Further results on the deformation of thin shells

Further Results on the Deformation of Thin Shells *)

Résultats supplémentaires sur la déformation des coques minces

Weitere Resultate bei der Deformation dünner Schalen

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Introduction

In order to be able to describe approximately the deformation of a thin shell by means of a two-dimensional mathematical model, the changes in midsurface normal curvature and torsion, referred to as bending strains in the shell literature, are required in addition to the midsurface in-plane (tangential) strains. The definition of these bending strains has been the subject of much discussion during the last ten years. In fact, the problem of the definition of the bending strain tensor of linear shell theory remained an unsettled issue until KOITER introduced his curvature change tensor [1, 2, 3], a tensor which was independently also arrived at by NAGHDI [4, 5, 6, 7] and which was subsequently shown to be a "best" definition [8].

Physical components of changes of curvature and torsion for the non-orthogonal coordinate case were defined in terms of derivatives of the rotation vector [9, 10, 11, 12, 13]. However, no general vectorial definition in terms of the rotation vector has been given for the curvature change tensor, even though the rotation vector has been defined in a general form [14, 15].

As long as the question of bending strains was an open issue, the form of the compatibility equations of linear shell theory, equations first derived by GOL'DENVEIZER [9], could not become a settled issue. This question has also been finalized and agreed upon recently [3, 4, 5, 6].

*) The results presented in this paper were obtained in the course of research sponsored by the National Research Council of Canada, Grant No. A-2736, and the Defence Research Board of Canada, Grant No. 6301-11.

The purpose of this paper is to define the curvature change tensor in a general vectorial form, using derivatives of the rotation vector. In analogy with the definitions of unique symmetric and skew-symmetric tangential strain components, where derivatives of the displacement vector are used, the presently accepted curvature change tensor of linear shell theory is defined as the unique symmetric bending strain tensor whereas the unique skew-symmetric components are shown to be expressible in terms of the symmetric tangential strain components. The results obtained are compared with those of others, where possible, and with our earlier work [13, 16]. In particular, the unsymmetric physical changes in torsion, discussed earlier [11, 12, 13], are expressed directly in terms of the mixed components of the unsymmetric bending strain tensor, and the difference between these torsion components is expressed in terms of symmetric physical tangential and bending strain components. The difference between the second fundamental tensors of the deformed and undeformed midsurface, a quantity which is used by some authors to define curvature changes in linear shell theory [15] and is now generally accepted as the curvature change tensor of nonlinear shell theory [17, 18, 19], is compared with the curvature change tensor of linear shell theory.

Using the partial mixed-derivatives of the displacement and rotation vectors, the equations of compatibility of linear thin shell theory are obtained and compared with the currently accepted form [3, 4, 5]. Finally, the tangential strains of the parallel surface are written in a new form and compared with previous results [16, 20].

Throughout the paper, emphasis is placed on the use of direct vector notation and vector algebra. However, where required, the component form of tensors is used with the usual summation convention and index notation and with repeated Greek and Latin indices implying summation over the range 1, 2, and 1, 2, 3, respectively. Partial differentiation and the covariant derivative with respect to the metric of the undeformed midsurface are indicated by a comma and a vertical stroke in front of a subscript, respectively. Where a midsurface quantity is designated by a lower case letter, the corresponding variable for the parallel surface is denoted by the same upper case letter, by the same lower case letter with a superscript (z), or by a new lower case letter. A bar above a quantity indicates a vector whereas a bar below two indices denotes suspension of the summation convention for the underlined indices. Parentheses around indices are used to designate physical components of a tensor whereas a caret ($\hat{}$) and tilde ($\tilde{}$) above a symbol denote unique symmetric and skew-symmetric components of a tensor, respectively. Wherever the symbol $1 \rightleftarrows 2$ appears next to an expression, a corresponding equation is obtained by an interchange of the indices 1 and 2. The outward surface normal is positive and the right-hand rule is used in defining the positive direction of vectors resulting from cross products. All starred quantities refer to the deformed state and convected coordinates are used to describe the deformation of surfaces. For the sake of brevity, frequent reference will be made to previous work presented elsewhere [13, 16, 21]. Symbols are defined where they first appear in the text and are summarized for convenience under Notations.

I. Review of Geometry of Shell Space

Let $\bar{r}=\bar{r}(x^{\alpha})$ denote the position vector of an arbitrary point, P, on the midsurface of a shell of constant thickness, h, and $x^{\alpha}\,(\alpha=1,2)$ be a set of arbitrary (real) Gaussian surface coordinates. Then, the partial derivatives of the position vector with respect to the surface coordinates are defined as the covariant base vectors, $\bar{a}_{\alpha}=\bar{r}_{,\alpha}$, which in turn are used to define the contravariant base vectors, $\bar{a}^{\alpha}$, the covariant and contravariant unit tangent vectors, $\bar{t}_{\alpha}$ and $\bar{t}^{\alpha}$, the magnitudes of the base vectors, $A_{\alpha}=|\bar{a}_{\alpha}|$, the unit normal to the midsurface, $\bar{n}$, the components of the permutation tensor of the midsurface, $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$, the surface metric components, $a_{\alpha\beta}$ and $a^{\alpha\beta}$, the determinant $a=|a_{\alpha\beta}|$, and the Christoffel symbols of the second kind, $\Gamma^{\gamma}_{\alpha\beta}$, $\Gamma^{3}_{\alpha\beta}$, $\Gamma^{\alpha}_{3\beta}$. The components of the second fundamental tensor of the midsurface, $b_{\alpha\beta}$ and b^{α}_{β}, the components of the third fundamental tensor, $c_{\alpha\beta}$, the normal and tangential curvatures, κ_{α} and $(\kappa_g)_{\alpha}$, the geodesic torsion, $(\tau_g)_{\alpha}$, as well as the radii of normal curvatures and torsion R_{α} and $R_{\alpha\beta}$, for the parametric lines are defined as in [13, 16, 21]. The angle between the tangents to the surface coordinates at their intersection and the infinitesimal distance along any curve, C, on the midsurface are designated by ψ and ds, respectively. The coordinate in the surface normal direction is Cartesian and is denoted by $z=x^3\,(-h/2\leqq z\leqq h/2)$.

