Simplified method of analysis for elliptical paraboloidal shallow shells under the action of concentrated loads

Simplified Method of Analysis for Elliptical Paraboloidal Shallow Shells under the Action of Concentrated Loads

Méthode simplifiée de calcul des voiles minces en forme de paraboloïde elliptique, soumis à des charges concentrées

Vereinfachtes Berechnungsverfahren für elliptisch-paraboloidförmige, schwachgekrümmte Schalen unter Einzellasten

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Many authors have studied the stress analysis of shallow shells under the action of a concentrated load. In 1946, E. REISSNER [1] obtained solutions for the case of shallow spherical shells, while V. Z. JIGANTU [2], in 1956, obtained the solution for shallow parabolic shells of revolution. In 1949, V. Z. VLASOV [3] gave the solution for elliptical paraboloidal shallow shells in the form of double trigonometrical series whose convergence has been proved to be rather slow, thereby obscuring its practical value for engineering computations. A. R. RZHANITZYN [4] and H. C. CHAWUSOV [5] attempted to obtain simplified solutions based on the membrane theory of shells, but for regions close to the concentrated load, the more exact theory, with consideration of bending moments and twisting moments of thin shells, is to be preferred for the analysis in order to obtain good results. In this paper simplified formulas, with accompanying tables, will be presented, which will be found to be quite convenient in application for the design of elliptical paraboloidal shallow shells under the action of concentrated loads.

I. Fundamental Equations of the Analysis

As could have been observed, the action of a concentrated load on the vertical displacement and internal stresses of the shell damp out rapidly as the distance from the point of application of the load increases. It will thus be more convenient to confine our study of the problem to the case where this distance is grater than $6/\sqrt[4]{K}$ (K will be defined later) from the boundary. As will be shown later, the effect of the latter will then be quite negligible and we could, for simplicity, assume that the shell possesses a boundary which is at an infinite distance from the load. As for the case where the concentrated load is close to the boundary, we can easily make a correction by the method of images, as has been done in the case of flat plates [6].

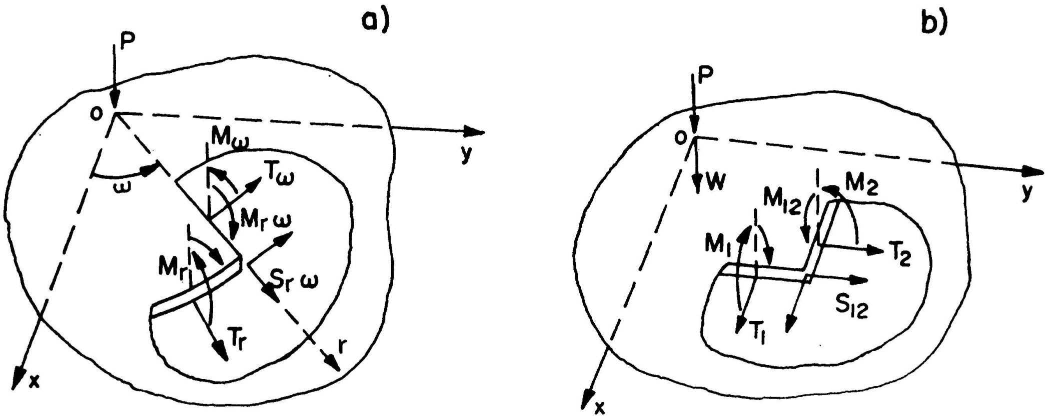

Fig. 1.

With the notations given in Fig. 1 [1]), the problem is reduced to the determination of the stress function φ and the vertical displacement w of the shell that satisfy the following differential equations:

$$D\Delta^2 w - \Delta_k \varphi = P\delta(0,0), \qquad \Delta^2\varphi + E\delta\Delta_k w = 0, \tag{1}$$

in which $$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \Delta_k = k_2\frac{\partial^2}{\partial x^2} + k_1\frac{\partial^2}{\partial y^2},$$

δ is the thickness of the shell, P is the concentrated load and $\delta(0,0)$ is the Dirac delta function. In order to avoid some divergent definite integrals that would appear in the process of derivation, we make use of the relations

$$\frac{\partial^2\varphi}{\partial y^2} = T_1, \qquad \frac{\partial^2\varphi}{\partial x^2} = T_2 \tag{2}$$

to transform the set of Eqs. (1) into the following:

$$D\Delta^2 w - (k_1 T_1 + k_2 T_2) = P\delta(0,0), \qquad \Delta(T_1+T_2) + E\delta\Delta_k w = 0,$$
$$\frac{\partial^2 T_1}{\partial x^2} = \frac{\partial^2 T_2}{\partial y^2}. \tag{3}$$

As we have assumed for the present case that the boundary is at an infinite distance from the load and as we know that all internal forces and all displacements of the shell damp out as the distance from the load increases, then

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[1]) As Q_1 and Q_2 are not important, we omit them from the figure.

the boundary conditions would be such that w, T_1, T_2 as well as all their derivatives of whatsoever order, could be regarded as zero at infinity.

If we make the following double Fourier cosine integral transformations:

$$\begin{aligned}\overline{w}(\xi,\eta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)\cos(\xi x)\cos(\eta y)\,dx\,dy = 4\int_0^\infty\int_0^\infty w\cos(\xi x)\cos(\eta y)\,dx\,dy,\\ \overline{T}_1 &= 4\int_0^\infty\int_0^\infty T_1\cos(\xi x)\cos(\eta y)\,dx\,dy, \qquad \overline{T}_2 = 4\int_0^\infty\int_0^\infty T_2\cos(\xi x)\cos(\eta y)\,dx\,dy.\end{aligned} \tag{4}$$

and note that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\,\delta(0,0)\cos(\xi x)\cos(\eta y)\,dx\,dy = P$$

we obtain, after substituting them in (3):

$$\begin{aligned} D(\xi^2+\eta^2)^2\overline{w} - (k_1\overline{T}_1 + k_2\overline{T}_2) &= P,\\ (\xi^2+\eta^2)(\overline{T}_1+\overline{T}_2) + E\delta(k_2\xi^2+k_1\eta^2)\overline{w} &= 0,\\ \xi^2\overline{T}_1 &= \eta^2\overline{T}_2.\end{aligned} \tag{5}$$

