Dynamic analysis of floor systems

Dynamic Analysis of Floor Systems

Calcul dynamique des planchers

Schwingungsberechnung von Decken

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Introduction

In this paper a method is presented for analysis of floor systems based on a similar technique by SAIBEL and D'APPOLONIA [3] for the treatment of continuous beams. The problem is considered as a two-dimensional plate problem with the slab of the floor system resting on supporting columns, and solution of a fourth order partial differential equation complying with the necessary boundary condition is involved. For solution of the equation the normal mode method is employed together with some properties of orthogonal functions.

A simple frequency equation is derived in terms of the number of "symmetrically situated" columns supporting a floor system. Numerical solutions for the equation, which is of cubic form, are obtained by trial and error methods for a few cases and from the data obtained an approximate formula is derived. A general frequency equation is also derived in terms of the stiffness number for floor system with elastic interior columns. Finally, a brief analysis of mode shape due to free and forced vibrations of floor system is included.

An ideal elastic structure is one in which no internal damping forces exist. Such a structure may vibrate for an infinitely long period of time without the application of external forces, other than those required to initiate the motion. The vibration can occur in any one of several natural modes in which each point in the structure executes harmonic motion about a position of static equilibrium, every point passing through its position of equilibrium at the same instant and reaching its peak position at the same instant. Thus the frequency of oscillation is the same at every point and this is the natural frequency of the system in the particular mode involved, which is conveniently expressed in terms of the deflected configuration of the structure in the extreme position.

The equilibrium of a dynamical system may be conveniently expressed by application of any one of the following basic principles: 1. Newton's laws of equilibrium; 2. D'Alembert's principle and the associated principle of virtual work; 3. Hamilton's principle and the associated equations of Lagrange. Since physical structures are always more or less continuous, exact governing equations must be in the form of differential or integral equations in which the independent space variables may vary continuously. However, in all but exceptional cases, solution of these equations is not possible unless approximations are made with respect to the actual boundary conditions and to the mode of solution of the equations. The validity of the results obtained will depend directly on how well the assumed mathematical model portrays the actual structure. In this respect structural damping of the real structure plays a dominant role and hence confirmation of theoretical concepts by actual structural records is necessary.

Existing Methods of Analysis for Floor Systems

The most notable contributions to analysis of floor-systems have been made by Bleich [1] and Rogers [2]. Bleich's method, essentially, is to reduce the complex structural system to a number of simpler structures, the natural frequency of which can be easily found. The simpler structures are termed complementary systems.

Consider the simplified beam and girder floor system shown in Fig. 1. If the moment of inertia of the main girder is assumed to be infinitely great, ($I_0 = \infty$), the problem resolves itself into a system of independently vibrating floor beams which may be termed System A. The normal shapes and frequencies of each of the floor beams can be readily determined. On the other hand if it is assumed that all the floor beams are infinitely rigid, ($I_B = \infty$), and the main girder remains flexible, a complementary system, (System B), is formed. The normal mode shapes and frequencies of this system must also be determined before the whole system can be analysed.

For the complementary system Bleich has shown that

$$X = X_A + X_B, \tag{1}$$

where X = shape function of the entire structure,
X_A = shape function of system A,
X_B = shape function of system B.

Eq. (1) holds true regardless of the actual nature of the loads as long as the members forming the structure can be considered elastic. By expressing the shape functions of the complementary systems X_A and X_B in terms of a series expansion of the normal functions, and making use of the general theories of oscillation and properties of orthogonal functions the following set of equations is obtained [1, 2]:

$$\begin{aligned} C_{1A}\left[1-\frac{p_{1A}^2}{p^2}+\sum_k \frac{\beta_{1k}^2}{(p_{kB}^2/p^2)-1}\right]+C_{2A}\sum_k \frac{\beta_{1k}\beta_{2k}}{(p_{kB}^2/p^2)-1}+\cdots &= 0, \\ C_{1A}\sum_k \frac{\beta_{1k}\beta_{2k}}{(p_{kB}^2/p^2)-1}+C_{2A}\left[1-\frac{p_{2A}^2}{p^2}+\sum_k \frac{\beta_{2k}^2}{(p_{kB}^2/p^2)-1}\right]+\cdots &= 0. \end{aligned} \tag{2}$$

Eq. (2) is comprised of a set of i homogeneous, algebraic equations in the unknowns C_{iA}, and non-trivial solutions will exist only if the determinant of coefficients vanishes. This determinant is the frequency equation for the structure. For more general types of statically indeterminate structures application of the above method becomes highly complicated.

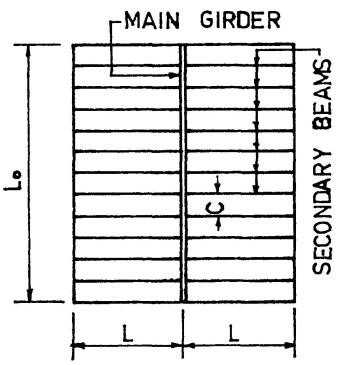

Fig. 1. Simplified floor-system.

