	1.457	2.873
1.4	1.673	2.778
1.2	2.090	2.610
1.0	2.468	2.468
0.9	2.645	2.404
0.8	2.816	2.346
0.7	2.981	2.291
0.6	3.140	2.241
0.5	3.295	2.194
0.4	3.444	2.150
0.3	3.589	2.109
0.2	3.730	2.070
0.1	3.865	2.033
0	4.000	2.000

ρ	A	B
0	4.000	2.000
-0.1	4.131	1.968
-0.2	4.255	1.938
-0.3	4.384	1.910
-0.4	4.502	1.883
-0.5	4.619	1.857
-0.6	4.736	1.834
-0.7	4.849	1.811
-0.8	4.959	1.789
-0.9	5.069	1.769
-1.0	5.175	1.749
-1.2	5.383	1.713
-1.4	5.583	1.681
-1.6	5.777	1.651
-1.8	5.964	1.623
-2.0	6.147	1.598
-2.5	6.580	1.544
-3.0	6.99	1.499
-4.0	7.75	1.40
-5.0	8.62	1.38
-7.0	9.62	1.30
-9.0	10.69	1.26

From the last definition it follows that

$$pL = \pi\sqrt{\left|\frac{P}{\pi^2 \frac{EI}{L^2}}\right|} = \pi\sqrt{\left|\frac{P}{P_e}\right|},$$

where P_e = Euler's Load.

Thus, the constants A and B in eq. (1) depend only upon the dimensionless quantity

$$\rho = \frac{P}{P_e}, \qquad \text{since } p\,L = \pi\sqrt{|\rho|}. \tag{2}$$

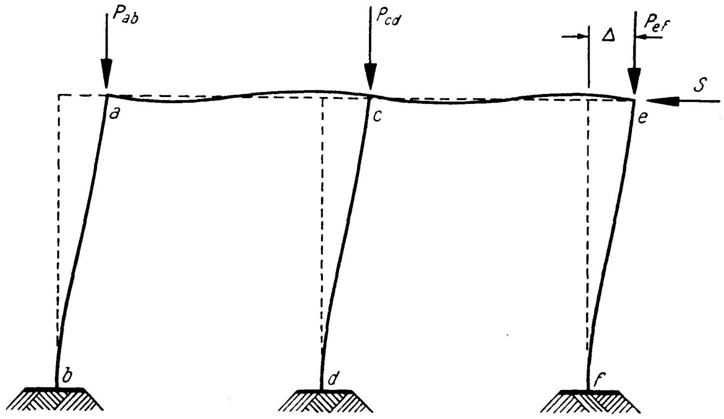

Fig. 3. Deflected Unstable Frame.

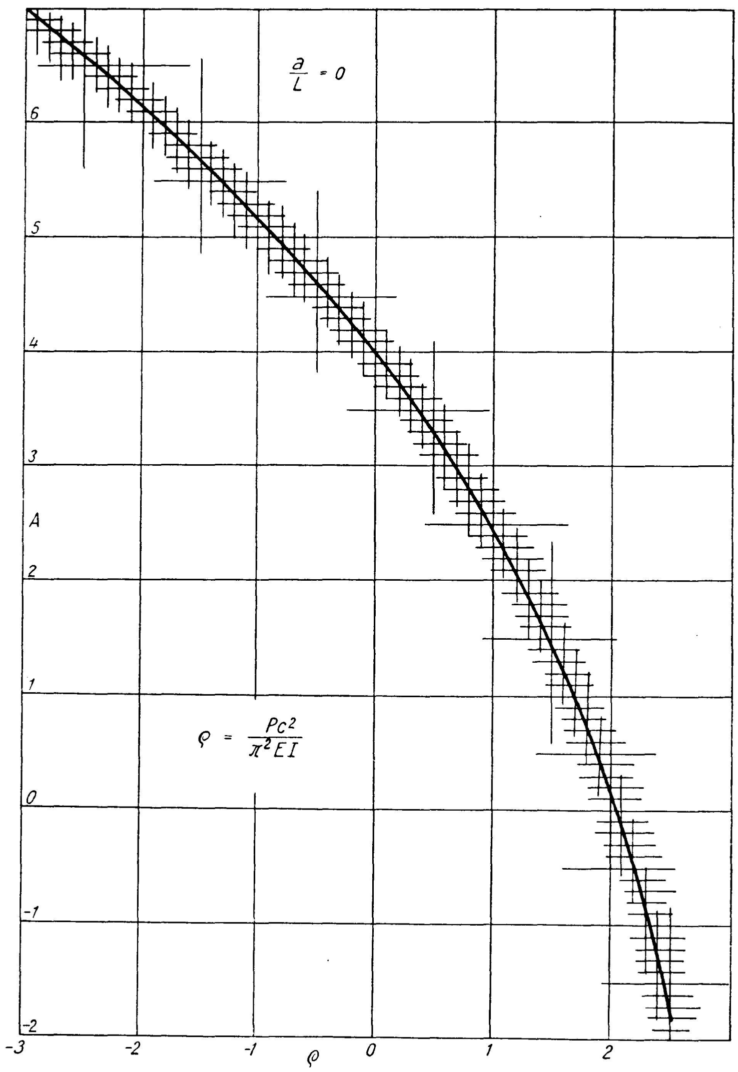

Fig. 4a. Values of A for $a/L = 0$.

The previously mentioned charts (see fig. 4) show A and B as functions of ρ. In fig. 4, a, b, are lengths of gussets, c is distance between gussets.