Solving $\overline{w}$, $\overline{T}_1$ and $\overline{T}_2$ simultaneously from Eqs. (5) and making the corresponding inverse transformation, we readily obtain expressions for w, T_1 and T_2 in the form of definite integrals which, by a further transformation of coordinate represented by

$$\xi = \rho\cos\theta, \qquad \eta = \rho\sin\theta,$$

take the following form:

$$w = \frac{P}{\pi^2 D}\int_0^{\pi/2}\int_0^\infty \frac{\rho\cos(\rho x\cos\theta)\cos(\rho y\sin\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}(k_2\cos^2\theta+k_1\sin^2\theta)^2}, \tag{6}$$

$$T_1 = -\frac{12P}{\delta^2\pi^2}\int_0^{\pi/2}\int_0^\infty \frac{\rho\sin^2\theta\,(k_2\cos^2\theta+k_1\sin^2\theta)\cos(\rho x\cos\theta)\cos(\rho y\sin\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}(k_2\cos^2\theta+k_1\sin^2\theta)^2}, \tag{7}$$

$$T_2 = -\frac{12P}{\delta^2\pi^2}\int_0^{\pi/2}\int_0^\infty \frac{\rho\cos^2\theta\,(k_2\cos^2\theta+k_1\sin^2\theta)\cos(\rho x\cos\theta)\cos(\rho y\sin\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}(k_2\cos^2\theta+k_1\sin^2\theta)^2}. \tag{8}$$

Neglecting the effect of Poisson's ratio, we make use of the known relations

$$M_1 = -D\frac{\partial^2 w}{\partial x^2}, \qquad M_2 = -D\frac{\partial^2 w}{\partial y^2}, \qquad M_{12} = -D\frac{\partial^2 w}{\partial x\,\partial y},$$

to obtain immediately

$$M_1 = \frac{P}{\pi^2}\int_0^{\pi/2}\int_0^\infty \frac{\rho^3\cos^2\theta\cos(\rho x\cos\theta)\cos(\rho y\sin\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}(k_2\cos^2\theta+k_1\sin^2\theta)^2}, \tag{9}$$

$$M_2 = \frac{P}{\pi^2}\int\limits_0^{\pi/2}\int\limits_0^{\infty}\frac{\rho^3 \sin^2\theta \cos(\rho x \cos\theta)\cos(\rho y \sin\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}(k_2\cos^2\theta+k_1\sin^2\theta)^2}, \tag{10}$$

$$M_{12} = -\frac{P}{\pi^2}\int\limits_0^{\pi/2}\int\limits_0^{\infty}\frac{\rho^3 \sin\theta\cos\theta \sin(\rho x \cos\theta)\sin(\rho y \sin\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}(k_2\cos^2\theta+k_1\sin^2\theta)^2}. \tag{11}$$

If we put S_{12} in the form:

$$S_{12} = \int\limits_0^{\pi/2}\int\limits_0^{\infty} f(\theta,\rho)\sin(\rho x\cos\theta)\sin(\rho y \sin\theta)\,d\rho\,d\theta$$

and make use of the relation

$$\frac{\partial T_1}{\partial x} = -\frac{\partial S_{12}}{\partial y} \quad \text{or} \quad \frac{\partial T_2}{\partial y} = -\frac{\partial S_{12}}{\partial x},$$

we obtain

$$S_{12} = -\frac{12\,P}{\delta^2\pi^2}\int\limits_0^{\pi/2}\int\limits_0^{\infty}\frac{\rho\cos\theta\sin\theta\,(k_2\cos^2\theta+k_1\sin^2\theta)\sin(\rho x\cos\theta)\sin(\rho y\sin\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}(k_2\cos^2\theta+k_1\sin^2\theta)^2}. \tag{12}$$

In the case where $k_1 = k_2 = k$, we further transform x, y into polar coordinates r, ω and if we take into account the fact that we have here an axis-symmetrical case, then it is not difficult to obtain

$$w = \frac{P}{\pi^2 D}\int\limits_0^{\pi/2}\int\limits_0^{\infty}\frac{\rho\cos(\rho r\cos\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}k^2}, \tag{13}$$

$$T_r = -\frac{12\,P\,k}{\delta^2\pi^2}\int\limits_0^{\pi/2}\int\limits_0^{\infty}\frac{\rho\sin^2\theta\cos(\rho r\cos\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}k^2}, \tag{14}$$

$$T_\omega = -\frac{12\,P\,k}{\delta^2\pi^2}\int\limits_0^{\pi/2}\int\limits_0^{\infty}\frac{\rho\cos^2\theta\cos(\rho r\cos\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}k^2}, \tag{15}$$

$$M_r = \frac{P}{\pi^2}\int\limits_0^{\pi/2}\int\limits_0^{\infty}\frac{\rho^3\cos^2\theta\cos(\rho r\cos\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}k^2}, \tag{16}$$

$$M_\omega = \frac{P}{\pi^2}\int\limits_0^{\pi/2}\int\limits_0^{\infty}\frac{\rho^3\sin^2\theta\cos(\rho r\cos\theta)\,d\rho\,d\theta}{\rho^4+\frac{12}{\delta^2}k^2}, \tag{17}$$

$$M_{r\omega} = 0, \qquad S_{r\omega} = 0. \tag{18}$$

Thus we succeed in expressing the vertical displacement and internal stresses of the shell under the action of a concentrated load in the form of definite integrals and the solution of the problem is thus reduced to the evaluation of these definite integrals.