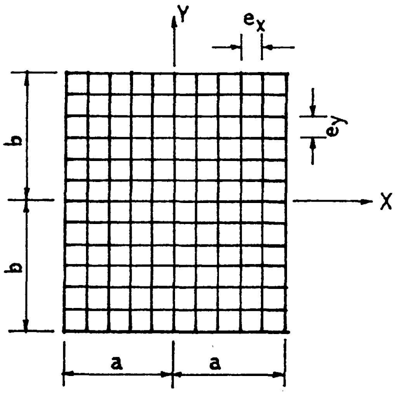

Fig. 2. Grid floor-system.

For the type of structure shown in Fig. 1. Rogers considers the dynamic shear at the end of each floor beam as a dynamic load on the main girder. By a simple approximation these concentrated loads are converted into an equivalent uniformly distributed load represented by:

$$w(x',t) = \frac{E\,I\,\lambda_n^3}{e}(\cot\lambda_n L - \coth\lambda_n L)\,y_0(x',t). \tag{3}$$

The problem is then solved for a forced vibration of the main girder as in Eq. (3). The governing equation is

$$E_0 I_0 \frac{\partial^4 y_0}{\partial x'^4} + w_0 \frac{\partial^2 y_0}{\partial t^2} = w(x',t). \tag{4}$$

After consideration of the boundary conditions of the main girder the frequency equation is given by:

$$p_n^2 = n^4 p_0^2 + \frac{E\,I\,\lambda_n^3}{w_0\,e}(\cot\lambda_n L - \coth\lambda_n L), \tag{5}$$

where n designates the particular mode.

A more general case of a grid-floor system with beams running in the x and y directions, as shown in Fig. 2., has also been investigated by Rogers. If, in any given direction the beams are assumed to have the same flexural rigidity, EI; the same spacing, e; and the same mass per unit length, w; the free vibration of the grid is expressed, approximately, by the equation:

$$\frac{E_x I_x}{w_x e_x} \frac{\partial^4 y}{\partial x^4} + \frac{E_y I_y}{w_y e_y} \frac{\partial^4 y}{\partial y^4} + \frac{\partial^2 y}{\partial t^2} = 0. \qquad (6)$$

For the grid shown in Fig. 2 with simply-supported edges, its fundamental frequency is given by:

$$p^2 = \frac{\pi^4 D_y}{16\, b^4 w_y} + \frac{\pi^4 D_x}{16\, a^4 w_x} \qquad (7)$$

and the mode by.

$$y = C \cos \lambda x \cos \lambda' y \,(A \sin p_{11} t + B \cos p_{11} t). \qquad (8)$$

Conventional analysis treats a floor-system as a set of vibrating beams, the mass of the floor slab being either neglected completely or distributed on to the beams. Thus the system is left vibrating as a free body and the effects of interior supporting columns are neglected. But the interior columns and the floor slab mass have significant effects, particularly, on the natural periods and mode shapes of the floor system. These effects are considered in the following sections.

Floor Systems on Rigid Columns

Consider the floor system shown in Fig. 3, consisting of a rectangular slab $ABCD$ of constant thickness, h; modulus of elasticity, E; Poisson's ratio, μ; deflexion Z; mass per unit area, W; and lengths a and b along the x and y axes respectively. It is internally supported by any arbitrary number, N, of rigid columns.

Taking A as the origin the various columns are located by the coordinates

$$(c_1, d_1),\ (c_2, d_2),\ \ldots\ (c_N, d_N).$$

For dynamic analysis of the floor system the following assumptions are made:

1. No deformation occurs in the middle plane of the slab. This plane remains central during bending.
2. Points of the slab lying initially on the normal to its middle plane remain so after bending.
3. The normal stresses perpendicular to the plane of the slab can be neglected.
4. The material of the slab is isotropic.
5. The floor system is taken as a flat slab.
6. The edges of the floor system are assumed simply-supported and the contact surface of the slab with each interior supporting column is taken as a single point.

Assumptions (1), (2), (3) and (4) are those usually adopted in the theory of thin plates. Assumption (5) is reasonable because most of the stiffening power comes from the interior columns. Assumption (6) ignores the rotational resistance but not the translationary resistance of the boundary walls and columns on the grounds that the flexural resistance offered by these members is relatively small. The assumption of point contact at the interior columns is also reasonable unless dropheads are used and it simplifies the analysis considerably.

