Lateral and torsional buckling of thin-walled beams

Lateral and Torsional Buckling of Thin-Walled Beams

Déversement et flambage par torsion des poutres à parois minces

Kippen und Biegedrillknicken dünnwandiger Träger

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Introduction

A problem of occasional, and perhaps increasing, interest is that of determining the buckling strength of thin-walled members when subjected to longitudinal compression, to longitudinal bending, or to combinations of these. In the usual formulation of these problems [1][1]), it is assumed that the cross sections do not deform in their own planes during buckling and, therefore, that the buckling mode is characterized by lateral bending of the entire member or by a combination of lateral bending and torsion. Even with this simplifying assumption, the determination of the loading at which the member will fail often is an inordinately tedious task for unusual or irregular sections. The use of a computer in making this determination is therefore justified, and it is also true that a more comprehensive theory may, with a computer, be employed.

The purpose of the present paper is to report on a method for determining critical loads which, in addition to general bending and torsion during buckling, includes the effect of deformation of the cross sections in their own planes. The method has been developed primarily for use with an electronic digital computer.

Members must be prismatic and are considered as comprising a sequence of longitudinal strips or elements, each of which extends the entire length of the member (Fig. 1). There may be as many elements as necessary to represent or to approximate the cross section. Thickness and material properties may vary from strip to strip as well as within strips. Thus, certain composite con-

[1]) Numbers in brackets refer to items in the List of References.

structions may be treated. Likewise, buckling which occurs when all or a portion of the cross section is stressed above the proportional limit may be handled if a suitable stress-strain law (e. g., the tangent modulus law) is postulated. The cross section may be open or closed and need not have an axis of symmetry. For clarity, the member and its elements are assumed to be simply-supported at the two ends. However, the procedure may be extended to certain other end conditions.

If the initial curvature of the member before buckling may be neglected, each of the elements may be treated as a flat plate. On the other hand, if initial curvature cannot be neglected, each strip may be treated either as a segment of a conical shell, a segment of a cylinder, or a sector of a circular plate, depending upon its inclination to the plane of the neutral axis. Because of limitations of space, equations are presented only for the case in which initial curvature may be neglected.

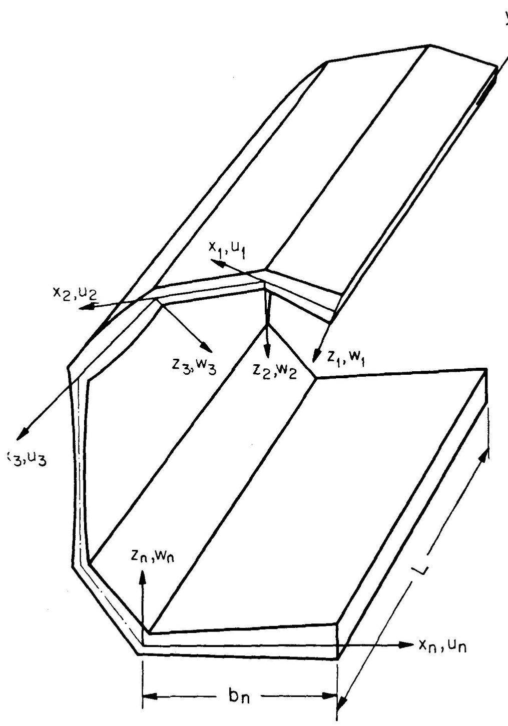

Fig. 1. Typical thin-walled section.

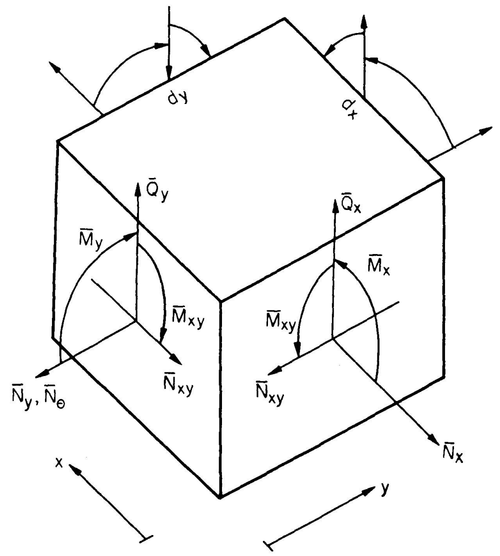

Fig. 2. Differential element showing tractions.