II. Formulas of the Analysis

A. For the Case $k_1 = k_2 = k$

We take advantage of the mathematical formulas

$$\int_0^{\pi/2} \cos(\rho r \cos\theta)\, d\theta = \frac{\pi}{2} J_0(\rho r), \qquad \int_0^{\infty} \frac{\rho J_0(\rho r)}{\rho^4+1} d\rho = -kei(r) \tag{I}$$

to obtain readily from Eq. (13) that

$$w = -\frac{\sqrt{3}\, P}{E \pi \delta^2 k} kei(\bar{r}), \tag{19}$$

in which

$$\bar{r} = \sqrt[4]{\frac{12}{\delta^2}}\, k^2 r.$$

From the relationships

$$M_r = -D \frac{d^2 w}{dr^2}, \qquad M_\omega = -D \frac{1}{r} \frac{dw}{dr}$$

we obtain

$$M_r = \frac{P}{2\pi} \left[ker(\bar{r}) - \frac{kei'(\bar{r})}{\bar{r}} \right], \tag{20}$$

$$M_\omega = \frac{P}{2\pi} \frac{kei'(\bar{r})}{\bar{r}}. \tag{21}$$

Furthermore, it can be proved mathematically that

$$\int_0^{\pi/2} \sin^2\theta \cos(\rho r \cos\theta)\, d\theta = \frac{\pi}{2r} J_1(\rho r), \qquad \int_0^{\infty} \frac{J_1(\rho r)}{\rho^4+1} d\rho = \frac{1}{r} + ker'(r), \tag{II}$$

With these relationships we obtain from Eqs. (14) that

$$T_r = -\frac{\sqrt{3}\, P}{\pi \delta} \left[\frac{1}{\bar{r}^2} + \frac{ker'(\bar{r})}{\bar{r}} \right]. \tag{22}$$

From the known relations

$$T_r = \frac{1}{r} \frac{d\varphi}{dr}, \qquad T_\omega = \frac{d^2 \varphi}{dr^2}$$

we have

$$T_\omega = \frac{d}{dr}(r\, T_r).$$

Thus we obtain

$$T_\omega = \frac{\sqrt{3}\, P}{\pi \delta} \left[\frac{1}{\bar{r}^2} + kei(\bar{r}) + \frac{ker'(\bar{r})}{\bar{r}} \right]. \tag{23}$$

We find that Eqs. (18)—(23) are exactly the same as those given by E. Reissner [1] for shallow spherical shells. That is to say, the same set of equations can be used for elliptical paraboloidal shallow shells of equal radii of curvature in both directions as well as for shallow spherical shells.

As formulas (18)—(23) are quite simple, they will be suggested for design work and tables have been prepared to facilitate the computation. Thus we have

$$T_r = -\frac{\sqrt{3}\,P}{\pi\,\delta} f_1(\bar r), \qquad M_r = \frac{P}{2\pi} f_3(\bar r),$$
$$T_\omega = -\frac{\sqrt{3}\,P}{\pi\,\delta} f_2(\bar r), \qquad M_\omega = \frac{P}{2\pi} f_4(\bar r), \qquad w = \frac{\sqrt{3}\,P}{\pi\,E\,\delta^2 k} f_5(\bar r), \tag{24}$$

in which $f_1(\bar r) - f_5(\bar r)$ can be found in Table 1.

B. For the Case $k_1 \neq k_2$

It is somewhat difficult to perform the integrations of Eqs. (6)—(12) directly, but for the point immediately beneath the concentrated load (i. e. for the point $x=0$, $y=0$), the integrations can be easily performed so that we obtain

$$w(0,0) = \frac{\sqrt{3}\,P}{4\,E\,\delta^2\sqrt{k_1 k_2}}, \qquad T_1(0,0) = T_2(0,0) = -\frac{\sqrt{3}\,P}{8\,\delta},$$
$$M_1(0,0) = M_2(0,0) = \infty, \qquad M_{12}(0,0) = S_{12}(0,0) = 0. \tag{25}$$

For any other point on the shell, the integrations can be performed in the following manner.

Expand the denominator within the double integrations of (6)—(12) into series:

$$\frac{1}{\rho^4 + \frac{12}{\delta^2}(k_2\cos^2\theta + k_1\sin^2\theta)^2} = \frac{1}{(\rho^4+K) + \left[\frac{12}{\delta^2}(k_2\cos^2\theta + k_1\sin^2\theta)^2 - K\right]} =$$
$$\frac{1}{\rho^4+K}[1 - L + L^2 - L^3 + \cdots], \tag{26}$$

in which
$$L = \frac{\frac{12}{\delta^2}(k_2\cos^2\theta + k_1\sin^2\theta)^2 - K}{\rho^4+K}$$

and K is a constant so chosen as to ensure the condition of convergence of the series i. e. we must make $|L| < 1$.

Here we choose $K = \frac{6}{\delta^2}(k_1^2 + k_2^2)$ so that whatever the ratio of k_1/k_2 may be, the series will converge. When $k_1 = k_2$, we have $L = 0$ so that the series (26) is reduced to its first term only, and Eqs. (6)—(12) are reduced to (13)—(18).

Let
$$\bar\rho = \frac{\rho}{\sqrt[4]{K}}, \qquad \lambda = \frac{k_1}{k_2},$$

then
$$\frac{1}{\rho^4+\frac{12}{\delta^2}(k_2\cos^2\theta+k_1\sin^2\theta)^2}=\frac{1}{K(\bar{\rho}^4+1)}[1-L+L^2-\cdots], \tag{27}$$

in which
$$L=\frac{2(\lambda-1)}{\lambda^2+1}\,\frac{1}{\bar{\rho}^4+1}\left[-\frac{1+\lambda}{2}+2\sin^2\theta+(\lambda-1)\sin^4\theta\right].$$

If we substitute (27) in (6)—(12) and attempt to integrate first with respect to θ, we shall meet the following two types of integration:

$$H_{2n}=\int_0^{\pi/2}\cos(\bar{\rho}\,\bar{x}\cos\theta)\cos(\bar{\rho}\,\bar{y}\sin\theta)\sin^{2n}\theta\,d\theta, \tag{28}$$

$$D_{2n}=\int_0^{\pi/2}\sin(\bar{\rho}\,\bar{x}\cos\theta)\sin(\bar{\rho}\,\bar{y}\sin\theta)\cos\theta\sin^{2n+1}\theta\,d\theta, \tag{29}$$

in which
$$\bar{x}=\sqrt[4]{K}\,x,\qquad \bar{y}=\sqrt[4]{K}\,y,\qquad n=0,1,2,\ldots.$$

These can be evaluated without difficulty if we differentiate a certain number of times, with respect to $\bar{x}$ or $\bar{y}$, the following mathematical identity:

$$\int_0^{\pi/2}\cos(\bar{\rho}\,\bar{x}\cos\theta)\cos(\bar{\rho}\,\bar{y}\sin\theta)\,d\theta=\frac{\pi}{2}J_0(\bar{\rho}\,\bar{r}), \tag{III}$$

in which $\bar{r}=\sqrt[4]{K}\,r$.