As is well-known, an arbitrary vector, $\bar{v}$, in 3 dimensions may be expressed in terms of three linearly independent vectors and may therefore be written as

$$\bar{v} = v^{\alpha}\bar{a}_{\alpha}+v^{3}\bar{n} = v_{\alpha}\bar{a}^{\alpha}+v_{3}\bar{n} = v^{(\alpha)}t_{\alpha}+v^{(3)}\bar{n}, \tag{1}$$

where v^{α} and v_{α} are the contravariant and covariant surface components, $v^3=v_3$ the surface normal components, and $v^{(\alpha)}$ and $v^{(3)}$ the physical components of $\bar{v}$. Because of the special "normal" coordinate system used, $v^3=v_3=v^{(3)}$. The partial derivative of $\bar{v}$ is given by

$$\begin{aligned}\bar{v}_{,\alpha} &= (v^{\lambda}|_{\alpha}-b^{\lambda}_{\alpha}v^{3})\,\bar{a}_{\lambda}+(v^{3}_{,\alpha}+b_{\lambda\alpha}v^{\lambda})\,\bar{n},\\ &= (v_{\lambda|\alpha}-b_{\alpha\lambda}v_{3})\,\bar{a}^{\lambda}+(v_{3,\alpha}+b^{\lambda}_{\alpha}v_{\lambda})\,\bar{n},\end{aligned} \tag{2a, b}$$

where

$$v^{\lambda}|_{\alpha} = v^{\lambda}_{,\alpha}+\Gamma^{\lambda}_{\mu\alpha}v^{\mu}; \qquad v_{\lambda|\alpha}=v_{\lambda,\alpha}-\Gamma^{\mu}_{\lambda\alpha}v_{\mu}. \tag{3a, b}$$

The partial derivatives of the base vectors

$$\bar{a}_{\alpha,\beta} = \Gamma^{\lambda}_{\alpha\beta}\bar{a}_{\lambda}+b_{\alpha\beta}\bar{n}; \qquad \bar{a}^{\alpha}_{,\beta} = -\Gamma^{\alpha}_{\beta\lambda}\bar{a}^{\lambda}+b^{\alpha}_{\beta}\bar{n}; \qquad \bar{n}_{,\alpha} = -b^{\lambda}_{\alpha}\bar{a}_{\lambda} \tag{4a, b, c}$$

are used to define the covariant derivatives

$$\bar{a}_{\alpha|\beta} = \bar{a}_{\alpha,\beta}-\Gamma^{\lambda}_{\alpha\beta}\bar{a}_{\lambda} = b_{\alpha\beta}\bar{n}; \qquad \bar{a}^{\alpha}|_{\beta} = \bar{a}^{\alpha}_{,\beta}+\Gamma^{\alpha}_{\beta\lambda}\bar{a}^{\lambda} = b^{\alpha}_{\beta}\bar{n}. \tag{5a, b}$$

Use will be made of Ricci's lemma i. e. the fact that the metric and permutation tensors are treated as constants in covariant differentiation, thus

$$a_{\alpha\beta|\lambda} = a^{\alpha\beta}|_{\lambda} = \varepsilon_{\alpha\beta|\lambda} = \varepsilon^{\alpha\beta}|_{\lambda} = 0\,. \tag{6}$$

The position vector of a point Q, defined by the intersection of the positive surface normal and a parallel surface at a distance z from the midsurface, is designated by $\bar{R}$ and is given by

$$\bar{R} = \bar{R}\,(x^{\alpha}, z) = \bar{r} + z\,\bar{n}\,. \tag{7}$$

Partial derivatives of $\bar{R}$ with respect to x^{α} are defined as the base vectors of the z surface, $\bar{g}_{\alpha} = \bar{R}_{,\alpha}$, which are used in defining [16, 21] the contravariant base vectors, $\bar{g}^{\alpha}$, the metric components, $g_{\alpha\beta}$ and $g^{\alpha\beta}$, the determinant $g = |g_{\alpha\beta}|$, the magnitudes of the base vectors $H_{\alpha} = |\bar{g}_{\alpha}|$, and the covariant unit tangent vectors of the z surface, $\bar{T}_{\alpha}$. The base vectors of the parallel surface are expressible in terms of midsurface base vectors as

$$\bar{g}_{\alpha} = \bar{a}_{\alpha} - z\,b_{\alpha}^{\lambda}\,\bar{a}_{\lambda} = \mu_{\alpha}^{\lambda}\,\bar{a}_{\lambda}\,, \tag{8}$$

where μ_{α}^{λ} is defined by [4]

$$\mu_{\alpha}^{\lambda} = \delta_{\alpha}^{\lambda} - z\,b_{\alpha}^{\lambda}\,, \tag{9}$$

and where $\delta_{\alpha}^{\lambda}$ is the Kroenecker delta. The determinant of μ_{α}^{λ} can be shown to be [4]

$$\mu = |\mu_{\alpha}^{\lambda}| = \sqrt{\frac{g}{a}} = 1 - 2\,z\,H + z^2\,K\,, \tag{10}$$

where H and K are the mean and Gaussian curvature of the midsurface, respectively.

II. Deformation of Surface

2.1. Tangential Strains

During deformation, the point P moves to a new position P^*. Let the vector $\overline{PP^*}$ be denoted by $\bar{u}$ and the position vector of P^* by $\bar{r}^*$, such that (see Fig. 1)

$$\bar{r}^* = \bar{r}\,(x^{\alpha}) + \bar{u}\,(x^{\alpha})\,, \tag{11}$$

where $\bar{u}$ is the displacement vector, expressed in terms of its contravariant, covariant, and physical components by

$$\bar{u} = u^{\alpha}\,\bar{a}_{\alpha} + u^3\,\bar{n} = u_{\alpha}\,\bar{a}^{\alpha} + u_3\,\bar{n} = u^{(\alpha)}\,\bar{t}_{\alpha} + w\,\bar{n}\,. \tag{12}$$

The base vectors of the deformed surface, $\bar{a}_{\alpha}^*$, are obtained from Eq. (11) as

$$\bar{a}_{\alpha}^* = \bar{r}_{,\alpha}^* = \bar{r}_{,\alpha} + \bar{u}_{,\alpha} = \bar{a}_{\alpha} + \bar{u}_{,\alpha}\,, \tag{13}$$

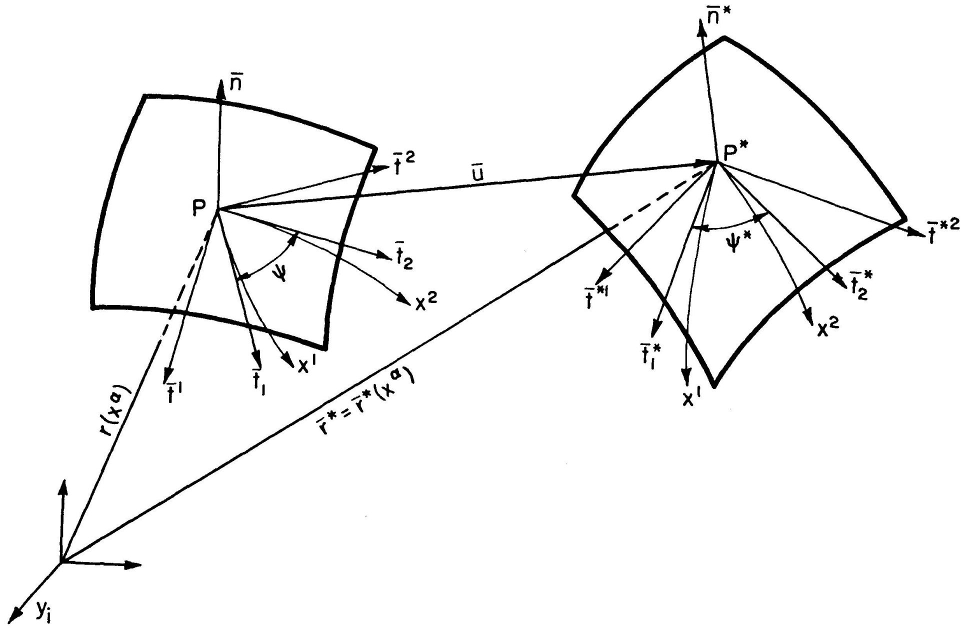

Fig. 1. Deformation of a Surface.