If the ends of the member are displaced in the transverse direction by the relative amount Δ, as shown in fig. 2, so that the chord of the member is rotated through a positive angle Δ/L, eq. (1) becomes

$$M_{ab} = K\left[A\,\theta_a + B\,\theta_b - (A+B)\frac{\Delta}{L}\right]. \tag{3}$$

Consider a frame, such as shown in fig. 3. The axial loads, P_{ab}, P_{cd}, P_{ef}, on the columns are assumed to be known, and the lateral displacement, Δ, is specified arbitrarily. It is required to find the magnitude of the horizontal force, S, which is required to produce the displacement, Δ. Since Δ may be taken as small as one wishes, S can be made so small that its effect upon the flexural stiffness of the girders is negligible. The basic slope deflection constants therefore apply to the girders, and the moments at the ends of the girders are:

$$\begin{aligned} M_{ac} &= K_{ac}(4\,\theta_a + 2\,\theta_c),\\ M_{ca} &= K_{ac}(4\,\theta_c + 2\,\theta_a),\\ M_{ce} &= K_{ce}(4\,\theta_c + 2\,\theta_e),\\ M_{ec} &= K_{ce}(4\,\theta_e + 2\,\theta_c). \end{aligned} \tag{4}$$

By eq. (3), the moments at the upper ends of the columns are

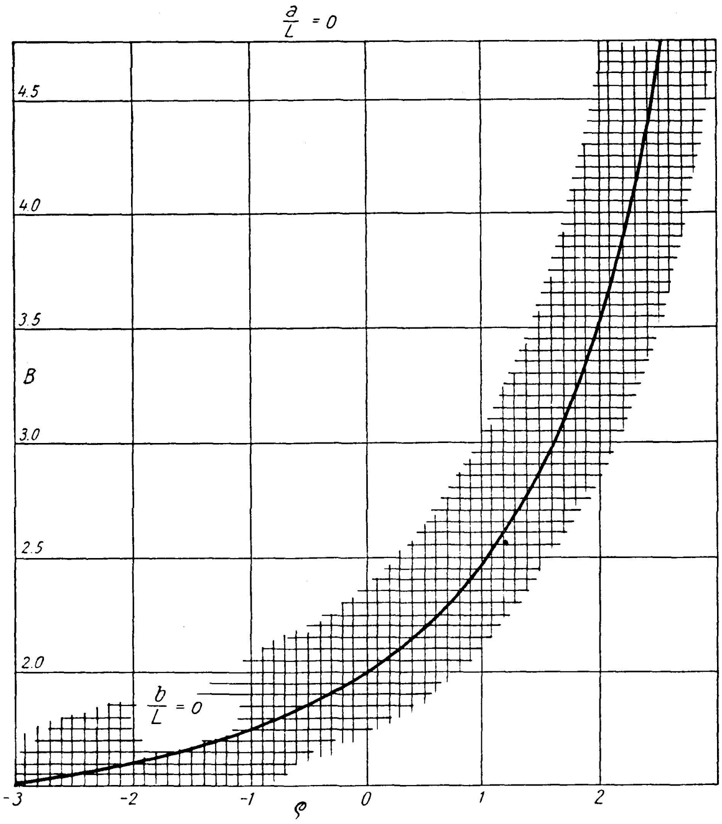

Fig. 4b. Values of B for $a/L = 0$.

$$
\begin{aligned}
M_{ab} &= K_{ab}\left[A_{ab}\,\theta_a - (A_{ab}+B_{ab})\frac{\Delta}{L_{ab}}\right], \\
M_{cd} &= K_{cd}\left[A_{cd}\,\theta_c - (A_{cd}+B_{cd})\frac{\Delta}{L_{cd}}\right], \\
M_{ef} &= K_{ef}\left[A_{ef}\,\theta_e - (A_{ef}+B_{ef})\frac{\Delta}{L_{ef}}\right],
\end{aligned}
\tag{5}
$$

where the A's and B's may be taken from fig. 4 or from similar charts.

At each joint, the sum of the moments must be zero and therefore

$$
\begin{aligned}
&M_{ab} + M_{ac} = 0, \\
&M_{ca} + M_{cd} + M_{ce} = 0, \\
&M_{ec} + M_{ef} = 0.
\end{aligned}
\tag{6}
$$

Substituting eqs. (4) and (5) into eqs. (6) yields

$$
\begin{aligned}
(A_{ab}K_{ab}+4\,K_{ac})\,\theta_a + 2\,K_{ac}\,\theta_c &= K_{ab}\,(A_{ab}+B_{ab})\frac{\Delta}{L_{ab}}, \\
(A_{cd}K_{cd}+4\,K_{ac}+4\,K_{ce})\,\theta_c + 2\,K_{ac}\,\theta_a + 2\,K_{ce}\,\theta_e &= K_{cd}\,(A_{cd}+B_{cd})\frac{\Delta}{L_{cd}}, \\
(A_{ef}K_{ef}+4\,K_{ce})\,\theta_e + 2\,K_{ce}\,\theta_c &= K_{ef}\,(A_{ef}+B_{ef})\frac{\Delta}{L_{ef}}.
\end{aligned}
\tag{7}
$$

Taking Δ as unity, eqs. (7) are to be solved for the θ's. This may be done either by iteration or by a formal algebraic procedure, e.g., substitution and elimination, determinants, etc.