Thus we obtain

$$H_{2n}=\frac{\pi}{2}\left[\overline{r^{2n}}_{n}\frac{J_n(\bar{\rho}\,\bar{r})}{(\bar{\rho}\,\bar{r})^n}+\overline{r^{2n}}_{n-1}\frac{J_{n-1}(\bar{\rho}\,\bar{r})}{(\bar{\rho}\,\bar{r})^{n-1}}+\cdots+\overline{r^{2n}}_{0}J_0(\bar{\rho}\,\bar{r})\right], \tag{30}$$
$$(n=0,1,2\ldots\ \text{altogether } n+1 \text{ terms})$$

and
$$D_{2n}=\frac{\pi}{2}\left[\overline{s^{2n}}_{n+1}\frac{J_{n+1}(\bar{\rho}\,\bar{r})}{(\bar{\rho}\,\bar{r})^{n+1}}+\overline{s^{2n}}_{n}\frac{J_n(\bar{\rho}\,\bar{r})}{(\bar{\rho}\,\bar{r})^n}+\cdots+\overline{s^{2n}}_{0}J_0(\bar{\rho}\,\bar{r})\right], \tag{31}$$
$$(n=0,1,2\ldots\ \text{altogether } n+2 \text{ terms}),$$

in which $\overline{r^{2n}}_{n-i}$, $\overline{s^{2n}}_{n-i}$ are definite numerical coefficients [2]).

On integration upon ρ, we shall obtain the following two types of integration:

$$\frac{I_{mn}}{\bar{r}^m}=\frac{1}{\bar{r}^m}\int_0^\infty\frac{\bar{\rho}}{(\bar{\rho}^4+1)^{1+n}}\,\frac{J_m(\bar{\rho}\,\bar{r})}{\bar{\rho}^m}\,d\bar{\rho}, \tag{32}$$

$$\frac{K_{mn}}{\bar{r}^m}=\frac{1}{\bar{r}^m}\int_0^\infty\frac{\bar{\rho}^3}{(\rho^4+1)^{1+n}}\,\frac{J_m(\bar{\rho}\,\bar{r})}{\bar{\rho}^m}\,d\bar{\rho}, \tag{33}$$

in which $m,n=0,1,2\ldots$

[2]) For detailed accounts of these coefficients see reference [7].

After a series of mathematical operations upon (I) and (II), these can again be integrated into the following:

$$
\begin{aligned}
I_{00} &= -kei(\bar{r}), \\
\frac{I_{10}}{\bar{r}} &= \frac{1}{\bar{r}}\left[\frac{1}{\bar{r}} + ker'(\bar{r})\right], \\
\frac{I_{20}}{\bar{r}^2} &= \frac{1}{\bar{r}^2}\left[\frac{1}{2} - \frac{2}{\bar{r}}(-I_{00})' + ker(\bar{r})\right], \\
\frac{I_{30}}{\bar{r}^3} &= \frac{1}{\bar{r}^3}\left[\frac{\bar{r}}{2^2 \cdot 2!} - \frac{4}{\bar{r}}\left(\frac{2}{\bar{r}} I_{10} - I_{00}\right) + (-I_{00})'\right], \\
&\cdots\cdots\cdots\cdots \\
\frac{I_{n0}}{\bar{r}^n} &= \frac{1}{\bar{r}^n}\left[\frac{\bar{r}^{n-2}}{2^{n-1}(n-1)!} - \frac{2(n-1)}{\bar{r}}\left(\frac{2(n-2)}{\bar{r}} I_{n-2\,0} - I_{n-3\,0}\right)\right. \\
&\quad \left. + \left(\frac{2(n-3)}{\bar{r}} I_{n-3\,0} - I_{n-4\,0}\right)\right] \quad (n \geqq 4), \\
&\cdots\cdots\cdots\cdots \\
\frac{I_{nm}}{\bar{r}^n} &= \frac{1}{\bar{r}^n}\left[\left(1 - \frac{-(n-2)}{4m}\right) I_{n\,m-1} - \frac{\bar{r}}{4m} I'_{n\,m-1}\right] \quad (m \geqq 1), \\
K_{00} &= ker(\bar{r}), \\
\frac{K_{10}}{\bar{r}} &= (-I_{00})' \frac{1}{\bar{r}}, \\
&\cdots\cdots\cdots\cdots \\
\frac{K_{n0}}{\bar{r}} &= \frac{1}{\bar{r}^n}\left(\frac{2(n-1)}{\bar{r}} I_{n-1\,0} - I_{n-2\,0}\right) \quad (n \geqq 2), \\
\frac{K_{nm}}{\bar{r}^n} &= \frac{1}{\bar{r}^n}\left[\left(1 - \frac{-(n-4)}{4m}\right) K_{n\,m-1} - \frac{\bar{r}}{4m} K'_{n\,m-1}\right] \quad (m \geqq 1).
\end{aligned} \tag{34}
$$

Thus the vertical displacement and internal forces of the shell can be finally expressed in the form of series composed of certain combinations of Thompson functions such as those given in (34). For instance, we have

$$
\begin{aligned}
M_{12} = -\frac{P}{2\pi}\Bigg\{&\left(d_{00} K_{00} + d_{10}\frac{K_{10}}{\bar{r}}\right) - \frac{2(\lambda-1)}{\lambda^2+1}\left(d_{01} K_{01} + d_{11}\frac{K_{11}}{\bar{r}} + d_{21}\frac{K_{21}}{\bar{r}^2} + d_{31}\frac{K_{31}}{\bar{r}^3}\right) \\
&+ \cdots + (-1)^n\left[\frac{2(\lambda-1)}{\lambda^2+1}\right]^n\left(d_{0n} K_{0n} + d_{1n}\frac{K_{1n}}{\bar{r}} + \cdots + d_{2n+1\,n}\frac{K_{2n+1\,n}}{\bar{r}^{2n+1}}\right) + \cdots\Bigg\},
\end{aligned} \tag{35}
$$

in which d_{ij} are definite numerical coefficients[3]).