which, when using Eq. (2) to express $\overline{u}_{,\alpha}$ in terms of its components, may be written in the form

$$\overline{a}_\alpha^* = l^\lambda_{.\alpha}\,\overline{a}_\lambda + \varphi_\alpha \overline{n} = \overline{a}_\alpha + \gamma_{\lambda\alpha}\overline{a}^\lambda + \varphi_\alpha \overline{n}, \tag{14}$$

where the unsymmetric tensors, $l^\lambda_{.\alpha}$ and $\gamma_{\lambda\alpha}$ are defined by [17, 4]

$$l^\lambda_{.\alpha} = \delta^\lambda_\alpha + \overline{u}_{,\alpha}\cdot\overline{a}^\lambda = \delta^\lambda_\alpha + u^\lambda|_\alpha - b^\lambda_\alpha w, \tag{15a}$$

$$\gamma_{\lambda\alpha} = u_{\lambda|\alpha} - b_{\lambda\alpha} w \tag{15b}$$

and

$$\varphi_\alpha = w_{,\alpha} + b^\lambda_\alpha u_\lambda. \tag{15c}$$

From Eqs. (13) and (14) the derivative of the displacement vector is written as

$$\overline{u}_{,\alpha} = \gamma_{\lambda\alpha}\overline{a}^\lambda + \varphi_\alpha \overline{n}, \tag{16}$$

from which

$$\gamma_{\beta\alpha} = \overline{u}_{,\alpha}\cdot\overline{a}_\beta. \tag{17}$$

The unique symmetric and skew-symmetric components of this unsymmetric strain tensor are defined by

$$\hat{\gamma}_{\alpha\beta} = \tfrac{1}{2}(\gamma_{\beta\alpha} + \gamma_{\alpha\beta}) = \tfrac{1}{2}(\overline{u}_{,\alpha}\cdot\overline{a}_\beta + \overline{u}_{,\beta}\cdot\overline{a}_\alpha) = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} w, \tag{18a}$$

$$\tilde{\gamma}_{\alpha\beta} = \tfrac{1}{2}(\gamma_{\beta\alpha} - \gamma_{\alpha\beta}) = \tfrac{1}{2}(\overline{u}_{,\alpha}\cdot\overline{a}_\beta - \overline{u}_{,\beta}\cdot\overline{a}_\alpha) = \tfrac{1}{2}(u_{\beta|\alpha} - u_{\alpha|\beta}) = \tfrac{1}{2}(u_{\beta,\alpha} - u_{\alpha,\beta}). \tag{18b}$$

From Eqs. (18a, b), $\hat{\gamma}_{\alpha\beta}$ and $\tilde{\gamma}_{\alpha\beta}$ are easily recognized as the usual symmetric tangential strain tensor and the skew-symmetric rotation tensor of linear shell theory [2, 4, 11, 12]

$$\hat{\gamma}_{\alpha\beta} \equiv \epsilon_{\alpha\beta}; \qquad \tilde{\gamma}_{\alpha\beta} \equiv \omega_{\alpha\beta}. \tag{18c}$$

Using Eqs. (18) the expression for $\overline{u}_{,\alpha}$, Eq. (16), may be rewritten as

$$\overline{u}_{,\alpha} = (\hat{\gamma}_{\alpha\lambda} + \tilde{\gamma}_{\alpha\lambda})\overline{a}^\lambda + \varphi_\alpha \overline{n} = (\epsilon_{\alpha\lambda} + \omega_{\alpha\lambda})\overline{a}^\lambda + \varphi_\alpha \overline{n} \tag{19}$$

and from a comparison of Eqs. (15a, b) and (18), the following relation established

$$l_{\lambda\alpha} = a_{\lambda\alpha} + \gamma_{\lambda\alpha} = a_{\alpha\lambda} + \epsilon_{\alpha\lambda} + \omega_{\alpha\lambda}. \tag{20}$$

The physical extensional tangential strain components are related to the tensorial components by [11, 13]

$$\epsilon_{(11)} \equiv \epsilon_{(1)} = \frac{\epsilon_{11}}{a_{11}} = \frac{\epsilon_{11}}{(A_1)^2} = \frac{1}{A_1}\frac{\partial \bar{u}}{\partial x^1}\cdot \bar{t}_1 \quad 1 \rightleftarrows 2, \tag{21a}$$

whereas the total change in angle between the tangents drawn to the parametric lines, defined as the physical shear strain, is given by [11, 12, 13]

$$\hat{\gamma}_{(12)} = \gamma_{(1)} + \gamma_{(2)} = \frac{2\,\epsilon_{12}}{\sqrt{a}} - \cot\psi\,(\epsilon_{(1)} + \epsilon_{(2)}), \tag{21b}$$

where $$\gamma_{(1)} = \frac{1}{A_1}\frac{\partial \bar{u}}{\partial x^1}\cdot \bar{t}^2 = \frac{A_2 \sin\psi}{A_1} l^2_{.1} \quad 1 \rightleftarrows 2. \tag{21c}$$

As is clear from Eq. (21c), the unsymmetric physical shear strain components are expressible directly in terms of the mixed components of the unsymmetric tangential strain tensor, $l^{\alpha}_{.\beta}$, whereas the sum of these shear strain components is given in terms of symmetric strain components, Eq. (21b). In closing this section on tangential strains, it should be noted that $\epsilon_{\alpha\beta}$ is comprised of the linear terms in the expression $\frac{1}{2}(a^*_{\alpha\beta} - a_{\alpha\beta})$.

2.2. Rotation Vector and Bending Strains

Since deformation of a body in the neighbourhood of a point is composed of translation, rotation, and change in shape (straining) [22, 23], it is not surprising that in addition to displacements, described by the displacement vector $\bar{u}$, rotations and the rotation vector, $\bar{\Omega}$, are also required in the study of the linear deformation of the midsurface of shells. The contravariant components of the rotation vector, ω^i, are defined in terms of the components of the skew-symmetric rotation tensor, ω_{ij}, of linear deformation theory by

$$\omega^i = \tfrac{1}{2}\,\varepsilon^{ijk}\,\omega_{jk} = \tfrac{1}{2}\,\varepsilon^{ijk}\,u_{k|j}; \qquad \omega_{ij} = \tfrac{1}{2}(u_{j|i} - u_{i|j}), \tag{22}$$

where ε^{ijk} are the components of the permutation tensor in 3 space. Thus the components of rotation ω^α and ω^3 for the midsurface are expressed as [11, 13, 16]

$$\omega^\alpha = \frac{1}{2}\varepsilon^{\alpha\beta 3}\,\omega_{\beta 3}; \qquad \omega^1 = \frac{1}{2\sqrt{a}}[u_{3|2} - u_{2|3}]; \qquad \omega^2 = -\frac{1}{2\sqrt{a}}[u_{3|1} - u_{1|3}], \tag{23a}$$

$$\omega^3 = \frac{1}{2}\varepsilon^{3\alpha\beta}\,\omega_{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta}\,\omega_{\alpha\beta} = \frac{1}{2\sqrt{a}}[u_{2|1} - u_{1|2}], \tag{23b}$$

where $\varepsilon^{\alpha\beta 3}$ at the midsurface $(z=0)$ become $\varepsilon^{\alpha\beta}$. In a discussion of thin shell theory based on the Kirchhoff-Love hypothesis, the transverse shear strain components, $\epsilon_{\alpha 3}$, are assumed to be zero and thus

$$\epsilon_{\alpha 3} = \tfrac{1}{2}(u_{\alpha|3} + u_{3|\alpha}) = 0; \qquad \therefore\ u_{\alpha|3} = -u_{3|\alpha}. \tag{24}$$