Having the values of the joint rotations and Δ, the shears in the columns may be computed. For this purpose, consider, for example, column ab as a free body. (See fig. 2.) Taking moments about either end, one obtains

$$
S_{ab} = \frac{1}{L_{ab}}\,(M_{ab}+M_{ba}+P_{ab}\Delta). \tag{8}
$$

The end moments are

$$
\begin{aligned}
M_{ab} &= K_{ab}\left[A_{ab}\,\theta_a - (A_{ab}+B_{ab})\frac{\Delta}{L_{ab}}\right], \\
M_{ba} &= K_{ab}\left[B_{ab}\,\theta_a - (A_{ab}+B_{ab})\frac{\Delta}{L_{ab}}\right].
\end{aligned}
\tag{9}
$$

Substituting eqs. (9) in eq. (8) yields

$$
S_{ab} = \frac{1}{L_{ab}}\left(\theta_a - 2\frac{\Delta}{L_{ab}}\right)(A_{ab}+B_{ab}) + \frac{P_a\Delta}{L_{ab}}. \tag{10}
$$

Similarly

$$
S_{cd} = \frac{1}{L_{cd}}\left(\theta_c - 2\frac{\Delta}{L_{cd}}\right)(A_{cd}+B_{cd}) + \frac{P_c\Delta}{L_{cd}}, \tag{11}
$$

$$
S_{ef} = \frac{1}{L_{ef}}\left(\theta_e - 2\frac{\Delta}{L_{ef}}\right)(A_{ef}+B_{ef}) + \frac{P_e\Delta}{L_{ef}}. \tag{12}
$$

For the complete frame, the external force associated with the linear displacement, $\varDelta$, clearly is

$$S = S_{ab} + S_{cd} + S_{ef} \tag{13}$$

and, in view of eqs. (10, (11), and (12), the shear stiffness or spring rate of the frame may be computed using the specified unit value of $\varDelta$ and the computed values of the joint rotations.

In this manner, the spring rate of each frame is found when the columns are subjected to a given set of vertical loads.

It should be noted that the same approach may be used for frames containing any number of columns. Also, the modifications for conditions other than that of full fixity at the bases of the columns are easily deduced. For example, if the lower end of column $a\,b$ is pinned

$$M_{ba} = K_{ab}\left[A_{ab}\,\theta_b + B_{ab}\,\theta_a - (A_{ab} + B_{ab})\frac{\varDelta}{L_{ab}}\right] = 0$$

and

$$M_{ab} = K_{ab}\left[A_{ab}\,\theta_a + B_{ab}\,\theta_b - (A_{ab} + B_{ab})\frac{\varDelta}{L_{ab}}\right].$$

By the first of these equations

$$\theta_b = \frac{(A_{ab} + B_{ab})}{A_{ab}}\,\frac{\varDelta}{L_{ab}} - \frac{B_{ab}}{A_{ab}}\,\theta_a$$

and therefore

$$M_{ab} = K_{ab}\left(A_{ab} - \frac{B_{ab}^2}{A_{ab}}\right)\left(\theta_a - \frac{\varDelta}{L_{ab}}\right). \tag{14}$$

Corresponding modifications may be made in eqs. (7) and (10).

Characteristics of the Supporting System

The supporting system comprises the horizontal bracing system together with the end-walls which carry the bracing system. To these components of the bracing system we may add those frames which are less than critically loaded, i. e., those for which $S/\varDelta$ is a negative quantity, with S computed by eq. (13) or its generalization. The bracing system itself may be a truss, generally but not necessarily lying in a horizontal plane, a roof slab or other roof construction, flexural members (for example a set of continuous purlins), or a combination of such elements.

The supporting system, as defined above, may be treated as an elastic structure and, by use of the conventional techniques for computing deflections, the influence coefficients may be determined for deflections at the location of each critically loaded bent. These influence coefficients are designated a_{ij} and are defined as the deflection at the i-th point due to a unit load applied at the j-th point.

In a buckled configuration, the critically loaded frames exert a set of horizontal forces upon the supporting structure, these forces having the magnitudes

$$F_j = S_j y_j, \quad \text{where} \quad S_j = \frac{S}{\Delta}. \tag{15}$$

If the column loads throughout the entire structure are precisely such that the structure has reached its stability limit, then the structure will be in equilibrium in the buckled configuration. Defining the buckled configuration in terms of the displacements of the supporting structure, the condition of equilibrium is

$$y_i = \sum_{j=1}^{r} a_{ij} F_j, \quad i = 1, \ldots, r, \tag{16}$$

where r is the number of critically loaded frames. Using eq. (15), this may be written

$$\sum_{j=1}^{r} a_{ij} S_j y_j = y_i. \tag{17}$$

The Stability Criterion

Since the shearing stiffnesses, S_j, are functions of the column loads on the various frames, eqs. (17) define the magnitudes of the column loads for which the structure is just stable with respect to lateral or sidesway buckling. Thus, if the magnitudes of the axial loads are not specified *a priori*, eqs. (17) form an eigenvalue problem which yields the permissible values of these loads. The relations between the column loads and the shearing stiffnesses are, however, so transcendental that the direct determination of the permissible values of the column loads from eqs. (17) generally is not practical. One may note, in passing, that if the column loads are specified, eqs. (17) no longer form an eigenvalue problem but do, in fact, constitute an overdeterminate system.

Fortunately, from an engineering viewpoint, the central question frequently can be phrased in other ways such that obtaining an answer to the paraphrased buckling problem may be a much less tedious operation. If the central question is whether or not the given structure will buckle laterally under the specified loads, a modification of eqs. (17) is possible which puts these in the form of an eigenvalue problem for which the desired solution may be obtained in a relatively simple manner.