The convergence of the series is quite good and only a few terms are necessary for the ordinary scope of application i.e., for $0.5 \leqq \lambda \leqq 2$. For example, if we take the case $\lambda = 2$ (the closer is the value of λ to 1, the better will be the convergence) and compute the value of $w(0,0)$ and $T_1(0,0)$, we obtain

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[3]) For detailed accounts of these coefficients, see reference [7].

$$w(0,0) = \frac{\sqrt{3}\,P}{4\,E\,\delta^2\sqrt{k_1\,k_2}}\,\frac{1}{\sqrt{1.25}}(1+0.025+0.0687+0.00605+0.0138$$

$$+0.00081+\cdots) = \frac{\sqrt{3}\,P}{4\,E\,\delta^2\sqrt{k_1\,k_2}}\,\frac{1}{\sqrt{1.25}}[(1+0.025)+(0.0687+0.00605)$$

$$+(0.0138+0.00081)+\cdots] \approx \frac{\sqrt{3}\,P}{4\,E\,\delta^2\sqrt{k_1\,k_2}}\,\frac{1.114}{1.119},$$

$$T_1(0,0) = -\frac{\sqrt{3}\,P}{8\,\delta}\,\frac{8}{\sqrt{2.5}}(0.2185-0.0324+0.0144-\cdots) \approx -\frac{\sqrt{3}\,P}{8\,\delta}\,\frac{0.2005}{0.198}.$$

which deviate from the exact values given by formulas (25) by less than 0.5% for w and by about 1.26% for T_1.

However, formulas like that given by (35) are still too complicated for design work computations. As these formulas contain only one variable $\bar{r}$ and one parametric constant λ, empirical formulas could easily be derived from them. Computations by means of these formulas show that all values damp out insignificantly at $\bar{r}=6$. After a series of computations based upon these formulas for various values of $\bar{r}$ and λ, it is found that the variations are quite regular and empirical formulas could be formulated. For ordinary design work purposes, it is sufficient to know the values of the vertical displacement and internal forces of the shell along the x-axis, the y-axis and the line equally dividing the x- and y-axis before a design can be made. The empirical formulas suggested for the design work will then be, with $K=\frac{6}{\delta^2}(k_1^2+k_2^2)$ and $\lambda=\frac{k_1}{k_2}$:

1. *For $0 \leqq \bar{r} \leqq 4$*

a) Along the x-axis:

$$T_1 = -\frac{\sqrt{3}\,P}{\pi\,\delta}\,f_1(\bar{r}), \tag{36}$$

$$T_2 = -\frac{\sqrt{3}\,P}{\pi\,\delta}\left\{f_2(\bar{r})+(\lambda-1)\left[(0.07-0.02\,\lambda)\sin\frac{(3.8-\bar{r})\,\pi}{3.5}-0.008\right]\right\}, \tag{37}$$

$$M_1 = \frac{P}{2\,\pi}\left\{f_3(\bar{r})+0.04\,(\lambda-1)\sin\frac{(\bar{r}+0.8)\,\pi}{2.6}\right\}, \tag{38}$$

$$M_2 = \frac{P}{2\,\pi}\left\{f_4(\bar{r})+\frac{(\lambda-1)}{100}\left[\sin\frac{(0.5-\bar{r})\,\pi}{1.8}+2.6\,\lambda-6.9\right]\right\}, \tag{39}$$

$$w = \frac{P}{2\,\pi D\sqrt{K}}\left\{f_5(\bar{r})+(\lambda-1)\left[\frac{1}{7.1}\sin\frac{(0.5-\bar{r})\,\pi}{8.5}+0.04\,\lambda-0.01\right]\right\}, \tag{40}$$

$$M_{12} = S_{12} = 0. \tag{41}$$

b) Along the y-axis:

$$T_1 = -\frac{\sqrt{3}\,P}{\pi\delta} f_2(\bar{r}), \tag{42}$$

$$T_2 = -\frac{\sqrt{3}\,P}{\pi\delta} f_1(\bar{r}), \tag{43}$$

$$M_1 = \frac{P}{2\pi}\left\{f_4(\bar{r}) + 0.02\,(\lambda-1)\left[(\lambda-1)\sin\frac{(\bar{r}+0.5)\,\pi}{2.4} + \lambda\right]\right\}, \tag{44}$$

$$M_2 = \frac{P}{2\pi}\left\{f_3(\bar{r}) + \frac{(\lambda-1)}{100}\left[(10.4-3.6\,\lambda)\sin\frac{(\bar{r}-1.2)\,\pi}{3.2} + (1.2\,\lambda-2.8)\right]\right\}, \tag{45}$$

$$w = \frac{P}{2\pi D\sqrt{K}}\left\{f_5(\bar{r}) + (\lambda-1)\left[\frac{1}{12.5}\sin\frac{(\bar{r}+0.9)\,\pi}{7} + 0.022\,\lambda - 0.013\right]\right\}, \tag{46}$$

$$M_{12} = S_{12} = 0. \tag{47}$$

c) Along the line equally dividing the x- and y-axis:

$$T_1 = -\frac{\sqrt{3}\,P}{2\pi\delta} f_5(\bar{r}), \tag{48}$$

$$T_2 = -\frac{\sqrt{3}\,P}{2\pi\delta}\left\{f_5(\bar{r}) + (\lambda-1)\,(0.036-0.008\,\lambda)\sin\frac{(4-\bar{r})\,\pi}{3}\right\}, \tag{49}$$

$$M_1 = \frac{P}{2\pi}\left\{f_6(\bar{r}) + (\lambda-1)\left[\frac{1}{28}\sin\frac{(2.2-\bar{r})\,\pi}{5.2} + \frac{1}{77}\right]\right\}, \tag{50}$$

$$M_2 = \frac{P}{2\pi}\left\{f_6(\bar{r}) + (\lambda-1)\,[0.035-0.018\,(\lambda-1)]\,\frac{\bar{r}-3.2}{2.7}\right\}, \tag{51}$$

$$M_{12} = \frac{P}{2\pi} f_7(\bar{r}), \tag{52}$$

$$S_{12} = -\frac{\sqrt{3}\,P}{\pi\delta} f_8(\bar{r}), \tag{53}$$

$$w = \frac{P}{2\pi D\sqrt{K}}\left\{f_5(\bar{r}) + (\lambda-1)\left[\frac{1}{22}\sin\frac{(3.25-\bar{r})\,\pi}{5.5} + \frac{\lambda-1}{50}\right]\right\}, \tag{54}$$