Theory

The member is assumed to be subjected to a combination of axial load and pure bending moments in a specified ratio. These produce longitudinal membrane forces, N_0 (average stress times thickness), which may vary over the cross section.

For a flat plate element subjected to initial longitudinal membrane forces,

N_0, due to axial load and primary bending, the differential equations of equilibrium in the buckled state may be taken as [1]

$$
\begin{aligned}
&\frac{\partial \bar{N}_x}{\partial x}+\frac{\partial \bar{N}_{xy}}{\partial y}-N_0\left(\frac{\partial^2 \bar{v}}{\partial x\,\partial y}-\frac{\partial^2 \bar{u}}{\partial y^2}\right)=0, \\
&\frac{\partial \bar{N}_y}{\partial y}+\frac{\partial \bar{N}_{xy}}{\partial x}=0, \qquad \frac{\partial \bar{Q}_x}{\partial x}+\frac{\partial \bar{Q}_y}{\partial y}+N_0\frac{\partial^2 \bar{w}}{\partial y^2}=0, \\
&\frac{\partial \bar{M}_{xy}}{\partial x}-\frac{\partial \bar{M}_y}{\partial y}+\bar{Q}_y=0, \qquad \frac{\partial \bar{M}_{xy}}{\partial y}-\frac{\partial \bar{M}_x}{\partial x}+\bar{Q}_x=0.
\end{aligned}
\tag{1}
$$

The relations between the stress resultants and displacements are given by [1]

$$
\begin{aligned}
\bar{N}_x &= B\left(\frac{\partial \bar{u}}{\partial x}+\nu\frac{\partial \bar{v}}{\partial y}\right), & \bar{M}_x &= -D\left(\frac{\partial^2 \bar{w}}{\partial x^2}+\nu\frac{\partial^2 \bar{w}}{\partial y^2}\right), \\
\bar{N}_y &= B\left(\frac{\partial \bar{v}}{\partial y}+\nu\frac{\partial \bar{u}}{\partial x}\right), & \bar{M}_y &= -D\left(\frac{\partial^2 \bar{w}}{\partial y^2}+\nu\frac{\partial^2 \bar{w}}{\partial x^2}\right), \\
\bar{N}_{xy} &= C\left(\frac{\partial \bar{u}}{\partial y}+\nu\frac{\partial \bar{v}}{\partial x}\right), & \bar{M}_{xy} &= D(1-\nu)\frac{\partial^2 \bar{w}}{\partial x\,\partial y},
\end{aligned}
\tag{2}
$$

where

$$B=\frac{E\,h}{1-\nu^2}, \qquad C=\frac{E\,h}{2(1+\nu)}, \qquad D=\frac{E\,h^3}{12(1-\nu^2)}.$$

It is convenient to introduce two new dependent variables; namely, the slope in the x-direction denoted by $\bar{S}$, and the effective or Kirchhoff shear, $\bar{V}_x$:

$$\bar{S}=\frac{\partial \bar{w}}{\partial x}, \qquad \bar{V}_x=\bar{Q}_x-\frac{\partial \bar{M}_{xy}}{\partial y}. \tag{3}$$

Since we have assumed a member simply-supported at its ends, we may now assume the dependent variables to vary harmonically with y; thus,

$$\bar{u}(x,y)=u(x)\sin\frac{\pi y}{L}$$

and similarly for $\bar{w}$, $\bar{S}$, $\bar{N}_x$, $\bar{N}_y$, $\bar{M}_x$, $\bar{M}_y$, $\bar{Q}_x$, and $\bar{V}_x$; while

$$\bar{v}(x,y)=v(x)\cos\frac{\pi y}{L}$$

and similarly for $\bar{N}_{xy}$, $\bar{M}_{xy}$ and $\bar{Q}_y$.