The central question may be replaced temporarily by the problem of determining the stiffness of a supporting structure which would prevent buckling of the loaded bents. In particular, it is desired to find the factor, μ, by which the stiffness of the existing structure must be multiplied to provide the required support. If the stiffness is multiplied by this factor, it follows that the influence coefficients are reduced in the same ratio. The pertinent equations now are deduced from eqs. (17) and are written

$$\sum_{j=1}^{r} \frac{a_{ij}}{\mu} S_j y_j = y_i, \qquad i = 1, \ldots, r \tag{18}$$

or

$$\sum_{j=1}^{r} a_{ij} S_j y_j = \mu y_i, \quad i = 1, \ldots, r. \tag{19}$$

These are the equations of an eigenvalue problem in which μ is the eigenvalue to be determined. The relative values of the y's which satisfy eqs. (18) may be found either by the formal methods for determining an eigenvalue and an eigenvector or by an iterative procedure. Generally the iterative procedure is more attractive.

In using the iterative technique, one assumes an initial set of values for the y_j's and substitutes into eqs. (19). This operations yields a second, and ordinarily more correct, set of numbers proportional to the deflections at the several points. With the initial set and the computed set normalized to unity at the same point, the sets are compared point for point. If the agreement is satisfactory, no further iterations will be necessary, and the value of μ may be computed. If the agreement is not satisfactory, the iterative procedure is repeated a sufficient number of cycles until satisfactory agreement is obtained.

At this stage, the deflections may be assumed to have converged to their correct relative values. The value of μ is now computed as the ratio of two successive values of the deflection at a particular point by the formula

$$\mu = \frac{y_i^{(n+1)}}{y_i^{(n)}}, \tag{20}$$

in which $y_i^{(n+1)}$ is the computed value of the last cycle before normalizing.

Recalling that μ is the factor by which the stiffness of the supporting structure must be multiplied to provide precisely the proper support for the imposed loads, the stability or instability of the actual structure under the prescribed or assumed pattern of loads now can be deduced from the computed value of μ since,

$$\begin{aligned} &\text{for stability} && \mu \leqq 1, \\ &\text{and, for instability} && \mu > 1. \end{aligned} \tag{21}$$

Example 1

As a simple illustrative example, we consider a structure consisting of four identical steel frames, with a steel bracing system and masonry end walls as shown in fig. 5. For the purpose of obtaining a simple example, it is assumed that all columns are identical and that they carry equal axial loads. The problem is to determine the magnitude of the column loads at which a lurching mode of collapse will occur. The transverse horizontal deflections, a_{ij}, of the bracing system due to unit loads applied in turn at the location of each frame have been computed by the Maxwell-Mohr Method for trussed systems and are listed in the table below. Shear deformation of the masonry end walls is so small as to be practically negligible, accounting for only 0.1 per cent to 0.3 per cent of the deflection. No difficulty would be encountered in including the deformations of less stiff end constructions such as braced frames or metal covered frames.

Table 2. Numerical Example. Deflections of Bracing System Due to Unit Loads, or Influence Coefficients, a_{ij}

Location of Unit Load	Influence Coefficients *)			
	Frame 1	Frame 2	Frame 3	Frame 4
Frame 1	127.266	98.200	67.279	46.180
Frame 2	98.200	194.440	135.584	67.279
Frame 3	67.279	135.584	197.415	117.885
Frame 4	46.180	67.279	117.885	127.885

*) Influence Coefficients, a_{ij}, are in units of 10^{-6} in./lb.

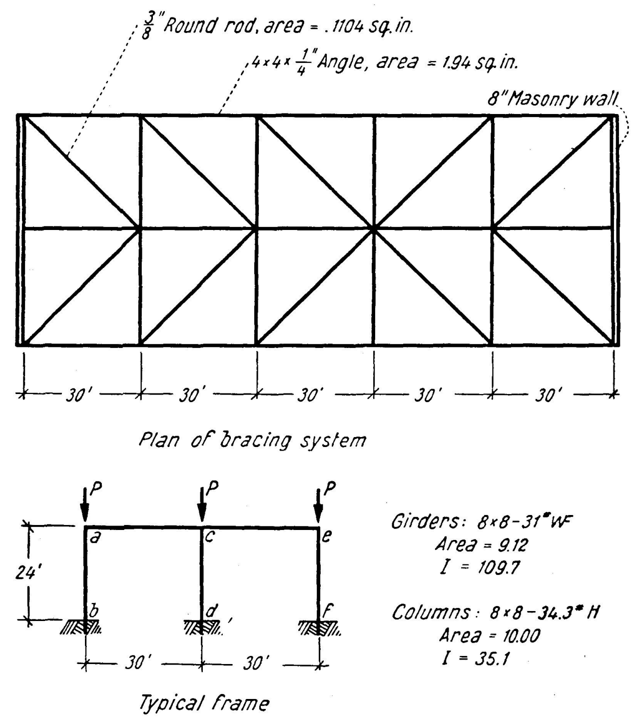

Fig. 5. Illustrative Example.