2. *For* $4 < \bar{r} \leqq 6$

a) Along the x-axis:

$$T_1 = -\frac{\sqrt{3}\,P}{\pi\delta} f_1(\bar{r}), \tag{55}$$

$$T_2 = -\frac{\sqrt{3}\,P}{\pi\delta}\{f_2(\bar{r}) + (\lambda-1)\,(0.005-0.003\,\bar{r})\}, \tag{56}$$

$$w = \frac{P}{2\pi D\sqrt{K}}\{f_5(\bar{r}) + (\lambda-1)\,[0.011\,\bar{r} - 0.107]\}. \tag{57}$$

b) Along the y-axis:

$$T_1 = -\frac{\sqrt{3}\,P}{\pi\delta} f_2(\bar{r}), \tag{58}$$

$$T_2 = -\frac{\sqrt{3}\,P}{\pi\delta} f_1(\bar{r}), \tag{59}$$

$$w = \frac{P}{2\pi D\sqrt{K}}\{f_5(\bar{r}) + (\lambda - 1)(0.167 - 0.02\,\bar{r})\}. \tag{60}$$

c) Along the line equally dividing the x- and y-axis:

$$T_1 = -\frac{\sqrt{3}\,P}{2\pi\delta} f_5(\bar{r}), \tag{61}$$

$$T_2 = -\frac{\sqrt{3}\,P}{2\pi\delta} f_5(\bar{r}), \tag{62}$$

$$w = \frac{P}{2\pi D\sqrt{K}} f_5(\bar{r}). \tag{63}$$

$$S_{12} = -\frac{\sqrt{3}\,P}{\pi\delta} f_8(\bar{r}), \tag{64}$$

As far as the values of M_1, M_2 and M_{12} are concerned, it is sufficient to take a linear variation within the interval $4 < \bar{r} \leqq 6$ from the value at $\bar{r} = 4$ to that of zero at $\bar{r} = 6$. S_{12} is equal to zero along the x-axis as well as along the y-axis.

It should be noted that the axes are so chosen that $k_1 \geqq k_2$ and that the scope of application is then $1 \leqq \lambda \leqq 2$.

The values of $f_6(\bar{r}) - f_8(\bar{r})$ can be found for different values of $\bar{r}$ from Table 2. They are defined by the following formulas:

$$\begin{aligned} f_6(\bar{r}) &= \frac{ker(\bar{r})}{2}, \\ f_7(\bar{r}) &= \frac{kei'(\bar{r})}{\bar{r}} - \frac{ker(\bar{r})}{2}, \\ f_8(\bar{r}) &= \frac{1}{\bar{r}^2} + \frac{ker'(\bar{r})}{\bar{r}} + \frac{kei(\bar{r})}{2}. \end{aligned} \tag{65}$$

Formulas (24) and (36)—(64) are only applicable when the concentrated load is at a distance greater than $6/\sqrt[4]{K}$ from the boundary (i.e. when $\bar{r} = 6$). For the case where the concentrated load is close to the corner or to the boundary of the shell, then the method of images should serve as a correction to these formulas and it will be applied in a similar manner to that used for flat plates in reference [6].

III. Some Further Applications of the Formulas

A. Circular Line Load Formulas for Elliptical Paraboloidal Shallow Shells with Equal Radii of Curvature in Both Directions

As the circular line load can be regarded as a ring of concentrated loads, we obtain from Eq. (19) that

$$w = -\frac{q\,a\sqrt{3}}{\pi E\,\delta^2 k}\int_0^{2\pi} I_m K_0\left(\sqrt{i}\sqrt[4]{\frac{12\,k^2}{\delta^2}}\sqrt{a^2+r^2-2\,a\,r\cos\omega}\right)d\omega, \qquad (66)$$

in which q is the intensity of the circular line load per unit length of the ring. From the mathematical identity:

$$\begin{aligned} r \leqq a:\ & K_0(\sqrt{a^2+r^2-2\,a\,r\cos\omega}) = I_0(r)\,K_0(a) + 2\sum_{n=1}^{\infty} I_n(r)\,K_n(a)\cos(n\,\omega), \\ r \geqq a:\ & K_0(\sqrt{a^2+r^2-2\,a\,r\cos\omega}) = I_0(a)\,K_0(r) + 2\sum_{n=1}^{\infty} I_n(a)\,K_n(r)\cos(n\,\omega), \end{aligned} \qquad (\Lambda I)$$

Eq. (66) can be integrated as:

$$\begin{aligned} r \leqq a:\quad & w = -\frac{\sqrt{12}\,q\,a}{E\,\delta^2 k}\left[ber(\bar r)\,kei(\bar a) + bei(\bar r)\,ker(\bar a)\right], \\ r \geqq a:\quad & w = -\frac{\sqrt{12}\,q\,a}{E\,\delta^2 k}\left[ber(\bar a)\,kei(\bar r) + bei(\bar a)\,ker(\bar r)\right], \\ r = 0:\quad & w(0,0) = -\frac{\sqrt{12}\,q\,a}{E\,\delta^2 k}\,kei(\bar a), \end{aligned} \qquad (67)$$

in which $\bar a = \sqrt[4]{\frac{12}{\delta^2}}\,k^2 a$. Hence it follows that

$$\begin{aligned} r \leqq a:\quad & M_r = q\,a\left[kei(\bar a)\,ber''(\bar r) + ker(\bar a)\,bei''(\bar r)\right], \\ & M_\omega = q\,a\frac{1}{\bar r}\left[ber'(\bar r)\,kei(\bar a) + bei'(\bar r)\,ker(\bar a)\right]; \\ r \geqq a:\quad & M_r = q\,a\left[ber(\bar a)\,kei''(\bar r) + bei(\bar a)\,ker''(\bar r)\right], \\ & M_\omega = q\,a\frac{1}{\bar r}\left[ber(\bar a)\,kei'(\bar r) + bei(\bar a)\,ker'(\bar r)\right]; \\ r = 0:\quad & M_r(0,0) = M_\omega(0,0) = \frac{q\,a}{2}\,ker(\bar a). \end{aligned} \qquad (68)$$

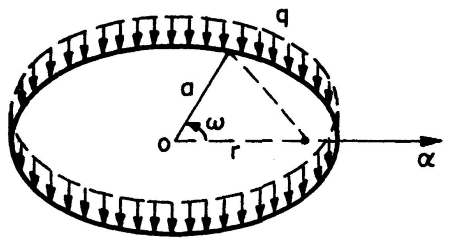

Fig. 2. Circular line loads.