It is readily seen that

$$V_x=Q_x+\frac{\pi}{L}M_{xy}.$$

Substitution of the assumed form of the dependent variables into (1) and (2) reduces these equations to

$$\frac{dN_x}{dx}-\frac{\pi}{L}N_{xy}+N_0\frac{\pi}{L}\left(\frac{dv}{dx}-\frac{\pi}{L}u\right)=0, \qquad \frac{dN_{xy}}{dx}+\frac{\pi}{L}N_y=0,$$
$$\frac{dQ_x}{dx}-\frac{\pi}{L}Q_y-N_0\left(\frac{\pi}{L}\right)^2 w=0, \qquad \frac{dM_{xy}}{dx}-\frac{\pi}{L}M_y+Q_y=0, \tag{4}$$
$$\frac{dM_x}{dx}+\frac{\pi}{L}M_{xy}-Q_x=0$$

and
$$N_x=B\left(\frac{du}{dx}-\nu\frac{\pi}{L}v\right), \qquad M_x=-D\left[\frac{dS}{dx}-\nu\left(\frac{\pi}{L}\right)^2 w\right],$$
$$N_y=B\left(\nu\frac{du}{dx}-\frac{\pi}{L}v\right), \qquad M_y=D\left[\left(\frac{\pi}{L}\right)^2 w-\nu\frac{dS}{dx}\right], \tag{5}$$
$$N_{xy}=C\left(\frac{\pi}{L}u+\frac{dv}{dx}\right), \qquad M_{xy}=D(1-\nu)\left(\frac{\pi}{L}\right)S.$$

From (5) we obtain

$$N_y=\nu N_x-B\frac{\pi}{L}(1-\nu^2)\,v; \qquad M_y=\nu M_x+D(1-\nu^2)\left(\frac{\pi}{L}\right)^2 w. \tag{6}$$

Eliminating Q_y between the third and fourth of (4) yields

$$\frac{dQ_x}{dx}+\frac{\pi}{L}\frac{dM_{xy}}{dx}-\left(\frac{\pi}{L}\right)^2 M_y-N_0\left(\frac{\pi}{L}\right)^2 w=0. \tag{7}$$

The last of (4) may be written as

$$\frac{dM_x}{dx}-\left(Q_x+\frac{\pi}{L}M_{xy}\right)+2\frac{\pi}{L}M_{xy}=0. \tag{8}$$

From the foregoing we obtain the following system of eight first-order simultaneous differential equations:

$$\begin{aligned}
\frac{dw}{dx}&=S, \qquad \frac{dS}{dx}=-\frac{M_x}{D}+\nu\left(\frac{\pi}{L}\right)^2 w,\\
\frac{du}{dx}&=\frac{N_x}{B}+\nu\frac{\pi}{L}v, \qquad \frac{dv}{dx}=\frac{N_{xy}}{C}-\frac{\pi}{L}u,\\
\frac{dN_x}{dx}&=\frac{\pi}{L}\left(1-\frac{N_0}{C}\right)N_{xy}+2\left(\frac{\pi}{L}\right)^2 N_0 u,\\
\frac{dN_{xy}}{dx}&=B\left(\frac{\pi}{L}\right)^2(1-\nu^2)\,v-\nu\left(\frac{\pi}{L}\right)N_x,\\
\frac{dV_x}{dx}&=\nu\left(\frac{\pi}{L}\right)^2 M_x+\left(\frac{\pi}{L}\right)^2\left[D(1-\nu^2)\left(\frac{\pi}{L}\right)^2+N_0\right]w,\\
\frac{dM_x}{dx}&=V_x-2D(1-\nu)\left(\frac{\pi}{L}\right)^2 S.
\end{aligned} \tag{9}$$

It may be noticed that no derivatives of properties appear in (9). Nevertheless, these equations are valid for strips in which the thickness and material properties may vary with x.