If the system is considered equivalent to a simply-supported rectangular slab having the same dimensions and distribution of elastic properties as the actual system, and subjected to time-dependent concentrated loads, where the interior columns occur, the governing differential equation is given by:

$$D\,\nabla^2\nabla^2(Z) + W\frac{\partial^2 Z}{\partial t^2} = f(x,y,t). \tag{9}$$

The term $f(x,y,t)$ represents the applied load which, in this instance, is the reaction offered by the interior columns. If this term becomes zero then Eq. 9 becomes the governing equation of free vibration for a similar slab simply-supported at the boundaries and having no columns.

Since the whole system is vibrating freely the column reactions must depend upon the frequency of the system. If R_i represents the amplitude of the time-dependent column reaction P_i, acting at the point c_i, d_i then:

$$P_i = R_i \cos(\omega t - \theta), \tag{10}$$

where ω = frequency of the system

and θ = phase angle of the motion

and values of P_1 and θ should be determined through the initial time conditions. The concentrated column reaction, P_1, can be expressed as an equivalent distributed load in terms of the normal functions (3) by:

$$f(x,y,t) = \sum_{i=1}^{N} R_i \cos(\omega t - \theta) \sum_m \sum_n W\,\bar{Z}_{mn}(x,y)\,\bar{Z}_{mn}(c_i, d_i), \tag{11}$$

where $i = 1, 2, 3, \ldots N$ depending on the number of interior columns.

Substituting Eq. 11 into Eq. 9 the solution will take the form

$$Z(x,y,t) = \sum_m \sum_n \bar{Z}_{mn}(x,y)\, q_{mn}(t), \tag{12}$$

which leads to

$$\begin{aligned} &\sum_m \sum_n [W\, p_{mn}^2\, \bar{Z}_{mn}(x,y)\, q_{mn}(t) + W\,\bar{Z}_{mn}(x,y)\, q_{mn}(t)] = \\ &\qquad \sum_i^N R_i \cos(\omega t - \theta) \sum_m \sum_n W\,\bar{Z}_{mn}(x,y)\,\bar{Z}_{mn}(c_i, d_i). \end{aligned} \tag{13}$$

If the coefficients of $\overline{Z}_{mn}(x, y)$ are matched to get the equation on the time function $q_{mn}(t)$, then:

$$q_{mn}(t) + p_{mn}^2 q_{mn}(t) = \sum_i^N R_i \cos(\omega t - \theta)\, \overline{Z}_{mn}(c_i, d_i). \tag{14}$$

The solution of Eq. 14 takes the form:

$$q_{mn}(t) = D_{mn} \cos(\omega t - \theta), \tag{15}$$

where

$$D_{mn} = \frac{\sum_i^N R_i \overline{Z}_{mn}(c_i, d_i)}{p_{mn}^2 - \omega^2}. \tag{16}$$

Since the deflexion at each point of column support must equal zero, then for a column support at (c_j, d_j)

$$Z(c_j, d_j, t) = \sum_m \sum_n \overline{Z}_{mn}(c_j, d_j)\, D_{mn} \cos(\omega t - \theta) = 0 \tag{17}$$

or

$$\sum_m \sum_n \frac{\sum_i^N R_i \overline{Z}_{mn}(c_i, d_i)\, Z_{mn}(c_j, d_j)}{(p_{mn}^2 - \omega^2)} = 0. \tag{18}$$

There will be as many equations of type (18) as there are interior supports, each containing as many terms as there are interior supports, and all these equations are linear and homogeneous in the constants, R. By setting the determinant equal to zero the frequencies of the floor system can be obtained, the least value giving the fundamental natural frequency.

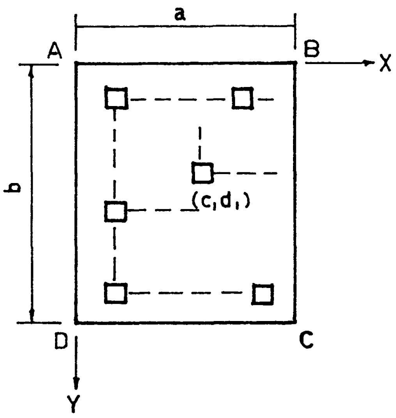

Fig. 3.

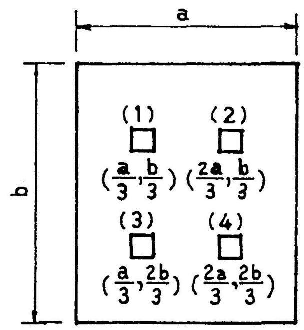

Fig. 4.

Actual solution of Eq. 18, may become intractable if more than four interior columns are considered, or more than two terms of the series. But if the columns are symmetrically spaced about the centrelines of the floor system the amount of computation is greatly reduced.

Consider, for example, the floor system with four symmetrically placed interior columns shown in Fig. 4. For brevity, the column positions are numbered (1), (2), (3) and (4) respectively.