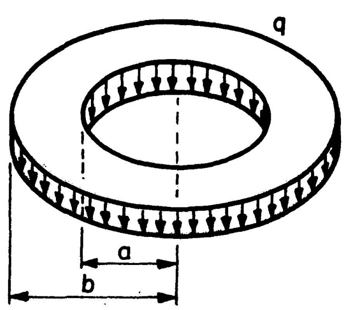

Fig. 3. Circular ring loads.

As $\varphi = \int T_r r\, dr$, we therefore obtain from (22)

$$\varphi = -\frac{P}{2\pi k}[\ln(\bar{r}) + ker(\bar{r})]. \tag{69}$$

Here we neglect the constant of integration of φ as it does not affect the internal stresses of the shell. For the case of circular line loads, we then have

$$\begin{aligned}\varphi = -\frac{qa}{2\pi k}\int_0^{2\pi}\Bigg[&Re\, K\left(\sqrt{i}\sqrt[4]{\frac{12}{\delta^2}}k^2\sqrt{a^2+r^2-2ar\cos\omega}\right)\\ &+\ln(\sqrt{a^2+r^2-2ar\cos\omega})\Bigg]d\omega.\end{aligned} \tag{70}$$

From the mathematical formula

$$\frac{d}{dr}\left[\int_0^{2\pi}\ln(\sqrt{a^2+r^2-2ar\cos\omega})\,d\omega\right] = \begin{cases}0 & (a>r)\\ \dfrac{2\pi}{r} & (a<r)\end{cases} \tag{V}$$

we obtain through Eqs. (22) and (23)

$$\begin{aligned}
r \leqq a:\quad & T_r = -\frac{\sqrt{12}\,qa}{\delta\bar{r}}[ber'(\bar{r})\,ker(\bar{a}) - bei'(\bar{r})\,kei(\bar{a})],\\
& T_\omega = \frac{\sqrt{12}\,qa}{\delta}[-ker(\bar{a})\,ber''(\bar{r}) + kei(\bar{a})\,bei''(\bar{r})];\\
r \geqq a:\quad & T_r = -\frac{\sqrt{12}\,qa}{\delta\bar{r}}\left[ber(\bar{a})\,ker'(\bar{r}) - bei(\bar{a})\,kei'(\bar{r}) + \frac{1}{\bar{r}}\right],\\
& T_\omega = \frac{\sqrt{12}\,qa}{\delta}\left[-ber(\bar{a})\,ker''(\bar{r}) + bei(\bar{a})\,kei''(\bar{r}) + \frac{1}{\bar{r}^2}\right];\\
r = 0:\quad & T_r(0,0) = T_\omega(0,0) = \frac{\sqrt{3}\,qa}{\delta}kei(\bar{a}).
\end{aligned} \tag{71}$$

B. Circular Ring Load Formulas for Elliptical Paraboloidal Shallow Shells with Equal Radii of Curvature in Both Directions

These formulas may be readily obtained by integrating a from a to b as is clear from Fig. 3. Thus we obtain

$$\begin{aligned}
r \geqq b:\quad & w = -\frac{q}{E\delta k^2}[kei(\bar{r})\,B_r + ker(\bar{r})\,B_i],\\
& M_r = \frac{q\delta}{\sqrt{12}\,k}[kei''(\bar{r})\,B_r + ker''(\bar{r})\,B_i],\\
& M_\omega = \frac{q\delta}{\sqrt{12}\,k\bar{r}}[kei'(\bar{r})\,B_r + ker'(\bar{r})\,B_i],\\
& T_r = -\frac{q}{k\bar{r}}\left[ker'(\bar{r})\,B_r - kei'(\bar{r})\,B_i + \frac{1}{2\bar{r}}(\bar{b}^2-\bar{a}^2)\right],\\
& T_\omega = \frac{q}{k}\left[B_i\,kei''(\bar{r}) - B_r\,ker''(\bar{r}) + \frac{1}{2\bar{r}^2}(\bar{b}^2-\bar{a}^2)\right];
\end{aligned} \tag{72}$$

$$
\begin{aligned}
a \leqq r \leqq b: \quad w &= -\frac{q}{E\,\delta\,k^2}\{\bar{a}\,[ker\,(\bar{r})\,ber'\,(\bar{a}) - ker\,(\bar{r})\,bei'\,(\bar{a})] \\
&\quad + \bar{b}\,[bei\,(\bar{r})\,kei'\,(\bar{b}) - ker'\,(\bar{b})\,ber\,(\bar{r})] - 1\}, \\
M_r &= \frac{q\,\delta}{\sqrt{12}\,k}\{\bar{a}\,[ber'\,(\bar{a})\,ker''\,(\bar{r}) - bei'\,(\bar{a})\,kei''\,(\bar{r})] \\
&\quad + \bar{b}\,[kei'\,(\bar{b})\,bei''\,(\bar{r}) - ker'\,(\bar{b})\,ber''\,(\bar{r})]\}, \\
M_\omega &= \frac{q\,\delta}{\sqrt{12}\,k\,\bar{r}}\{\bar{a}\,[ber'\,(\bar{a})\,ker'\,(\bar{r}) - bei'\,(\bar{a})\,kei'\,(\bar{r})] \\
&\quad + \bar{b}\,[kei'\,(\bar{b})\,bei'\,(\bar{r}) - ker'\,(\bar{b})\,ber'\,(\bar{r})]\}, \\
T_r &= -\frac{q}{k\,\bar{r}}\Big\{-\bar{a}\,[bei'\,(\bar{a})\,ker'\,(\bar{r}) + ber'\,(\bar{a})\,kei'\,(\bar{r})] \\
&\quad + \bar{b}\,[kei'\,(\bar{b})\,ber'\,(\bar{r}) + ker'\,(\bar{b})\,bei'\,(\bar{r})] + \frac{1}{2\,\bar{r}}\,(\bar{r}^2 - \bar{a}^2)\Big\}, \\
T_\omega &= \frac{q}{k}\Big\{\bar{a}\,[bei'\,(\bar{a})\,ker''\,(\bar{r}) + ber'\,(\bar{a})\,kei''\,(\bar{r})] \\
&\quad - \bar{b}\,[kei'\,(\bar{b})\,ber''\,(\bar{r}) + ker'\,(\bar{b})\,bei''\,(\bar{r})] - \frac{1}{2\,\bar{r}^2}\,(\bar{r}^2 + \bar{a}^2)\Big\};
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
r \leqq a: \quad w &= -\frac{q}{E\,\delta\,k^2}[ber\,(\bar{r})\,K_i + bei\,(\bar{r})\,K_r], \\
M_r &= \frac{q\,\delta}{\sqrt{12}\,k}\,[ber''\,(\bar{r})\,K_i + bei''\,(\bar{r})\,K_r], \\
M_\omega &= \frac{q\,\delta}{\sqrt{12}\,k\,\bar{r}}[ber'\,(\bar{r})\,K_i + bei'\,(\bar{r})\,K_r], \\
T_r &= -\frac{q}{k\,\bar{r}}[ber'\,(\bar{r})\,K_r - bei'\,(\bar{r})\,K_i], \\
T_\omega &= \frac{q}{k}[bei''\,(\bar{r})\,K_i - ber''\,(\bar{r})\,K_r];
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
r = 0: \quad w\,(0,0) &= -\frac{q}{E\,\delta\,k^2}K_i, \\
M_r\,(0,0) &= M_\omega\,(0,0) = \frac{q\,\delta}{2\sqrt{12}\,k}K_r, \\
T_r\,(0,0) &= T_\omega\,(0,0) = \frac{q}{2\,k}K_i,
\end{aligned}
$$