In addition to (9), we require transformations at the common edge or joint between two successive strips as shown in Fig. 3. It is easily seen that these transformations are

$$\begin{aligned} S^+ &= S^-, & N_{xy}^+ &= N_{xy}^-, \\ v^+ &= v^-, & M_x^+ &= M_x^-, \\ u^+ &= u^- \cos\alpha + w^- \sin\alpha, & w^+ &= w^- \cos\alpha - u^- \sin\alpha, \\ N_x^+ &= N_x^- \cos\alpha + V_x^- \sin\alpha, & V_x^+ &= V_x^- \cos\alpha - N_x^- \sin\alpha. \end{aligned} \tag{10}$$

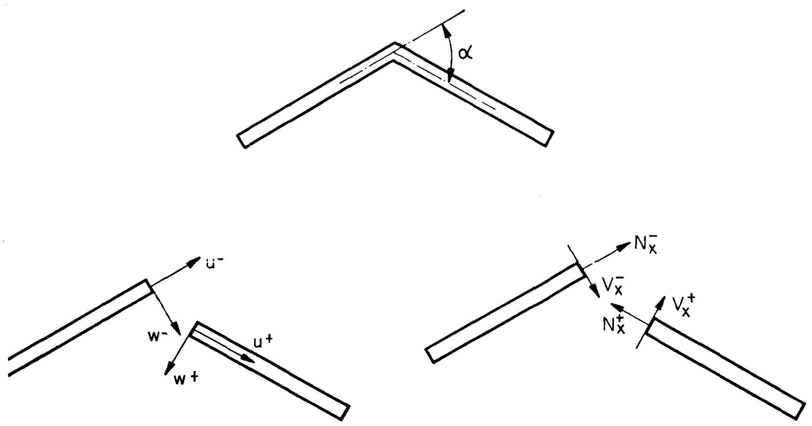

Fig. 3. Transformations at Joint.

Eqs. (9) form a system of eighth order which, when specified values of N_0 are inserted, and with arbitrary initial values, may be integrated numerically from the edge $x_1 = 0$ to the opposite edge. This may be done using any suitable technique, such as the Runge-Kutta fourth-order process [2], applying the transformations (10) where necessary. These differential equations are subject to four boundary conditions at the edge $x_1 = 0$ and four additional boundary conditions at the opposite edge of the section, $x_n = b_n$. For unsupported sections such as shown in Fig. 1, the conditions at $x_1 = 0$ clearly are

$$N_x = N_{xy} = V_x = M_x = 0. \tag{11}$$

No difficulty arises, however, if more complicated conditions exist at the initial boundary such as, for example, an elastically built-in edge.

Assuming a section such as shown in Fig. 1, for which (11) apply, we observe that the four components of displacement at the boundary $x_1 = 0$ are not known a priori but, if N_0 has its critical values throughout, must be such as to satisfy the correct boundary conditions at the opposite edge.

In order to investigate the question of satisfying the four terminal boundary conditions, we transform our problem into a linear combination of initial value problems. Taking, as an example, the section with a free edge at $x_1 = 0$, we integrate (9) four times with four linearly independent sets of initial values for the displacements, taking these, for convenience, according to Table 1 and taking the initial values of the stress resultants according to (11) for each case.

Table 1

Case	Initial Values			
	u	v	w	S
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

Now, each of the four cases yields numerical values for the complete set of eight dependent variables at the opposite edge as listed in Table 2.