Due to the symmetry of the system, the frequency equation, is given by Eq. 19.

$$\left.\begin{aligned}
&\sum_m \sum_n \frac{R_1 \bar{Z}_{mn}(1)\,\bar{Z}_{mn}(1)}{(p_{mn}^2-\omega^2)} + \sum_m \sum_n \frac{R_2 \bar{Z}_{mn}(1)\,\bar{Z}_{mn}(2)}{(p_{mn}^2-\omega^2)} \\
&\qquad + \sum_m \sum_n \frac{R_3 \bar{Z}_{mn}(1)\,\bar{Z}_{mn}(3)}{(p_{mn}^2-\omega^2)} + \sum_m \sum_n \frac{R_4 \bar{Z}_{mn}(1)\,\bar{Z}_{mn}(4)}{(p_{mn}^2-\omega^2)} = 0, \\
&\sum_m \sum_n \frac{R_1 \bar{Z}_{mn}(2)\,\bar{Z}_{mn}(1)}{(p_{mn}^2-\omega^2)} + \sum_m \sum_n \frac{R_2 \bar{Z}_{mn}(2)\,\bar{Z}_{mn}(2)}{(p_{mn}^2-\omega^2)} \\
&\qquad + \sum_m \sum_n \frac{R_3 \bar{Z}_{mn}(2)\,\bar{Z}_{mn}(3)}{(p_{mn}^2-\omega^2)} + \sum_m \sum_n \frac{R_4 \bar{Z}_{mn}(2)\,\bar{Z}_{mn}(4)}{(p_{mn}^2-\omega^2)} = 0, \\
&\sum_m \sum_n \frac{R_1 \bar{Z}_{mn}(3)\,\bar{Z}_{mn}(1)}{(p_{mn}^2-\omega^2)} + \sum_m \sum_n \frac{R_2 \bar{Z}_{mn}(3)\,\bar{Z}_{mn}(2)}{(p_{mn}^2-\omega^2)} \\
&\qquad + \sum_m \sum_n \frac{R_3 \bar{Z}_{mn}(3)\,\bar{Z}_{mn}(3)}{(p_{mn}^2-\omega^2)} + \sum_m \sum_n \frac{R_4 \bar{Z}_{mn}(3)\,\bar{Z}_{mn}(4)}{(p_{mn}^2-\omega^2)} = 0, \\
&\sum_m \sum_n \frac{R_1 \bar{Z}_{mn}(4)\,\bar{Z}_{mn}(1)}{(p_{mn}^2-\omega^2)} + \sum_m \sum_n \frac{R_2 \bar{Z}_{mn}(4)\,\bar{Z}_{mn}(2)}{(p_{mn}^2-\omega^2)} \\
&\qquad + \sum_m \sum_n \frac{R_3 \bar{Z}_{mn}(4)\,\bar{Z}_{mn}(3)}{(p_{mn}^2-\omega^2)} + \sum_m \sum_n \frac{R_4 \bar{Z}_{mn}(4)\,\bar{Z}_{mn}(4)}{(p_{mn}^2-\omega^2)} = 0.
\end{aligned}\right\} \quad (19)$$

Since the relative positions of the columns are known the corresponding normal functions can be written as

$$\bar{Z}_{mn}(1) = C \sin\frac{m\pi}{3}\sin\frac{n\pi}{3}, \qquad \bar{Z}_{mn}(2) = C \sin\frac{2m\pi}{3}\sin\frac{n\pi}{3},$$

$$\bar{Z}_{mn}(3) = C \sin\frac{m\pi}{3}\sin\frac{2n\pi}{3}, \qquad \bar{Z}_{mn}(4) = C \sin\frac{2m\pi}{3}\sin\frac{2n\pi}{3}.$$

Because of the rapid convergence of the series in Eq. 18 it is sufficient to consider values of m and $n \leqq 6$.

It can be shown that all of the interior column reactions, R, have the same absolute value and differ only in sign. The following possibilities exist:

$$\begin{aligned}
&1.\ R_1 = R_2 = R_3 = R_4, \\
&2.\ R_1 = -R_2 = -R_3 = R_4, \\
&3.\ R_1 = R_2 = -R_3 = -R_4, \\
&4.\ R_1 = -R_2 = R_3 = -R_4.
\end{aligned} \qquad (20)$$