in which $B_i = -\bar{b}\,ber'\,(\bar{b}) + \bar{a}\,ber'\,(\bar{a})$, $\quad B_r = \bar{b}\,bei'\,(\bar{b}) - \bar{a}\,bei'\,(\bar{a})$,

$K_i = -\bar{b}\,ker'\,(\bar{b}) + \bar{a}\,ker'\,(\bar{a})$, $\quad K_r = \bar{b}\,kei'\,(\bar{b}) - \bar{a}\,kei'\,(\bar{a})$,

$$
\bar{a} = \sqrt[4]{\frac{12}{\delta^2}}\,k^2\,a, \qquad \bar{b} = \sqrt[4]{\frac{12}{\delta^2}}\,k^2\,b, \qquad \bar{r} = \sqrt[4]{\frac{12}{\delta^2}}\,k^2\,r.
$$

Appendix

Table 1

$\bar{r}$	$f_i(\bar{r})$				
	$f_1(\bar{r})$	$f_2(\bar{r})$	$f_3(\bar{r})$	$f_4(\bar{r})$	$f_5(\bar{r})$
0.00	0.396	0.396	∞	∞	0.782
0.25	0.382	0.364	0.510	1.004	0.746
0.50	0.360	0.311	0.190	0.666	0.672
0.75	0.334	0.251	0.027	0.478	0.584
1.00	0.305	0.190	−0.066	0.352	0.495
1.25	0.277	0.133	−0.118	0.263	0.410
1.50	0.248	0.083	−0.144	0.197	0.331
1.75	0.222	0.041	−0.156	0.150	0.262
2.00	0.197	0.006	−0.152	0.110	0.202
2.50	0.153	−0.043	−0.129	0.060	0.111
3.00	0.118	−0.067	−0.098	0.031	0.051
3.50	0.091	−0.075	−0.067	0.015	0.016
4.00	0.070	−0.073	−0.042	0.006	−0.002
5.00	0.043	−0.055	−0.012	−0.0002	−0.011
6.00	0.029	−0.036	−0.002	−0.001	−0.007

Table 2

$\bar{r}$	$f_i(\bar{r})$		
	$f_6(\bar{r})$	$f_7(\bar{r})$	$f_8(\bar{r})$
0.00	∞	0.000	0.000
0.25	0.757	0.247	0.009
0.50	0.427	0.239	0.025
0.75	0.252	0.225	0.041
1.00	0.143	0.209	0.058
1.25	0.073	0.190	0.072
1.50	0.026	0.171	0.083
1.75	−0.003	0.153	0.091
2.00	−0.021	0.131	0.096
2.50	−0.035	0.095	0.098
3.00	−0.034	0.064	0.093
3.50	−0.026	0.041	0.083
4.00	−0.018	0.024	0.071
5.00	−0.006	0.006	0.049
6.00	0.000	−0.001	0.032

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Summary

Simplified calculation formulas for determining the vertical displacement and internal forces of elliptical paraboloidal shallow shells under the action of a concentrated load are given in the paper. For the case of shallow shells of unequal radii of curvature, the formulas finally presented are of the empirical type and, with the aid of tables, they are quite convenient for application in design work. Formulas for circular line loads and circular ring loads are also given in the paper for the case of equal radii of curvature in both directions of the shallow shell.

Résumé

Les auteurs présentent des formules simplifiées pour le calcul des déplacements verticaux et des sollicitations des voiles minces en forme de paraboloïde elliptique, soumis à une charge concentrée. Lorsque les rayons de courbure sont différents, les formules finalement données sont du type empirique et, à l'aide de tables, elles peuvent être utilisées pour des études. Pour les voiles présentant des mêmes rayons de courbure dans les deux directions, les auteurs donnent également les formules relatives à des charges appliquées le long d'une circonférence ou sur un anneau circulaire.

Zusammenfassung

Es wird ein vereinfachtes Berechnungsverfahren für die Bestimmung der vertikalen Verschiebungen und der inneren Kräfte von elliptisch-paraboloidförmigen, schwachgekrümmten Schalen unter einer Einzellast behandelt. Für den Fall der schwachgekrümmten Schalen mit ungleichen Krümmungsradien werden empirische Berechnungsformeln aufgestellt; sie können mit Hilfe der Tabellen in der Baupraxis sehr einfach verwendet werden. Die Formeln für eine kreisförmige Linienlast und für eine Ringbelastung werden nur für Schalen mit gleichen Krümmungsradien in beiden Richtungen angegeben.

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