The Euler load of each column is

$$P_e = \frac{\pi^2 E I}{L^2} = \frac{\pi^2 \times 30 \times 10^6 \times 35.1}{(288)^2} = 125{,}303 \text{ lbs.}$$

For a first trial we assume that the axial load on each column is

$$P = 2.2\,P_e = 275{,}667 \text{ lbs.}$$

Then, for each column, from fig. 4 or from the table,

$$\begin{aligned} \rho &= 2.2, \\ A &= -0.519, \\ B &= 3.901, \\ A+B &= 3.382, \\ L &= 288. \end{aligned}$$

Since $\theta_b = \theta_d = \theta_f = 0$, and $\theta_a = \theta_e$, the bending moments at the ends of all members may be written by use of eqs. (4) and (5) as

$$\begin{aligned} M_{ab} = M_{ef} &= 3.565 \times 10^6 \left(-0.519\,\theta_c - \frac{3.382}{288} \right), \\ M_{ba} = M_{fe} &= 3.656 \times 10^6 \left(3.901\,\theta_a - \frac{3.382}{288} \right), \\ M_{cd} &= 3.656 \times 10^6 \left(-0.519\,\theta_c - \frac{3.382}{288} \right), \\ M_{dc} &= 3.656 \times 10^6 \left(3.901\,\theta_c - \frac{3.382}{288} \right), \\ M_{ac} = M_{ec} &= 9.142 \times 10^6 (4\,\theta_a + 2\,\theta_c), \\ M_{ca} = M_{ce} &= 9.142 \times 10^6 (4\,\theta_c + 2\,\theta_a). \end{aligned}$$

The equilibrium equations at the joints are

$$\begin{aligned} M_{ab} + M_{ac} &= 0, \\ M_{ca} + M_{cd} + M_{ce} &= 0. \end{aligned}$$

Substituting the expressions for the moments as given above yields the pair of simultaneous equations

$$\begin{aligned} 34.670\,\theta_a + 18.284\,\theta_c &= 0.042933, \\ 36.568\,\theta_a + 71.238\,\theta_c &= 0.042933. \end{aligned}$$

The solution of these equations is

$$\begin{aligned} \theta_a &= 1.2621 \times 10^{-3} \text{ radians}, \\ \theta_c &= -0.04524 \times 10^{-3} \text{ radians}. \end{aligned}$$

The moments at the ends of the several members are obtained by substituting these values into the expressions previously listed, and are

$$M_{ab} = M_{ef} = 3.656 \times 10^6 \left(-0.519 \times 1.2621 \times 10^{-3} - \frac{3.382}{288} \right) = -45{,}328,$$

$$M_{ba} = 3.656 \times 10^6 \left(3.901 \times 1.2621 \times 10^{-3} - \frac{3.382}{288} \right) = -24{,}933,$$

$$M_{cd} = 3.656 \times 10^6 \left(0.519 \times 0.04525 \times 10^{-3} - \frac{3.382}{288} \right) = -42{,}847,$$

$$M_{dc} = 3.656 \times 10^6 \left(-3.901 \times 0.04524 \times 10^{-3} - \frac{3.382}{288} \right) = -43{,}578,$$

$$M_{ac} = 9.142 \times 10^6 (4 \times 1.2621 \times 10^{-3} - 2 \times 0.04524 \times 10^{-3}) = 45{,}325,$$

$$M_{ca} = 9.142 \times 10^6 (4 \times 0.04524 \times 10^{-3} + 2 \times 1.2621 \times 10^{-3}) = 21{,}422.$$

By eq. (8), the column shears are

$$S_{ab} = S_{ef} = \frac{1}{L_{ab}} (M_{ab} + M_{ba} + P\,\Delta)$$

$$= \frac{1}{288} (-45{,}328 - 24{,}933 + 275{,}667) = 713 \text{ lbs.}$$

$$S_{cd} = \frac{1}{288} (-43{,}578 - 42{,}847 + 275{,}667) = 657 \text{ lbs.}$$

For the complete frame under the assumed loads, the shear force is

$$S = S_{ab} + S_{cd} + S_{ef} = 2083 \text{ lbs.}$$

Since S is positive, the frame is unstable under the assumed loads unless externally braced.

Eq. (19) for the lateral deflections of the bracing system are

$$\Sigma a_{ij} S_j y_j = \mu y_i .$$

Using the values of a_{ij} previously listed, and the value of $S = 2083$ lbs. for each frame, these equations become

$$0.26510\, y_1 + 0.20455\, y_2 + 0.14014\, y_3 + 0.09619\, y_4 = \mu y_1,$$
$$0.20455\, y_1 + 0.40606\, y_2 + 0.27826\, y_3 + 0.14014\, y_4 = \mu y_2,$$
$$0.14014\, y_1 + 0.27826\, y_2 + 0.41122\, y_3 + 0.24555\, y_4 = \mu y_3,$$
$$0.09619\, y_1 + 0.14014\, y_2 + 0.24555\, y_3 + 0.26638\, y_4 = \mu y_4.$$

We now assume an arbitrary set of y's, substitute these into the left-hand sides of the above set of equations and compute a corresponding set of products μy_i. The corresponding set of y's may be normalized, and again substituted into the left-hand sides of the above equations to obtain a new set of results. This process may be repeated until an acceptable degree of convergence is attained. In assuming an initial set of y's, it is naturally advantageous to select values which have relative magnitudes as close to the final normalized set as possible. The process will converge, however, regardless of the correctness of the initial set.

For an initial set, we assume the values

$$y_1 = 0.600,$$
$$y_2 = 1.00,$$
$$y_3 = 1.00,$$
$$y_4 = 0.60.$$

Substituting into the equations we obtain

$$0.26510 \times 0.600 + 0.20455 \times 1.000 + 0.14014 \times 1.000 + 0.09619 \times 0.600 = 0.5615,$$
$$0.20455 \times 0.600 + 0.40606 \times 1.000 + 0.27826 \times 1.000 + 0.14014 \times 0.600 = 0.8911,$$
$$0.14014 \times 0.600 + 0.27826 \times 1.000 + 0.41122 \times 1.000 + 0.24555 \times 0.600 = 0.9209,$$
$$0.09619 \times 0.600 + 0.14014 \times 1.000 + 0.24555 \times 1.000 + 0.26638 \times 0.600 = 0.6032.$$

It is convenient to normalize these results, thus obtaining the following set of y's,

$$y_1 = \frac{0.5615}{0.9209} = 0.6097, \qquad y_2 = \frac{0.8911}{0.9209} = 0.9676,$$
$$y_3 = \frac{0.9209}{0.9209} = 1.000, \qquad y_4 = \frac{0.6032}{0.9209} = 0.6550.$$

These values are used to obtain a second calculated set of y's, and the process is repeated until convergence is obtained. The results of these iterative operations are presented in the following table.