With this approximation, and using Eqs. (3) and (15c) together with the fact that [24] $\Gamma^{\alpha}_{\beta 3} = \Gamma^{\alpha}_{3\beta} = -b^{\alpha}_{\beta}$, Eqs. (23a) become

$$\omega^{\alpha} = \varepsilon^{\alpha\beta}[w_{,\beta} + b^{\lambda}_{\beta} u_{\lambda}] = \varepsilon^{\alpha\beta}\varphi_{\beta}. \tag{25}$$

Using the three components of rotation thus derived from the skew-symmetric rotation tensor, a vector of rotation, $\bar{\Omega}$, was defined as [11, 13, 16]

$$\bar{\Omega} = \omega^{\alpha}\bar{a}_{\alpha} + \omega^{3}\bar{n} = -\Omega_{(2)}\bar{t}_1 + \Omega_{(1)}\bar{t}_2 + \Omega_{(3)}\bar{n}, \tag{26a}$$

which may also be written in the form

$$\bar{\Omega} = e^{\alpha\beta}\Omega_{(\alpha)}\bar{t}_{\beta} + \Omega_{(3)}\bar{n}, \tag{26b}$$

where the physical components of the rotation vector, $\Omega_{(\alpha)}$ and $\Omega_{(3)}$, are defined in terms of displacement components in [16], and where $e^{\alpha\beta}$ are the components of the permutation symbol related to the permutation tensor and defined by

$$\varepsilon^{\alpha\beta} = \frac{1}{\sqrt{a}}e^{\alpha\beta}; \qquad e^{12} = -e^{21} = 1; \qquad e^{11} = e^{22} = 0. \tag{26c}$$

Eq. (26b) suggests a form for writing the rotation vector in terms of its covariant components, Ω_{α},

$$\bar{\Omega} = \varepsilon^{\alpha\beta}\Omega_{\alpha}\bar{a}_{\beta} + \Omega_3\bar{n} \tag{27}$$

a form which was also used in [15].

A comparison of Eqs. (26) and (27) results in the following relations between the various rotation components

$$\omega^{\alpha} = \varepsilon^{\beta\alpha}\Omega_{\beta}; \qquad \omega^{3} = \Omega_3 = \Omega_{(3)} \equiv \Omega_n \equiv \Omega, \tag{28a, b}$$

$$\Omega_{\alpha} = \frac{\varepsilon_{\alpha\beta}\, e^{\lambda\beta}\,\Omega_{(\lambda)}}{\sqrt{a_{\underline{\beta\beta}}}}; \qquad \Omega_{(\alpha)} = e_{\alpha\beta}\,\varepsilon^{\lambda\beta}\sqrt{a_{\underline{\beta\beta}}}\,\Omega_{\lambda} \tag{28c, d}$$

or

$$\Omega_{\alpha} = \frac{\sqrt{a}\,\Omega_{(\alpha)}}{\sqrt{a_{\underline{\beta\beta}}}}\ (\alpha \neq \beta); \qquad \Omega_{(\alpha)} = \frac{\sqrt{a_{\underline{\beta\beta}}}}{\sqrt{a}}\Omega_{\alpha}\ (\alpha \neq \beta). \tag{28e, f}$$

Also from Eqs. (25) and (28a) one finds

$$\Omega_{\alpha} = -[w_{,\alpha} + b^{\lambda}_{\alpha}u_{\lambda}] = -\varphi_{\alpha}. \tag{29}$$

Using the definition for the vector of rotation, Eq. (27), the following relations are easily obtained

$$\bar{\Omega}\times\bar{n} = \Omega_{\alpha}\bar{a}^{\alpha}; \qquad \bar{\Omega}\times\bar{a}_{\alpha} = \omega_{\alpha\lambda}\bar{a}^{\lambda} - \Omega_{\alpha}\bar{n}, \tag{30a, b}$$

of which the latter is used in rewriting Eq. (19) in the form

$$\bar{u}_{,\alpha} = (\epsilon_{\alpha\lambda} + \omega_{\alpha\lambda})\bar{a}^{\lambda} + \varphi_{\alpha}\bar{n} = (\epsilon_{\alpha\lambda} + \omega_{\alpha\lambda})\bar{a}^{\lambda} - \Omega_{\alpha}\bar{n} = \epsilon_{\alpha\lambda}\bar{a}^{\lambda} + \bar{\Omega}\bar{a}_{\alpha}. \tag{31}$$

In analogy with the definition of $\hat{\gamma}_{\alpha\beta}=\epsilon_{\alpha\beta}$ and $\tilde{\gamma}_{\alpha\beta}=\omega_{\alpha\beta}$, the derivative of the rotation vector is now used to define symmetric and skew-symmetric bending strain components, $\hat{\eta}_{\alpha\beta}$ and $\tilde{\eta}_{\alpha\beta}$. Through the use of Eq. (2), the derivative of $\bar{\Omega}$ is written as [15]

$$\bar{\Omega}_{,\alpha} = \varepsilon^{\beta\lambda}\,\eta_{\beta\alpha}\,\bar{a}_\lambda + \zeta_\alpha\,\bar{n}, \tag{32}$$

where
$$\eta_{\beta\alpha} = \Omega_{\beta|\alpha} - \varepsilon_{\beta\lambda}\,b^\lambda_\alpha\,\Omega = \Omega_{\beta|\alpha} + b^\lambda_\alpha\,\omega_{\lambda\beta} = -[\varphi_{\beta|\alpha} + b^\lambda_\alpha\,\omega_{\beta\lambda}], \tag{33a}$$
$$\zeta_\alpha = \Omega_{,\alpha} + b_{\alpha\beta}\,\varepsilon^{\lambda\beta}\,\Omega_\lambda. \tag{33b}$$

From Eq. (32), the following relations are easily established

$$\bar{\Omega}_{,\alpha}\times\bar{n} = \eta_{\lambda\alpha}\,\bar{a}^\lambda; \quad \therefore\ \eta_{\beta\alpha} = \bar{\Omega}_{,\alpha}\times\bar{n}\cdot\bar{a}_\beta. \tag{34a, b}$$