Using the four possible arrangements of R-values shown in Eq. 20 four corresponding frequency equations can be obtained:

$$\frac{1}{l_{11}^2-\lambda^2}+\frac{1}{l_{15}^2-\lambda^2}+\frac{1}{l_{51}^2-\lambda^2}+\frac{1}{l_{55}^2-\lambda^2}=0, \tag{21}$$

$$\frac{1}{l_{22}^2-\lambda^2}+\frac{1}{l_{24}^2-\lambda^2}+\frac{1}{l_{42}^2-\lambda^2}+\frac{1}{l_{44}^2-\lambda^2}=0, \tag{22}$$

$$\frac{1}{l_{12}^2-\lambda^2}+\frac{1}{l_{14}^2-\lambda^2}+\frac{1}{l_{52}^2-\lambda^2}-\frac{1}{l_{54}^2-\lambda^2}=0, \tag{23}$$

$$\frac{1}{l_{21}^2-\lambda^2}+\frac{1}{l_{25}^2-\lambda^2}+\frac{1}{l_{41}^2-\lambda^2}+\frac{1}{l_{45}^2-\lambda^2}=0, \tag{24}$$

where $\quad l_{mn}^2=(m^2\alpha^2-n^2)^2, \quad K^2=\dfrac{\pi^4 D}{b^4 W}, \quad p_{mn}^2=K^2 l_{mn}^2,$

$$\alpha=\frac{b}{a}=\text{side ratio}, \quad \lambda=\frac{W}{K}=\text{Frequency number.}$$

The least value of the solution is given by Eq. 22 which determines the fundamental frequency of the floor system.

For the general case of a floor system with mn symmetrically located, infinitely rigid columns, with m columns in each row parallel to the x-axis and n columns per row, parallel to the y-axis (i.e. $i=m$, $j=n$), the frequency equation may be written as:

$$\frac{1}{l_{m,n}^2-\lambda^2}+\frac{1}{l_{m+2,n}^2-\lambda^2}+\frac{1}{l_{m,n+2}^2-\lambda^2}+\frac{1}{l_{m+2,n+2}^2-\lambda^2}=0. \tag{24}$$

Eq. 24 has as its parameters, K, α, m and n. For definite values of m and n, the equation, after expansion, becomes a cubic equation in the square of the frequency number, λ^2. It can be written as:

$$\begin{aligned}4(\lambda^2)^3 &- 3(\lambda^2)^2[l_{m,n}^2+l_{m,n+2}^2+l_{m+2,n}^2+l_{m+2,n+2}^2]\\ &+2(\lambda^2)[l_{m,n}^2 l_{m,n+2}^2+l_{m,n}^2 l_{m+2,n}^2+l_{m,n}^2 l_{m+2,n+2}^2\\ &\quad+l_{m,n+2}^2 l_{m+2,n}^2+l_{m,n+2}^2 l_{m+2,n+2}^2+l_{m+2,n}^2 l_{m+2,n+2}^2]\\ &-[l_{m,n}^2 l_{m,n+2}^2 l_{m+2,n}^2+l_{m,n}^2 l_{m,n+2}^2 l_{m+2,n+2}^2+l_{m,n+2}^2 l_{m+2,n}^2 l_{m+2,n+2}^2\\ &\quad+l_{m,n}^2 l_{m+2,n}^2 l_{m+1,n+1}^2]=0.\end{aligned} \tag{25}$$

Eq. 25 has been solved, by trial and error process, for different values of α, m and n to obtain the frequency numbers, λ, which are shown in Table 1. The relation between λ and the side-ratio, α, is shown in Fig. 5.

Table 1. Frequency number with change of side ratio (for rigid columns)

m	n	$\alpha=1.0$	$\alpha=1.1$	$\alpha=1.2$	$\alpha=1.3$	$\alpha=1.4$	$\alpha=1.5$
1	1	5.81	6.34	6.79	7.18	7.54	7.89
1	2	9.07	9.80	10.55	11.24	11.85	12.39
2	2	12.94	14.20	15.42	16.60	17.85	19.10
2	3	18.27	19.74	21.23	22.69	24.10	25.55
3	3	22.90	26.40	28.74	31.43	33.92	36.90
3	4	31.42	34.06	36.74	39.43	42.32	45.24

Actual numerical solution of the general frequency equation when applied to a particular case is tedious. For this reason an approximate expression has been worked out from the data given earlier in the paper. This is given below:

$$\lambda = 5.623\,(m\,x\,n)^{0.82}\,(\alpha - 1) + 5.689\,(m\,x\,n)^{0.66}. \tag{26}$$

By comparison with the results obtained from Eq. 25, the frequency values err, on the average, by 6.6% for α-values ranging from 1.0 to 1.4. If α is 1.5 then the error is 12%. But Eq. 26 is limited in its application, to differences in value between m and n not greater than one: i.e. $0 \leqq (n-m) \leqq 1$.