The deflections obtained in the fifth cycle evidently have converged to the correct values. As a matter of fact, it appears that the results of the second or third cycle would have been sufficiently good for design and analysis purposes.

Table 3. Example 1. Iterated Values of Deflections

	y_1	y_2	y_3	y_4
Trial Values	0.6000	1.0000	1.0000	0.6000
1st cycle	0.5615	0.8911	0.9209	0.6032
Normalized	0.6097	0.9676	1.0000	0.6550
2nd cycle	0.5627	0.8877	0.9267	0.6113
Normalized	0.6072	0.9579	1.0000	0.6597
3rd cycle	0.5605	0.8839	0.9248	0.6139
Normalized	0.6061	0.9558	1.0000	0.6638
4th cycle	0.5601	0.8834	0.9251	0.6146
Normalized	0.6054	0.9549	1.0000	0.6644
5th cycle	0.5599	0.8830	0.9249	0.6146

The required stiffness factor for the bracing system is now deduced by comparing, at any one of the four frame locations, the calculated result and its last previous value. Thus, using the results of fifth cycle at the third location

$$\mu_1 = \frac{0.9249}{1.0000} = 0.925.$$

It may be easily verified that the same value is obtained from the deflections at the other frames:

$$\text{at Frame 1,} \quad \mu = \frac{0.5599}{0.6054} = 0.925,$$

$$\text{at Frame 2,} \quad \mu = \frac{0.8830}{0.9549} = 0.925,$$

$$\text{at Frame 4,} \quad \mu = \frac{0.6146}{0.6644} = 0.925.$$

The results of the second cycle would have furnished a very satisfactory basis for determining the value of μ with an error of not more than two per-cent. Calculating μ at the several frames by comparing the results of the second cycle to the values which were used in that cycle gives:

$$\text{at Frame 1,} \quad \mu = \frac{0.5627}{0.6097} = 0.923,$$

$$\text{at Frame 2,} \quad \mu = \frac{0.8877}{0.9676} = 0.917,$$

$$\text{at Frame 3,} \quad \mu = \frac{0.9267}{1.0000} = 0.927,$$

$$\text{at Frame 4,} \quad \mu = \frac{0.6113}{0.6550} = 0.933.$$

The correct value of μ will lie[3]) between the upper and lower limits of this set and therefore

$$0.917 \leqq \mu \leqq 0.933.$$

Since a less stiff bracing system is needed to prevent a lurching collapse of the complete structure under the assumed pattern of axial loads, it is clear that our first estimate of 275,667 lbs. on each column is too low. The stiffness factor μ is 7.5 per cent less than unity. This percentage is certainly an upper limit to the additional load which the structure will support. For a second trial, we select a convenient load of 2.4 times the Euler load, or

$$P = 2.4 \times 125{,}303 = 300{,}727 \text{ lbs.}$$

and we will rely upon subsequent interpolation to obtain the critical loads of the actual structure.

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[3]) See H. A. Schwarz, "Gesammelte Werke", Vol. 1.

For each column,

$$\begin{aligned} \rho &= 2.4, \\ A &= -1.300, \\ B &= 4.383, \\ A+B &= 3.083, \\ L &= 288 \text{ inches.} \end{aligned}$$

When the tops of the frames are displaced a unit distance, the bending moments at the ends of the several members of each frame are

$$\begin{aligned} M_{ab} = M_{ef} &= 3.656\times 10^{6}\left(-1.300\,\theta_a - \frac{3.083}{288}\right), \\ M_{ba} = M_{fc} &= 3.656\times 10^{6}\left(\ \ 4.383\,\theta_a - \frac{3.083}{288}\right), \\ M_{cd} &= 3.656\times 10^{6}\left(-1.300\,\theta_c - \frac{3.083}{288}\right), \\ M_{dc} &= 3.656\times 10^{6}\left(\ \ 4.383\,\theta_c - \frac{3.083}{288}\right), \\ M_{ac} = M_{ec} &= 9.142\times 10^{6}\,(4\,\theta_a + 2\,\theta_c), \\ M_{ca} = M_{ce} &= 9.142\times 10^{6}\,(4\,\theta_c + 2\,\theta_a). \end{aligned}$$

The equilibrium equations at the joints are

$$\begin{aligned} M_{ab} + M_{ac} &= 0, \\ M_{ca} + M_{cd} + M_{ce} &= 0, \end{aligned}$$

and become

$$\begin{aligned} 31.815\,\theta_a + 18.284\,\theta_c &= 0.039137, \\ 36.568\,\theta_a + 68.383\,\theta_c &= 0.039137. \end{aligned}$$

The solution of this pair of equations is

$$\begin{aligned} \theta_a &= 1.3011\times 10^{-3} \text{ radians}, \\ \theta_c &= -0.1234\times 10^{-3} \text{ radians}. \end{aligned}$$