The unique symmetric and skew-symmetric components of this unsymmetric bending strain tensor, $\eta_{\beta\alpha}$, are defined by

$$\begin{aligned}\hat{\eta}_{\alpha\beta} &= \tfrac{1}{2}(\eta_{\beta\alpha}+\eta_{\alpha\beta}) = \tfrac{1}{2}(\bar{\Omega}_{,\alpha}\times\bar{n}\cdot\bar{a}_\beta + \bar{\Omega}_{,\beta}\times\bar{n}\cdot\bar{a}_\alpha) = \\ &= \tfrac{1}{2}(\Omega_{\beta|\alpha}+\Omega_{\alpha|\beta}+b^\lambda_\alpha\,\omega_{\lambda\beta}+b^\lambda_\beta\,\omega_{\lambda\alpha}),\end{aligned} \tag{35a}$$

$$\begin{aligned}\tilde{\eta}_{\alpha\beta} &= \tfrac{1}{2}(\eta_{\beta\alpha}-\eta_{\alpha\beta}) = \tfrac{1}{2}(\bar{\Omega}_{,\alpha}\times\bar{n}\cdot\bar{a}_\beta - \bar{\Omega}_{,\beta}\times\bar{n}\cdot\bar{a}_\alpha) = \\ &= \tfrac{1}{2}(\Omega_{\beta|\alpha}-\Omega_{\alpha|\beta}+b^\lambda_\alpha\,\omega_{\lambda\beta}-b^\lambda_\beta\,\omega_{\lambda\alpha}).\end{aligned} \tag{35b}$$

Upon recalling the definition of Koiter's curvature change tensor of linear shell theory, $\rho_{\alpha\beta}$ [1, 2, 3], it is clear that

$$\rho_{\alpha\beta} = -\hat{\eta}_{\alpha\beta} = -\tfrac{1}{2}[\bar{\Omega}_{,\alpha}\times\bar{n}\cdot\bar{a}_\beta + \bar{\Omega}_{,\beta}\times\bar{n}\cdot\bar{a}_\alpha]. \tag{36}$$

Thus the curvature change tensor of linear shell theory has been expressed in a new, vectorial form in terms of derivatives of the rotation vector. The method used in arriving at this definition is identical to that used in defining the symmetric tangential strain tensor in terms of derivatives of the displacement vector. To the best of the author's knowledge, this form of the curvature change tensor has not appeared in the literature heretofore, even though the form of the rotation vector and its derivatives, as used here, were given by CHERNYKH [15].

The skew-symmetric components, $\tilde{\eta}_{\alpha\beta}$, may be expressed in terms of the components of the tangential strain tensor as

$$\tilde{\eta}_{\alpha\beta} = \tfrac{1}{2}(b^\lambda_\alpha\,\epsilon_{\beta\lambda} - b^\lambda_\beta\,\epsilon_{\alpha\lambda}), \tag{37}$$

where the Codazzi-Mainardi relations, $b^\lambda_{\alpha|\beta} = b^\lambda_{\beta|\alpha}$, were used.

In order to compare these symmetric and skew-symmetric bending strain components with the physical components of change in normal curvature and torsion, $\varkappa_{(\alpha)}$ and $\tau_{(\alpha)}$, we recall an expression for the cross product of the derivative of the rotation vector and the unit normal given by [11, 16]

$$\bar{\Omega}_{,1}\times\bar{n} = \frac{A_1}{\sin\psi}[\bar{t}^1\,\varkappa_{(1)} + \bar{t}^2(\varkappa_{(1)}\cos\psi + \tau_{(1)}\sin\psi)] \quad 1 \rightleftarrows 2. \tag{38}$$

A comparison of Eqs. (34) and (38) results in

$$\eta_{11} = \hat{\eta}_{11} = (A_1)^2 \varkappa_{(1)}; \qquad \eta_{21} = \sqrt{a}\,(\tau_{(1)} + \varkappa_{(1)} \cot\psi) \quad 1 \rightleftarrows 2, \tag{39a}$$

$$\eta_{12} + \eta_{21} = 2\,\hat{\eta}_{12} = \sqrt{a}\,[\tau_{(1)} + \tau_{(2)} + \cot\psi\,(\varkappa_{(1)} + \varkappa_{(2)})] \quad 1 \rightleftarrows 2, \tag{39b}$$

whereas from the definition of the components $\tau_{(\alpha)}$ [16] one obtains

$$\bar{\Omega}_{,1} \cdot \bar{a}_1 = -(A_1)^2\, \tau_{(1)} \quad 1 \rightleftarrows 2. \tag{40}$$

From Eq. (32) the following relation is derived

$$\bar{\Omega}_{,\alpha} \cdot \bar{a}_\beta = \varepsilon_{\lambda\beta}\, \eta^{\lambda}_{.\alpha} = \varepsilon^{\lambda}_{.\beta}\, \eta_{\lambda\alpha} = \varepsilon^{\mu\lambda} a_{\lambda\beta}\, \eta_{\mu\alpha}, \tag{41}$$

which is used to write

$$\tau_{(1)} = \frac{A_2 \sin\psi}{A_1}\, \eta^{2}_{.1} \quad 1 \rightleftarrows 2. \tag{42}$$

The vectorial definition of the skew-symmetric components of the bending strain tensor, Eq. (35b), together with Eq. (38) leads to

$$\tilde{\eta}_{12} = \frac{\sqrt{a}}{2}\,[\tau_{(1)} - \tau_{(2)} + \cot\psi\,(\varkappa_{(1)} - \varkappa_{(2)})] = -\,\tilde{\eta}_{21}; \tag{43a}$$

$$\tilde{\eta}_{11} = \tilde{\eta}_{22} = 0 \tag{43b}$$

relations which may also be obtained directly from Eqs. (35b) and (39a) and from which, through the use of Eq. (37), one deduces

$$\tau_{(1)} - \tau_{(2)} = \frac{1}{\sin^2\psi}\left\{\frac{\epsilon_{(2)}}{R_{12}} - \frac{\epsilon_{(1)}}{R_{21}} + \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)[\hat{\gamma}_{(12)} + \cot\psi\,(\epsilon_{(1)} + \epsilon_{(2)})]\right\} + \cot\psi\,(\varkappa_{(2)} - \varkappa_{(1)}). \tag{44}$$

Although an expression for the sum $\tau = (\tau_{(1)} + \tau_{(2)})$ was given elsewhere [11, 13, 16], the difference between these unsymmetric physical components of change in torsion was not expressed explicitly heretofore.

The unit normal to the deformed midsurface, $\bar{n}^*$, is derived through the use of Eqs. (13), (19) and (31) and by neglecting nonlinear quantities [11, 13, 16] as

$$\bar{n}^* = \tfrac{1}{2}\, \varepsilon^{*\alpha\beta}\, \bar{a}^*_\alpha \times \bar{a}^*_\beta = \Omega_\lambda \bar{a}^\lambda + \bar{n} \tag{45a}$$

from which, using Eq. (30a), one obtains

$$\bar{n}^* - \bar{n} = \Omega_\lambda \bar{a}^\lambda = \bar{\Omega} \times \bar{n}. \tag{45b}$$