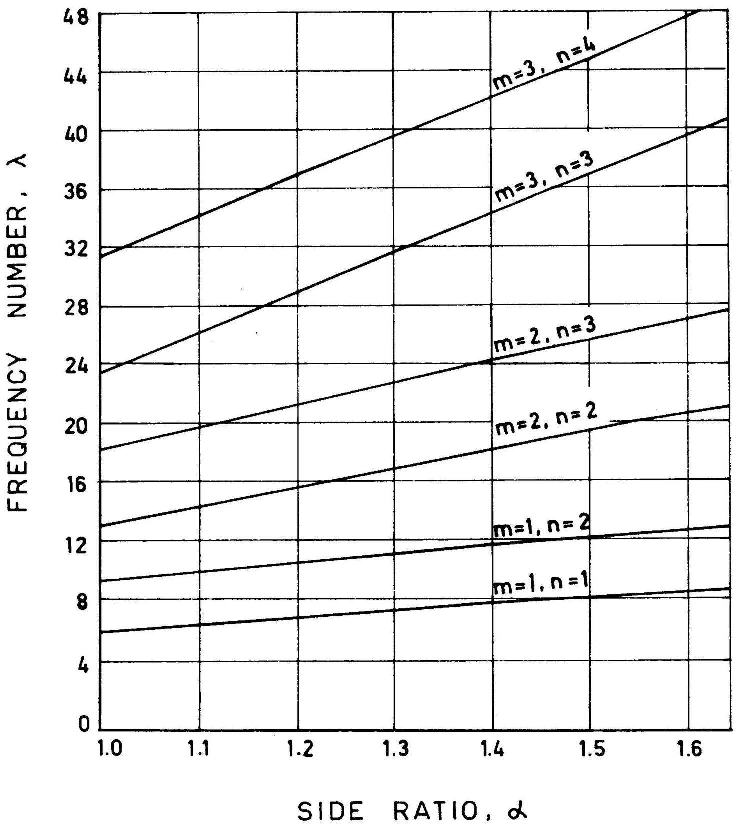

Fig. 5.

Floor System on Elastic Columns

When the interior supporting columns are elastic and have different spring constants, k, analysis can be performed using techniques similar to those shown in the preceding section.

Let the column at point x_i, y_i have a spring constant k_i. The column reaction P_i is represented by:

$$P_i = k_i\,Z\,(x_i, y_i, t)\,. \tag{27}$$

Hence it can be shown that the column reaction is given by:

$$R_i = k_i \sum_m \sum_n Z_{mn}\,(x_i, y_i)\,D_{mn} \tag{28}$$

or for support at j

$$R_j = k_j \sum_m \sum_n \frac{\sum_i^N R_i\, Z_{mn}(x_i, y_i)\, Z_{mn}(x_j, y_j)}{(p_{mn}^2 - \omega^2)}. \tag{29}$$

Again, a set of equations of type 29, and linear and homogeneous with respect to R, results. The spring constant will vary from zero for a fully flexible column to infinity for a perfectly rigid column. The two limiting cases have already been defined. For intermediate cases, either all the interior columns have the same degree of elasticity or different values of k. Where k is constant the analysis is considerably simplified, but if k varies, some approximations must be made in order to achieve a solution.

Dealing first with the case where k is constant, it has a dimension of length/force and should have a negative sine in order to ensure that a positive deflexion relates to an upward, or negative load.

Denoting
$$k\frac{(m+1)(n+1)}{W a b K^2} = -G, \tag{30}$$

where G can be termed the stiffness number of the floor system, the frequency equation may be written as:

$$1 + G\left[\frac{1}{l_{m,n}^2 - \lambda^2} + \frac{1}{l_{m,n+2}^2 - \lambda^2} + \frac{1}{l_{m+2,n}^2 - \lambda^2} + \frac{1}{l_{m+2,n+2}^2 - \lambda^2}\right] = 0. \tag{31}$$

Eq. 31 has been solved for floor systems having 1, 2, 4, 6, 9 and 12 symmetrically located interior columns and values of G equal to 1, 10, 10^2, 10^3, 10^4 and ∞ respectively. The results are grouped in Tables 2 to 7 and show that the frequency number, λ, is linearly related to the side ratio, α, for any particular floor system having a particular value of stiffness number, G. The main variation of frequency number with stiffness number lies within the range, $G=10$ to $G=10^3$ and as far as dynamic analysis is concerned, interior columns with G-values greater than 10^3 can be regarded as infinitely rigid, in the analysis.

It should be noted, however, that the value of the stiffness number depends on the value of the column spring constant and on the column numbers, m and n. Hence, in order to compare different floor systems with interior columns having the same spring constant, the stiffness number, G, must be divided by the corresponding number $(m+1)(n+1)$.

Where the spring constant, k_i, differs for every interior column the labour involved in solving Eq. 29 becomes prohibitive. Some simplification can be achieved by neglecting some of the terms $\frac{1}{l_{mn}^2 - \lambda^2}$, and this seems reasonable providing $l_{mn} \gg \lambda$, where λ can be estimated from the general equation 24. But the remaining equation may, even then, remain too laborious for solution. Besides no practical advantage seems to be gained by treating the interior columns as having differing spring constants.