Table 2

Case	Values at $x_n = b_n$							
	u	v	w	S	N_x	N_{xy}	M_x	V_x
1	$u^{(1)}$	$v^{(1)}$	$w^{(1)}$	$S^{(1)}$	$N_x^{(1)}$	$N_{xy}^{(1)}$	$M_x^{(1)}$	$V_x^{(1)}$
2	$u^{(2)}$	$v^{(2)}$	$w^{(2)}$	$S^{(2)}$	$N_x^{(2)}$	$N_{xy}^{(2)}$	$M_x^{(2)}$	$V_x^{(2)}$
3	$u^{(3)}$	$v^{(3)}$	$w^{(3)}$	$S^{(3)}$	$N_x^{(3)}$	$N_{xy}^{(3)}$	$M_x^{(3)}$	$V_x^{(3)}$
4	$u^{(4)}$	$v^{(4)}$	$w^{(4)}$	$S^{(4)}$	$N_x^{(4)}$	$N_{xy}^{(4)}$	$M_x^{(4)}$	$V_x^{(4)}$

If the values of N_0 correspond exactly to a critical combination of loads, it would be possible to find a linear combination of Cases 1 to 4 which would satisfy the desired specified boundary conditions at the edge of the last strip, $x_n = b_n$. For example, if this edge is free, a linear combination of the above solutions would be required such that

$$\begin{aligned} C_1 N_x^{(1)} + C_2 N_x^{(2)} + C_3 N_x^{(3)} + C_4 N_x^{(4)} &= 0, \\ C_1 N_{xy}^{(1)} + C_2 N_{xy}^{(2)} + C_3 N_{xy}^{(3)} + C_4 N_{xy}^{(4)} &= 0, \\ C_1 M_x^{(1)} + C_2 M_x^{(2)} + C_3 M_x^{(3)} + C_4 M_x^{(4)} &= 0, \\ C_1 V_x^{(1)} + C_2 V_x^{(2)} + C_3 V_x^{(3)} + C_4 V_x^{(4)} &= 0, \end{aligned} \tag{12}$$

in which the C's are relative constants.

Eqs. (12) are a set of homogeneous algebraic equations in the C's. A non-trivial solution exists only if the determinant of the coefficients of the unknown quantities vanishes. Hence, for a free edge, the stability criterion is the determinantal equation,

$$\begin{vmatrix} N_x^{(1)} & N_x^{(2)} & N_x^{(3)} & N_x^{(4)} \\ N_{xy}^{(1)} & N_{xy}^{(2)} & N_{xy}^{(3)} & N_{xy}^{(4)} \\ M_x^{(1)} & M_x^{(2)} & M_x^{(3)} & M_x^{(4)} \\ V_x^{(1)} & V_x^{(2)} & V_x^{(3)} & V_x^{(4)} \end{vmatrix} = 0, \tag{13}$$

where the numerical values of the elements are taken from Table 2. Other boundary conditions, such as a supported or elastically built-in edge would be handled in a similar manner.

When the member is not short and comprises plate elements having large width-to-thickness ratios, the mode of buckling associated with the critical loading may have more than one half-wave in the longitudinal direction. In this case, L in (9) should be replaced by L/m $(m=2,3,\dots)$ and the loading which satisfies (13) should be found for each m in the relevant range. The lowest of these calculated loadings then is the critical loading, and the associated m defines the mode of buckling.

One of the several possible procedures for determining the critical magnitudes of a combination of axial load and bending moments for a member with free edges may be outlined as follows:

1. Select trial values of axial load and moments, and compute N_0 for all points on the x_i-axes.
2. Starting with initial values of the dependent variables according to (11) and Table 1, integrate (9) four times using the transformations (10) as necessary, and thus evaluate the relevant quantities listed in Table 2 at the edge $x_n=b_n$.
3. Evaluate the determinant of (13). If zero, the trial values were correctly chosen. If not zero, a new set of trial values should be selected and the process repeated. If the value of the determinant is considered as a function of a loading parameter, for example the axial load, the lowest non-zero value of the loading parameter for which the determinant vanishes represents the critical loading condition. This point may be found by plotting, by interpolation or by a Newton procedure.
4. When the width-to-thickness ratio of one or more of the plate elements is such that the member may buckle in a mode having two or more half-waves in the longitudinal direction, the preceding steps should be repeated using a reduced length equal to $L/2$. If this calculation produces a buckling load less than obtained in the previous calculation, a third calculation should be made with a reduced length equal to $L/3$. This process should be continued until it is found that the use of successively shorter reduced lengths produces higher buckling loads.