The derivatives of $\bar{n}^*$, are given by

$$\bar{n}^*_{,\alpha} = (\Omega_{\lambda|\alpha} - b_{\alpha\lambda})\,\bar{a}^\lambda + b^\lambda_\alpha \Omega_\lambda \bar{n} = (\eta_{\lambda\alpha} - b^\mu_\alpha\, \omega_{\mu\lambda} - b_{\alpha\lambda})\,\bar{a}^\lambda + b^\lambda_\alpha \Omega_\lambda \bar{n} \tag{46}$$

and are used to define the components of the second fundamental tensor of the deformed midsurface, $b^*_{\alpha\beta}$, as

$$
\begin{aligned}
b^*_{\alpha\beta} &= -\tfrac{1}{2}(\bar{n}^*_{,\alpha}\cdot\bar{a}^*_\beta + \bar{n}^*_{,\beta}\cdot\bar{a}^*_\alpha) = \bar{n}^*\cdot\bar{a}^*_{\alpha,\beta} = -\bar{n}^*_{,\alpha}\cdot\bar{a}^*_\beta = -\bar{n}^*_{,\beta}\cdot\bar{a}^*_\alpha = \\
&= -\tfrac{1}{2}[\Omega_{\beta|\alpha} + \Omega_{\alpha|\beta} + b^\lambda_\alpha\,\omega_{\lambda\beta} + b^\lambda_\beta\,\omega_{\lambda\alpha} - b^\lambda_\alpha\,\epsilon_{\lambda\beta} - b^\lambda_\beta\,\epsilon_{\lambda\alpha} - 2\,b_{\alpha\beta}] = \qquad (47\text{a})\\
&= b_{\alpha\beta} - \eta_{\alpha\beta} + b^\lambda_\beta\,\epsilon_{\alpha\lambda} = b_{\alpha\beta} - \eta_{\beta\alpha} + b^\lambda_\alpha\,\epsilon_{\beta\lambda} = b_{\alpha\beta} - \hat{\eta}_{\alpha\beta} + \tfrac{1}{2}(b^\lambda_\alpha\,\epsilon_{\beta\lambda} + b^\lambda_\beta\,\epsilon_{\alpha\lambda}).
\end{aligned}
$$

Through the use of Eq. (45b), the expression for $b^*_{\alpha\beta}$ can also be written as

$$
\begin{aligned}
b^*_{\alpha\beta} &= -\tfrac{1}{2}[(\bar{\Omega}\times\bar{n})_{,\alpha}\cdot\bar{a}_\beta + (\bar{\Omega}\times\bar{n})_{,\beta}\cdot\bar{a}_\alpha + \bar{u}_{,\alpha}\cdot\bar{n}_{,\beta} + \bar{u}_{,\beta}\cdot\bar{n}_{,\alpha} + \bar{n}_{,\alpha}\cdot\bar{a}_\beta + \bar{n}_{,\beta}\cdot\bar{a}_\alpha] = \\
&= (\bar{\Omega}\times\bar{n})\cdot\bar{a}_{\alpha,\beta} + \bar{n}\cdot\bar{a}_{\alpha,\beta} + \bar{u}_{,\alpha\beta}\cdot\bar{n} = -[(\bar{\Omega}\times\bar{n})_{,\alpha}\cdot\bar{a}_\beta + \bar{u}_{,\beta}\cdot\bar{n}_{,\alpha} + \bar{n}_{,\alpha}\cdot\bar{a}_\beta] = \qquad (47\text{b})\\
&= -[(\bar{\Omega}\times\bar{n})_{,\beta}\cdot\bar{a}_\alpha + \bar{u}_{,\alpha}\cdot\bar{n}_{,\beta} + \bar{n}_{,\beta}\cdot\bar{a}_\alpha].
\end{aligned}
$$

Some authors [15] define a curvature change tensor of linear shell theory in the form

$$
\begin{aligned}
b^*_{\alpha\beta} - b_{\alpha\beta} &= \hat{\kappa}_{\alpha\beta} = -\tfrac{1}{2}[(\bar{\Omega}\times\bar{n})_{,\alpha}\cdot\bar{a}_\beta + (\bar{\Omega}\times\bar{n})_{,\beta}\cdot\bar{a}_\alpha + \bar{u}_{,\alpha}\cdot\bar{n}_{,\beta} + \bar{u}_{,\beta}\cdot\bar{n}_{,\alpha}] = \\
&= -[\hat{\eta}_{\alpha\beta} - \tfrac{1}{2}(b^\lambda_\alpha\,\epsilon_{\lambda\beta} + b^\lambda_\beta\,\epsilon_{\lambda\alpha}] = \rho_{\alpha\beta} + \tfrac{1}{2}(b^\lambda_\alpha\,\epsilon_{\lambda\beta} + b^\lambda_\beta\,\epsilon_{\lambda\alpha}).
\end{aligned} \qquad (48)
$$

This tensor is particularly useful in writing the z surface tangential strains in terms of midsurface variables.

The components of the third fundamental tensor of the deformed midsurface, $c^*_{\alpha\beta}$, are expressible in the form

$$
\begin{aligned}
c^*_{\alpha\beta} &= \bar{n}^*_{,\alpha}\cdot\bar{n}^*_{,\beta} = \bar{n}_{,\alpha}\cdot\bar{n}_{,\beta} + \bar{\Omega}_{,\alpha}\times\bar{n}\cdot\bar{n}_{,\beta} + \bar{\Omega}_{,\beta}\times\bar{n}\cdot\bar{n}_{,\alpha} = c_{\alpha\beta} - b^\rho_\alpha\,\eta_{\rho\beta} - b^\rho_\beta\,\eta_{\rho\alpha} = \\
&= c_{\alpha\beta} - [b^\rho_\alpha\,\hat{\eta}_{\beta\rho} + b^\rho_\beta\,\hat{\eta}_{\alpha\rho} + b^\lambda_\alpha b^\rho_\beta\,\epsilon_{\lambda\rho} - \tfrac{1}{2}\,b^\lambda_\rho\,(b^\rho_\alpha\,\epsilon_{\beta\lambda} + b^\rho_\beta\,\epsilon_{\alpha\lambda})].
\end{aligned} \qquad (49)
$$

2.3. Compatibility Equations

For the sake of completeness, the compatibility equations of linear shell theory are recorded. The mixed partial derivatives of the displacement and rotation vectors must be equal if deformation is to be continuous. This fact is used here, as it was used by others [9, 15, 25, 26], to derive the compatibility equations, i. e.

$$\bar{u}_{,\alpha\beta} = \bar{u}_{,\beta\alpha} \quad \therefore \quad \bar{u}_{,\alpha\beta} - \bar{u}_{,\beta\alpha} = \varepsilon^{\alpha\beta}\bar{u}_{,\alpha\beta} = 0, \qquad (50\text{a})$$

$$\bar{\Omega}_{,\alpha\beta} = \bar{\Omega}_{,\beta\alpha} \quad \therefore \quad \bar{\Omega}_{,\alpha\beta} - \bar{\Omega}_{,\beta\alpha} = \varepsilon^{\alpha\beta}\bar{\Omega}_{,\alpha\beta} = 0. \qquad (50\text{b})$$