Table 2. Influence of stiffness number and side ratio on frequency number

Floor System with Column Numbers: $m=1$, $n=1$						
Log G	α					
	1.0	1.1	1.2	1.3	1.4	1.5
0	2.23	2.42	2.63	2.87	3.12	3.40
1	3.46	3.62	3.80	3.98	4.19	4.41
2	5.32	5.72	6.07	6.38	6.67	6.98
3	5.75	6.27	6.70	7.09	7.45	7.79
4	5.80	6.33	6.78	7.17	7.53	7.87
∞	5.80	6.34	6.79	7.18	7.54	7.89

Table 3. Influence of stiffness number and side ratio on frequency number

Floor System with Column Numbers: $m=1$, $n=2$						
Log G	α					
	1.0	1.1	1.2	1.3	1.4	1.5
0	5.09	5.30	5.53	5.77	6.04	6.34
1	5.74	6.01	6.23	6.45	6.80	6.96
2	8.03	8.45	8.85	9.22	9.55	9.90
3	8.95	9.63	10.33	10.97	11.53	12.05
4	9.06	9.78	10.53	11.21	11.80	12.36
∞	9.07	9.80	10.55	11.24	11.85	12.39

Table 4. Influence of stiffness number and side ratio on frequency number

Floor System with Column Numbers: $m=2$, $n=2$						
Log G	α					
	1.0	1.1	1.2	1.3	1.4	1.5
0	8.08	8.90	9.80	10.80	11.90	13.20
1	8.55	9.35	10.23	11.17	12.23	13.37
2	10.90	11.73	12.60	13.50	14.50	15.52
3	12.60	13.80	14.95	16.10	17.32	18.52
4	12.88	14.13	15.37	16.55	17.80	19.07
∞	12.94	14.20	15.42	16.60	17.85	19.10

Table 5. Influence of stiffness number and side ratio on frequency number

Floor System with Column Numbers: $m=2$, $n=3$						
Log G	α					
	1.0	1.1	1.2	1.3	1.4	1.5
0	13.04	13.88	14.80	15.80	16.60	18.09
1	13.36	14.19	15.10	16.07	17.08	18.44
2	15.36	16.21	17.11	18.05	19.03	20.14
3	17.75	19.08	20.40	21.70	23.05	24.23
4	18.17	19.67	21.18	22.60	24.00	25.41
∞	18.27	19.74	21.23	22.69	24.10	25.55

Table 6. Influence of stiffness number and side ratio on frequency number

Floor System with Column Numbers: $m=3$, $n=3$						
Log G	α					
	1.0	1.1	1.2	1.3	1.4	1.5
0	18.20	19.92	22.00	24.18	26.63	29.20
1	18.28	20.06	22.20	24.40	26.80	29.50
2	20.02	21.80	23.80	25.90	28.30	30.34
3	22.43	25.30	27.60	29.90	32.40	34.94
4	22.85	26.30	28.70	31.30	33.90	36.40
∞	22.90	26.40	28.74	31.43	33.92	36.90

Table 7. Influence of stiffness number and side ratio on frequency number

Floor System with Column Numbers: $m=3$, $n=4$						
Log G	α					
	1.0	1.1	1.2	1.3	1.4	1.5
0	25.05	26.46	28.96	31.69	33.41	36.29
1	25.12	26.58	29.07	31.97	33.80	36.60
2	26.65	28.38	30.42	33.27	35.10	37.42
3	30.08	32.37	34.67	37.42	39.60	42.16
4	31.31	34.00	36.62	39.31	42.25	45.16
∞	31.42	34.06	36.74	39.43	42.32	45.24

Mode Shapes

The mode shapes of the floor system for free vibrations can be determined once the frequencies are known. The column reactions, R, are first determined within an arbitrary limit and these, when substituted into Eq. 12, give the following expression for the mode shape of the system:

$$Z(x,y,t) = \sum_m \sum_n \bar{Z}_{mn}(x,y) \frac{\sum_i^N R_i \bar{Z}_{mn}(c_i, d_i)}{(p_{mn}^2 - \omega^2)} (A\cos\omega t - B\sin\omega t). \qquad (32)$$

The arbitrary constants A and B in Eq. 32, should be evaluated using the initial time conditions.

If the floor system is subjected to forced vibration the general analytical principles involved can be illustrated by reference to the simple case of a single concentrated sinusoidal force, F.

If the force is

$$F = F_0 \sin f t, \qquad (33)$$

where f = frequency of the applied load, the distributed load has the form:

$$f(x, y, t) = F_0 \sin ft \sum_m \sum_n W \bar{Z}_{mn}(x, y) \bar{Z}_{mn}(u, v). \tag{34}$$

(u, v) being the point of application of the concentrated force.