The bending moments at the ends of the several members are evaluated as before by substituting these values of the angular displacements into the formulas for the moments, thus obtaining

$$\begin{aligned} M_{ab} = M_{cf} &= -45{,}321 \text{ lb.-ins.} \\ M_{ba} = M_{fc} &= -18{,}288 \text{ lb.-ins.} \\ M_{cd} &= -38{,}551 \text{ lb.-ins.} \\ M_{dc} &= -41{,}114 \text{ lb.-ins.} \\ M_{ac} = M_{ec} &= 45{,}322 \text{ lb.-ins.} \\ M_{ca} = M_{ce} &= 19{,}276 \text{ lb.-ins.} \end{aligned}$$

The shears in the columns are

$$S_{ab} = S_{ef} = \frac{1}{288}(-45{,}321 - 18{,}288 + 300{,}727) = 823 \text{ lbs.}$$

$$S_{cd} = \frac{1}{288}(-41{,}114 - 38{,}551 + 300{,}727) = 768 \text{ lbs.}$$

The sum of the column shears is

$$S = 823 + 768 + 823 = 2414 \text{ lbs.}$$

Each frame would, of course, be unstable with axial loads of 300,727 lbs. on each column unless additional bracing were provided.

Upon multiplying the influence coefficients, a_{ij}, by the shear stiffnesses of the frames, S_j, just computed, eqs. (19) for lateral deflection of the bracing system may be written,

$$\begin{aligned}
0.30722\,y_1 + 0.23705\,y_2 + 0.16241\,y_3 + 0.11148\,y_4 &= \mu\,y_1,\\
0.23705\,y_1 + 0.47059\,y_2 + 0.32247\,y_3 + 0.16241\,y_4 &= \mu\,y_2,\\
0.16241\,y_1 + 0.32247\,y_2 + 0.47656\,y_3 + 0.28457\,y_4 &= \mu\,y_3,\\
0.11148\,y_1 + 0.16241\,y_2 + 0.28457\,y_3 + 0.30871\,y_4 &= \mu\,y_4.
\end{aligned}$$

Beginning with an arbitrary set of y's, the normalized set would be found to converge to the values already obtained with the first trial values of loads. This is because all coefficients on the left hand sides of the above group of equations for the second trial are proportional to those for the first trial. The ratio of the coefficients of the second group to that of the first group is, in fact, merely equal to the ratio of the shear stiffness of the frames in the two cases. Consequently, when all frames are identical and identically loaded, only one iterative solution of the equations need be made; and the stiffness factor, μ_2, for any subsequent trial can be obtained by the formula

$$\mu_2 = \frac{S_2}{S_1}\mu_1,$$

where S_2 is the shear stiffness of the frames under the second set of loads, and S_1 is the shear stiffness of the frames under the loads for which μ_1 was required.

In the present example, therefore,

$$\mu_2 = \frac{2414}{2083}(0.925) = 1.072.$$

Since μ_2 is greater than unity, the structure will collapse under column loads of 300,727 lbs. unless the stiffness of the bracing system is increased by 7.2 per cent.

To determine the loads which the present structure is capable of supporting without failing in a lurching mode of collapse, it is probably sufficiently accurate to interpolate between the results obtained from the two trials which have been made. The value of the critical load obtained by linear interpolation is

$$P_{cr} = 288{,}453 \text{ lbs.} = 2.302\,P_e.$$

If desired, a third trial solution may be made in the neighborhood of $\rho = 2.302$.

Example 2

As a second illustrative example, we determine the critical magnitude of the three equal column loads at Frame 3 when the axial loads on the remaining columns have specified values. Let the specified value of the column loads at Frames 1, 2 and 4 be 275,667 lbs. per column.

For a first trial, we assume that the axial loads on the columns of Frame 3 are also 275,667 lbs. per column. The shear stiffness of each frame under these loads has been determined in Example 1 and was found to be 2083 lbs. per inch. The corresponding characteristic equations were found, in Example 1, to yield a value for the stiffness factor of

$$\mu_1 = 0.925.$$

For a second trial, we assume that the axial loads on the columns of Frame 3 are 300,727 lbs. per column. The shear stiffness of the frame under these loads was found, in Example 1, to be 2414 lbs. per inch. With

$$S_1 = S_2 = S_4 = 2083$$
$$\text{and } S_3 = 2414,$$

eqs. (19) become

$$\begin{aligned}
0.26510\,y_1 + 0.20455\,y_2 + 0.16241\,y_3 + 0.09619\,y_4 &= \mu\, y_1,\\
0.20455\,y_1 + 0.40606\,y_2 + 0.32247\,y_3 + 0.14014\,y_4 &= \mu\, y_2,\\
0.14014\,y_1 + 0.27826\,y_2 + 0.47656\,y_3 + 0.24555\,y_4 &= \mu\, y_3,\\
0.09616\,y_1 + 0.14014\,y_2 + 0.28457\,y_3 + 0.26638\,y_4 &= \mu\, y_4.
\end{aligned}$$

It may be noted that the coefficients of y_1, y_2 and y_4 on the left hand sides of these equations are the same as those used in the first trial solution of Example 1, while the coefficients of y_3 are the same as those used in the second trial solution of Example 1.

Results of iterative solution of these equations are given in the following table.