Using the definition for the covariant derivative of mixed and covariant components of second order tensors [24] as well as Eqs. (2), (3), (4), (6), (31), (32) and (35), the above statements, Eqs. (50), may be rewritten as

$$\varepsilon^{\alpha\beta}\bar{u}_{,\alpha\beta} = \varepsilon^{\alpha\beta}[(\epsilon_{\alpha\lambda|\beta} + \varepsilon_{\alpha\lambda}\,\zeta_\beta)\,\bar{a}^\lambda + (b^\lambda_\beta\,\epsilon_{\alpha\lambda} + \tilde{\eta}_{\alpha\beta})\,\bar{n}] = 0, \qquad (51\text{a})$$

$$\varepsilon^{\alpha\beta}\bar{\Omega}_{,\alpha\beta} = \varepsilon^{\alpha\beta}[(\varepsilon^{\mu\lambda}\,\eta_{\mu\alpha|\beta} - \zeta_\alpha\,b^\lambda_\beta)\,\bar{a}_\lambda + (\zeta_{\alpha|\beta} + b_{\lambda\beta}\,\varepsilon^{\mu\lambda}\,\eta_{\mu\alpha})\,\bar{n}] = 0, \qquad (51\text{b})$$

from which the following equations result

$$\varepsilon^{\alpha\beta}(\epsilon_{\alpha\lambda|\beta} + \varepsilon_{\alpha\lambda}\,\zeta_\beta) = \varepsilon^{\alpha\beta}(\epsilon_{\alpha\lambda|\beta} - \varepsilon_{\beta\lambda}\,\zeta_\alpha) = 0, \qquad (52\text{a})$$

$$\varepsilon^{\alpha\beta}(\tilde{\eta}_{\alpha\beta} + b^\lambda_\beta\,\epsilon_{\alpha\lambda}) = 0 \quad \therefore \quad \tilde{\eta}_{\alpha\beta} = \tfrac{1}{2}(b^\lambda_\alpha\,\epsilon_{\beta\lambda} - b^\lambda_\beta\,\epsilon_{\alpha\lambda}), \qquad (52\text{b})$$

$$\varepsilon^{\alpha\beta}(\varepsilon^{\mu\lambda}\,\eta_{\mu\alpha|\beta} - \zeta_\alpha\,b^\lambda_\beta) = 0; \qquad \varepsilon^{\alpha\beta}(\zeta_{\alpha|\beta} + b_{\lambda\beta}\,\varepsilon^{\mu\lambda}\,\eta_{\mu\alpha}) = 0. \qquad (53\text{a, b})$$

It is thus clear that Eq. (50a) leads to an identity, derived earlier (Eq. (37)), as well as the following useful relations

$$\zeta_\alpha = \varepsilon^{\lambda\mu}\,\epsilon_{\mu\alpha|\lambda}\,; \qquad \zeta_{\alpha|\beta} = \varepsilon^{\lambda\mu}\,\epsilon_{\mu\alpha|\lambda\beta}\,. \tag{54a, b}$$

With these relations and the identity (52b), Eqs. (53) are recast to read

$$\varepsilon^{\alpha\beta}\,[\varepsilon^{\mu\lambda}\,(\hat{\eta}_{\alpha\mu} + \tfrac{1}{2}\,b^\rho_\alpha\,\epsilon_{\mu\rho} - \tfrac{1}{2}\,b^\rho_\mu\,\epsilon_{\alpha\rho})_{|\beta} - \varepsilon^{\rho\mu}\,b^\lambda_\beta\,\epsilon_{\alpha\mu|\rho}] = 0\,, \tag{55a}$$

$$\varepsilon^{\alpha\beta}\,\varepsilon^{\lambda\mu}\,[\epsilon_{\alpha\mu|\lambda\beta} - b_{\lambda\beta}\,\hat{\eta}_{\alpha\mu}] = 0\,, \tag{55b}$$

which are the compatibility equations of linear shell theory. Eqs. (55) can be shown to be in complete agreement with corresponding results of others [3, 6]. Although the method of derivation used here has been employed previously in less general and/or physical component treatments [9, 15, 26] as well as for the derivation of the compatibility equations of a Cosserat surface [25], to the best of the author's knowledge, it has not been used in deriving the equations of compatibility in tensorial form and in terms of the now generally accepted strain measures of thin shell theory.

III. Deformation of Shell Space

The point Q on the parallel surface moves to the position Q^* during the deformation process. Let the position vector of Q^* be denoted by $\overline{R}^*$ and the displacement vector $\overline{QQ^*}$ by $\overline{U}$ (see Fig. 2), such that

$$\overline{R}^* = \overline{R}^*(x^\alpha, z) = \overline{R}(x^\alpha, z) + \overline{U}(x^\alpha, z) = \overline{r} + z\,\overline{n} + \overline{U} = \overline{r} + \overline{u} + z\,\overline{n}^*\,, \tag{56}$$

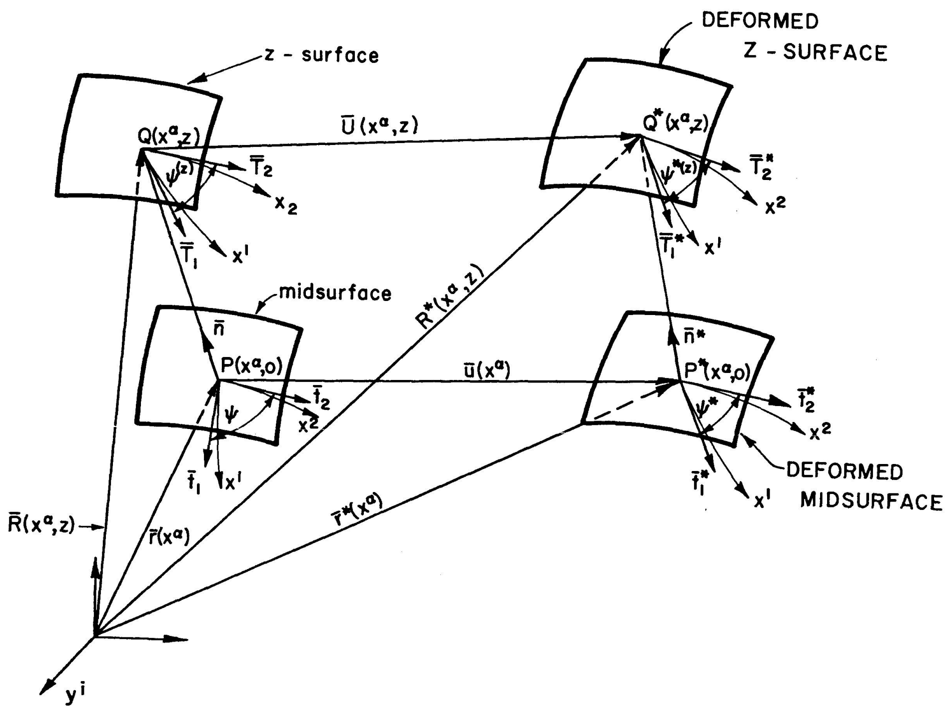

Fig. 2. Deformation of Shell Space.

from which, through the use of Eq. (45b)

$$\bar{U} = \bar{u} + z(\bar{n}^* - \bar{n}) = \bar{u} + z(\bar{\Omega} \times \bar{n}). \tag{57}$$

The base vectors of the deformed parallel surface, $\bar{g}^*_\alpha$, are written as

$$\bar{g}^*_\alpha = \bar{R}^*_{,\alpha} = \bar{R}_{,\alpha} + \bar{U}_{,\alpha} = \bar{g}_\alpha + \gamma^z_{\lambda\alpha}\bar{g}^\lambda - \Omega^z_\alpha \bar{n}, \tag{58}$$

where $$\bar{U}_{,\alpha} = \gamma^z_{\lambda\alpha}\bar{g}^\lambda - \Omega^z_\alpha \bar{n} = [\gamma_{\lambda\alpha} + z(\eta_{\lambda\alpha} + b^\rho_\alpha \omega_{\lambda\rho})]\bar{a}^\lambda - \mu^\lambda_\alpha \Omega_\lambda \bar{n}. \tag{59}$$