The distributed load in Eq. 34 is added to the r.h.s. of Eq. 9 to form the new governing differential equation. Its solution consists of a homogeneous part given by Eq. 32 and a particular solution to be determined. The particular solution is investigated by using Eq. 9, which yields the following:

$$q_{mn}(t) + p_{mn}^2 q_{mn}(t) = \sum_i^N F_i Z_{mn}(c_i, d_i) + F_0 \sin ft \bar{Z}_{mn}(u, v). \tag{35}$$

Solution of Eq. 35 for a steady-state forced vibration will be of the form:

$$q_{mn}(t) = D_{mn} \sin ft \tag{36}$$

with force P_i given by

$$P_i = R_i \sin ft. \tag{37}$$

Substituting from Eqs. 36, and 37 into Eq. 35 gives:

$$D_{mn} = \frac{\sum_i^N R_i \bar{Z}_{mn}(c_i, d_i) + F_0 \bar{Z}_{mn}(u, v)}{(p_{mn}^2 - f^2}. \tag{38}$$

By using the conditions of constraint at the interior supports

$$Z(c_i, d_i, t) = 0, \tag{39}$$

or

$$Z_{mn}(c_i, d_i) D_{mn} = 0 \tag{40}$$

the values of the coefficients R_i may be found from the equation:

$$\sum_m \sum_n \bar{Z}_{mn}(c_j, d_j) \frac{\sum_i^N R_i Z_{mn}(c_i, d_i) + F_0 \bar{Z}_{mn}(u, v)}{(p_{mn}^2 - f^2)} = 0. \tag{41}$$

N being the number of interior supports. This yields a set of N linear but non-homogeneous equations which can be solved explicitly for the R coefficients. With the R-values known, the constant D_{mn} can be found and, finally, the particular solution for the deflexion, Z, where:

$$Z(x, y, t) = \sum_m \sum_n \bar{Z}_{mn}(x, y) \frac{\sum_i^N R_i \bar{Z}_{mn}(c_i, d_i) + F_0 \bar{Z}_{mn}(u, v)}{(p_{mn}^2 - f^2)} \sin ft. \tag{42}$$

Nomenclature

a, $2a$	=	length of slab in x-direction
b, $2b$	=	length of slab in y-direction
e	=	spacing
E	=	modulus of elasticity
$E_0\ I_0$	=	flexural rigidity of main girder
$E_x\ I_x$	=	flexural rigidity of beam in x-direction
$E_y\ I_y$	=	flexural rigidity of beam in y-direction
e_x, e_y	=	spacing of beams in x and y directions respectively
F	=	applied sinusoidal force
f	=	frequency of applied load
$f(x, y, t)$	=	force function on slab
G	=	stiffness number
h	=	constant thickness of slab
k	=	spring constants of interior supporting columns
L	=	span length of secondary beams
L_0	=	span length of main girder
N	=	number of columns supporting the slab
P_i	=	reaction of column i
P_n	=	frequency of main girder in the nth mode
P_{mn}	=	natural frequency of slab
q_{mn}	=	generalised coordinate, function of time t only
R_i	=	amplitude of column reaction
W_0	=	mass per unit length of main girder
W_x, W_y	=	mass per unit length of beams in x, y direction respectively
W	=	mass per unit area of slab
X	=	shape function of entire structure
X_A, X_B	=	shape function of system A, B respectively
y_0	=	vertical deflexion of main girder
$Z_{mn}(x, y)$	=	shape function of slab in the (m, n)th mode
z	=	deflexion of slab
ω	=	frequency of floor system
θ	=	phase angle of vibration
μ	=	Poisson's Ratio
λ_n	=	frequency number

References

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Summary

This paper presents a method for analysis of floor-systems considered as plates resting on supporting columns. The fourth-order partial differential equation involved is solved by the normal mode method. A simple frequency equation in terms of the number of "symmetrically situated" columns is derived and some typical cases are solved numerically. The stiffness of the floor system on elastic interior columns is also considered and mode shapes due to free and forced vibrations briefly dealt with.

Résumé

Cet article présente une méthode de calcul des (systèmes de) planchers appuyés sur des colonnes. Les équations différentielles partielles du 4e ordre utilisées sont résolues par la méthode de la surface élastique.

Une équation d'oscillation simple en fonction du nombre de colonnes situées «symétriquement» est établie et quelques exemples typiques ont été résolus. En outre, on considère la rigidité du plancher appuyé sur des colonnes intérieures élastiques et on étudie brièvement la surface élastique due tant à une oscillation libre qu'à une oscillation entretenue.

Zusammenfassung

Dieser Beitrag zeigt die Berechnung von Decken, die auf Stützen ruhen. Die zugrunde liegende partielle Differentialgleichung vierter Ordnung ist durch die normale Biegefläche-Methode gelöst worden. Ebenso wird eine einfache Schwingungsgleichung in Abhängigkeit der Anzahl symmetrisch angeordneter Stützen hergeleitet und einige typische Fälle werden numerisch gelöst. Weiterhin wird die Steifigkeit der Decke auf elastischen Innenstützen, sowie die Biegefläche infolge freier und erzwungener Schwingungen berücksichtigt.