After the critical loading has been established, the associated mode shape or buckled form of the cross section may be readily found if desired. One may assign an arbitrary value to, say, C_1 and then solve any three of (12) for the remaining C's. Now, with the starting values $u=C_1$, $v=C_2$, $w=C_3$ and $S=C_4$, and with the starting values of the edge tractions according to (11), a final integration of (9) may be made. This calculation will yield the deformations and will also serve as a check since the terminal values of the edge tractions should vanish or approximately so.

If the initial curvature can not be neglected, equations for conical shells may be substituted for (1) and (2) with consequent changes in (9). Limitations on space prevent including these equations in the present paper.

Examples

1. Critical axial compressive stresses for several steel channel sections were found using this method, the computations being done on an electronic digital computer. A few of the results are shown in Table 3. Also listed are values of the critical stress computed by the method of Reference 3 which neglects deformation of the cross section. It may be noted that the critical stresses for the sections of Examples 3 and 4 are associated with buckling modes having respectively 6 and 5 half-waves in the longitudinal direction. The values calculated for single longitudinal buckles are also shown for comparison. The latter two examples show the marked reduction in critical stress due to deformation of the cross section and to buckling in a higher longitudinal mode.

Table 3

Example	Depth	Flange Width	Thickness		Length	Buckling Stress			Remarks
			Web	Flange		Present		Ref. 3	
	(in.)	(in.)	(in.)	(in.)	(in.)	(psi)	m	(psi)	
1	8	2	0.1	0.1	200	2469	1	2471	Euler Buckling
2	2	5	0.1	0.1	200	1776	1	1772	Torsional Buckling
3	8	2	0.025	0.025	50	1200	6	9884	See Fig. 4
3	8	2	0.025	0.025	50	8282	1	9884	
4	2	5	0.025	0.025	50	688	5	7088	See Fig. 5
4	2	5	0.025	0.025	50	4421	1	7088	See Fig. 5

2. The section shown in Fig. 6 is of steel with a length of 60 inches. The critical stress under axial compression was found, by the method described, to be 9325 psi. The deformed cross section at the critical load is shown in Fig. 7. The same section was checked for a bending condition in which the neutral axis was horizontal and the circular flange was in compression. The

critical elastic compressive stress at the extreme fiber was found to be 72,933 psi. The deformed cross section under the critical moment is shown in Fig. 8. In both cases, it was found that the curved section was adequately approximated by twelve straight segments.

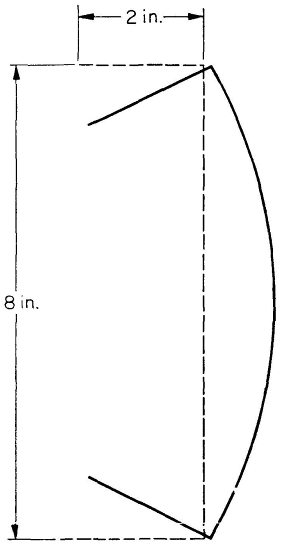

Fig. 4. Deformed cross section of buckled channel — example 3.

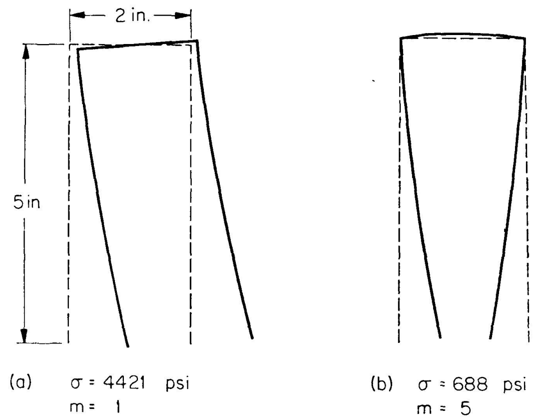

Fig. 5. Deformed cross section of buckled channel — example 4.