Comparison of the normalized values of the fourth cycle with the resulting values of the fifth cycle yields the magnitude of the stiffness factor,

$$\mu_2 = \frac{0.5710}{0.5830} = \frac{0.9126}{0.9316} = \frac{0.9797}{1.0000} = \frac{0.6471}{0.6605} = 0.980.$$

As a matter of fact, using Schwarz' inequalities, the second iteration yields

$$0.973 \leqq \mu_2 \leqq 0.986$$

Table 4. Example 2. Iterated Values of Deflections

	y_1	y_2	y_3	y_4
Trial Values	0.60	0.95	1.00	0.66
1st cycle	0.5793	0.9234	0.9771	0.6512
Normalized	0.5929	0.9450	1.0000	0.6665
2nd cycle	0.5770	0.9209	0.9863	0.6516
Normalized	0.5850	0.9337	1.0000	0.6607
3rd cycle	0.5720	0.9139	0.9806	0.6477
Normalized	0.5833	0.9320	1.0000	0.6605
4th cycle	0.5712	0.9128	0.9798	0.6472
Normalized	0.5830	0.9316	1.0000	0.6605
5th cycle	0.5710	0.9126	0.9797	0.6471

while the third iteration yields

$$0.978 \leqq \mu_2 \leqq 0.981.$$

This suggests that the required value of μ_2 accurate to two significant figures is 0.98. On this basis, two iterations would have been sufficient for design purposes.

The critical value of axial loads on the columns of Frame 3 may be obtained by extrapolation:

$$\begin{aligned} \mu_1 &= 0.925, & P_3 &= 275{,}667 \text{ lbs.}, \\ \mu_2 &= 0.980, & P_3 &= 300{,}727 \text{ lbs.}, \\ \mu_3 &= 1.000, & P_3 &= 309{,}840 \text{ lbs.}, \\ & & &= 2.473\ P_e. \end{aligned}$$

A third trial may be made in the neighborhood of $\rho_3 = 2.473$.

Conclusion

A method has been presented for determining the critical values of a lurching collapse of one-story framed structures. The method has been demonstrated by application to a structure consisting of four rectilinear continuous frames supported by a horizontal bracing truss lying in the plane of the upper members of the frames. The method may be applied to structures which include other types of frames, for example, pin-connected frames and frames in which the upper portions are roof trusses. Also, the bracing system need not be a truss, but may, for example, consist of diaphragms or of flexural members. Frames in the structure need not be identical or identically loaded. In fact, the method may be applied when some of the frames are less than critically loaded and therefore contribute to the stability of the critically loaded frames.

In some cases the critical loads may exceed the yield strengths of the columns. If the material of the columns does not have a sharply defined yield point, it is suggested that the tangent modulus be used in computing the flexural stiffnesses of the corresponding members.

It may be again mentioned that the shear stiffness of the frames can be determined by other methods in addition to the slope deflection method which was used in the examples. Those who are most familiar, for example, with the moment distribution method may prefer to use that approach. It may be mentioned also, that in the case of regular frames such as were used in the illustrative problem, the slope deflection equations and their solution may be reduced to a completely tabular scheme for each frame, and that such a scheme is particularly well suited to office use.

Summary

A theory of general three-dimensional instability of one-story framed building structures is presented. The individual parallel plane frames of the structure are assumed to be coupled by an elastic bracing system which, in turn, is supported by elastic walls or frames.

A lurching mode of general coolapse is investigated by a new method in which the prediction of stability or instability under a given set of loads is reduced to a small sequence of computationally simple problems. The stiffnesses of each frame under its applied loads are first determined by use of a generalized slope-deflection method or by an equivalent alternate method, and the deflection influence coefficients of the bracing system are also computed. These data are used to construct a set of simultaneous, linear, algebraic equations which may be put in matrix form if desired. Finally, an iterative process is applied to these equations or to their corresponding matrix to determine whether or not the structure buckle.

Résumé

L'auteur expose une théorie de l'instabilité générale à trois dimensions des constructions composées de portiques à un étage. Chaque portique parallèle est admis relié aux autres par un contreventement qui, de son côté, s'appuie sur une paroi ou une ossature élastique.

On examine le cas de ruine générale par instabilité latérale à l'aide d'une nouvelle méthode. Le détermination du critère de stabilité ou d'instabilité s'y réduit à une courte série de problèmes simples à résoudre. On détermine d'abord à l'aide d'une méthode des déformations généralisée, ou d'une autre méthode équivalente, la rigidité de chaque cadre sous les charges appliquées;

on calcule également les coefficients de déformation du contreventement. Ces bases servent à l'établissement d'un système d'équations algébriques linéaires qui peuvent être écrites aussi sous forme de matrice. A l'aide d'un procédé par itération, appliqué à ces équations ou à la matrice correspondante, on détermine enfin si la construction va flamber ou non.

Zusammenfassung

Es wird eine Theorie der allgemeinen, dreidimensionalen Instabilität einstöckiger Rahmenkonstruktionen beschrieben. Dabei wird von den einzelnen parallelen Rahmen angenommen, sie seien unter sich durch einen elastischen Verband gehalten, der sich seinerzeit auf elastische Wände oder Rahmen stützt.

Mit einer neuen Methode wird ein Versagen durch seitliches Ausweichen untersucht, wobei die Beurteilung der Stabilität oder Instabilität unter einer bestimmten Belastungsanordnung auf eine kleine Folge rechnerisch einfacher Operationen reduziert wird. Zuerst werden unter Anwendung einer verallgemeinerten Deformationsmethode oder einer ähnlichen Methode die Steifigkeiten eines jeden Rahmens unter seiner Nutzlast bestimmt. Ebenso werden die Formänderungskoeffizienten des querstützenden Verbandes bestimmt. Diese Ergebnisse werden zur Bildung eines Systems von simultanen, linearen Gleichungen verwendet, das allenfalls auch in Matrixenform angegeben werden kann. Durch ein Iterationsverfahren entscheidet man schließlich mit Hilfe dieser Gleichungen oder der entsprechenden Matrix, ob das Bauwerk stabil ist oder nicht.