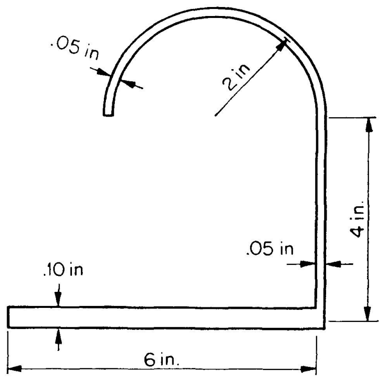

Fig. 6. Cross section for illustrative example.

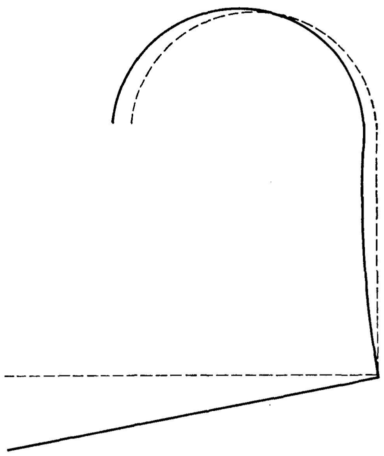

Fig. 7. Deformed cross section under critical compressive load.

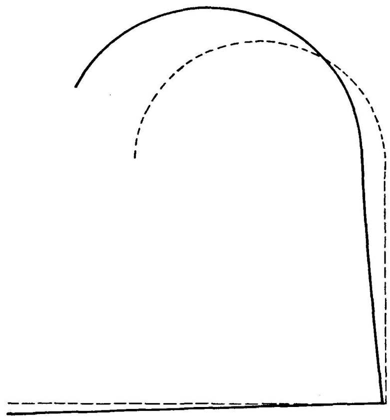

Fig. 8. Deformed cross section under critical bending moment.

References

1. Theory of Elastic Stability, by S. Timoshenko, New York, 1936.
2. Mathematical Methods for Digital Computers, edited by A. Ralston and H. Wilf, New York, 1960, p. 110.
3. Collected Papers of Stephen P. Timoshenko, New York, 1953, p. 559.

Summary

A general method is presented for determining critical magnitudes of bending moments and axial loads which cause lateral and torsional buckling of thin-walled beams having arbitrary cross sections. The theory upon which the method is based includes the effect of deformation of the cross sections in their own planes as well as warping out of their planes. This contrasts with the commonly used theory which presupposes that the cross sections do not deform in their own planes.

The method contemplates the use of a digital computer and is sufficiently simple to permit its employment in routine analysis.

Résumé

Les auteurs proposent une méthode générale pour la détermination des grandeurs critiques des moments de flexion et des charges axiales qui entraînent un déversement ou un flambage par torsion des poutres à parois minces ayant des sections droites quelconques. La théorie sur laquelle s'appuie cette méthode comprend l'effet de la déformation des sections droites dans leurs plans et les déplacements d'ensemble; elle diffère ainsi de la théorie habituellement utilisée, basée sur l'hypothèse de l'indéformabilité du contour des sections droites.

La méthode proposée envisage l'utilisation d'une calculatrice numérique. Elle est suffisamment simple pour pouvoir être utilisée dans une analyse de routine.

Zusammenfassung

Es wird eine allgemeine Methode zur Ermittlung der kritischen Biegemomente und Längskräfte dargestellt, die ein Kippen oder ein Biegedrillknicken dünnwandiger Träger mit beliebigem Querschnitt verursachen. Die als Grundlage verwendete Theorie berücksichtigt nicht nur die Gesamtverformungen, sondern auch die Veränderungen der Querschnittsform, dies im Gegensatz zu der allgemein gebräuchlichen Betrachtungsweise, welche die Erhaltung der Querschnittsform voraussetzt.

Die Methode kann auf einem Digitalrechner programmiert werden und ihre Einfachheit erlaubt die Anwendung bei routinemäßigen Berechnungen.