Thus the unsymmetric tangential strain tensor of the z surface, $\gamma^z_{\alpha\beta}$, is given by

$$\gamma^z_{\beta\alpha} = \bar{U}_{,\alpha} \cdot \bar{g}_\beta = \mu^\lambda_\beta [\gamma_{\lambda\alpha} + z\Omega_{\lambda|\alpha}] = [\gamma_{\lambda\alpha} + z(\eta_{\lambda\alpha} + b^\rho_\alpha \omega_{\lambda\rho})]\mu^\lambda_\beta \tag{60}$$

and is used to define the unique symmetric components, $\hat{\gamma}^z_{\alpha\beta}$,

$$\begin{aligned}
\hat{\gamma}^z_{\alpha\beta} \equiv E_{\alpha\beta} &= \tfrac{1}{2}(\gamma^z_{\beta\alpha} + \gamma^z_{\alpha\beta}) = \tfrac{1}{2}(\bar{U}_{,\alpha}\cdot\bar{g}_\beta + \bar{U}_{,\beta}\cdot\bar{g}_\alpha) = \tfrac{1}{2}\{[\bar{u}_{,\alpha} + z(\bar{\Omega}\times\bar{n})_{,\alpha}]\cdot\bar{g}_\beta \\
&\quad + [\bar{u}_{,\beta} + z(\bar{\Omega}\times\bar{n})_{,\beta}]\cdot\bar{g}_\alpha\} = \tfrac{1}{2}\mu^\lambda_\beta(\gamma_{\lambda\alpha} + z\Omega_{\lambda|\alpha}) + \tfrac{1}{2}\mu^\lambda_\alpha(\gamma_{\lambda\beta} + z\Omega_{\lambda|\beta}) = \\
&= \tfrac{1}{2}\{\mu^\lambda_\beta[\gamma_{\lambda\alpha} + z(\eta_{\lambda\alpha} + b^\rho_\alpha\omega_{\lambda\rho})] + \mu^\lambda_\alpha[\gamma_{\lambda\beta} + z(\eta_{\lambda\beta} + b^\rho_\beta\omega_{\lambda\rho})]\} = \\
&= \epsilon_{\alpha\beta} + z\,\hat{\eta}_{\alpha\beta} - \frac{z}{2}(b^\rho_\alpha\epsilon_{\beta\rho} + b^\rho_\beta\epsilon_{\alpha\rho}) - \frac{z^2}{2}[b^\rho_\alpha b^\delta_\beta\epsilon_{\delta\rho} + b^\delta_\alpha\hat{\eta}_{\beta\delta} \\
&\quad + b^\delta_\beta\hat{\eta}_{\alpha\delta} - \tfrac{1}{2}b^\rho_\delta(b^\delta_\alpha\epsilon_{\beta\rho} + b^\delta_\beta\epsilon_{\alpha\rho})],
\end{aligned} \tag{61a}$$

which, using the results obtained here, may be rewritten in various forms as

$$\begin{aligned}
E_{\alpha\beta} = \hat{\gamma}^z_{\alpha\beta} &= \epsilon_{\alpha\beta} + z\,\hat{\eta}_{\alpha\beta} - \frac{z}{2}(b^\rho_\alpha\epsilon_{\beta\rho} + b^\rho_\beta\epsilon_{\alpha\rho}) - \frac{z^2}{2}(b^\lambda_\alpha\Omega_{\lambda|\beta} + b^\lambda_\beta\Omega_{\lambda|\alpha}) = \\
&= \frac{1}{2}\left\{\mu^\lambda_\alpha\left[\epsilon_{\beta\lambda} + z\,\hat{\eta}_{\beta\lambda} + \frac{z}{2}(b^\mu_\beta\epsilon_{\lambda\mu} - b^\mu_\lambda\epsilon_{\beta\mu})\right] + \mu^\lambda_\beta\left[\epsilon_{\alpha\lambda} + z\,\hat{\eta}_{\alpha\lambda} + \frac{z}{2}(b^\mu_\alpha\epsilon_{\lambda\mu} - b^\mu_\lambda\epsilon_{\alpha\mu})\right]\right\}.
\end{aligned} \tag{61b}$$

As is known, $E_{\alpha\beta}$ is comprised of the linear terms in the expression $\tfrac{1}{2}(g^*_{\alpha\beta} - g_{\alpha\beta})$, i.e.

$$\begin{aligned}
E_{\alpha\beta} &= \frac{1}{2}(g^*_{\alpha\beta} - g_{\alpha\beta}) = \frac{1}{2}(a^*_{\alpha\beta} - a_{\alpha\beta}) - z(b^*_{\alpha\beta} - b_{\alpha\beta}) + \frac{z^2}{2}(c^*_{\alpha\beta} - c_{\alpha\beta}) = \\
&= \epsilon_{\alpha\beta} - z\,\hat{\kappa}_{\alpha\beta} - \frac{z^2}{2}(b^\rho_\alpha\eta_{\rho\beta} + b^\rho_\beta\eta_{\rho\alpha}) = \epsilon_{\alpha\beta} - z\,\hat{\kappa}_{\alpha\beta} + \frac{z^2}{2}(b^\lambda_\alpha\hat{\kappa}_{\lambda\beta} + b^\lambda_\beta\hat{\kappa}_{\lambda\alpha} - 2\,b^\lambda_\alpha b^\rho_\beta\epsilon_{\lambda\rho}),
\end{aligned} \tag{61c}$$

from which it is clear that [15]

$$\hat{\kappa}_{\alpha\beta} = b^*_{\alpha\beta} - b_{\alpha\beta} = -\frac{1}{2}\left[\frac{\partial}{\partial z}(g^*_{\alpha\beta} - g_{\alpha\beta})\right]_{z=0}. \tag{62}$$

The physical tangential strain components of the parallel surface are now expressible in terms of symmetric midsurface physical strains as

$$
\begin{aligned}
E_{(1)} = \frac{1}{H_1}\frac{\partial \bar{U}}{\partial x^1}\cdot \bar{T}_1 = \frac{E_{11}}{g_{11}} = \frac{E_{11}}{(A_1)^2 (l_1)^2} = \frac{1}{(l_1)^2}\Bigg\{ &\epsilon_{(1)} + z\left[\varkappa_{(1)} + \frac{\epsilon_{(1)}}{R_1}\right. \\
&\left. + \frac{1}{2R_{12}}[\hat{\gamma}_{(12)} + \cot\psi\,(\epsilon_{(2)} - \epsilon_{(1)})]\right] + z^2\left[\frac{\varkappa_{(1)}}{R_1} + \frac{1}{2R_{12}}[\tau + \cot\psi\,(\varkappa_{(2)} - \varkappa_{(1)}]\right. \\
&\left.\left. + \frac{1}{\sin^2\psi}\left(\frac{\epsilon_{(1)}}{R_{21}} - \frac{\epsilon_{(2)}}{R_{12}}\right) + \frac{1}{2\sin^2\psi}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)[\hat{\gamma}_{(12)} + \cot\psi\,(\epsilon_{(1)} + \epsilon_{(2)})]\right]\Bigg\} \quad 1 \rightleftarrows 2,
\end{aligned} \tag{63a}
$$

which is in complete agreement with corresponding expressions derived elsewhere [11, 16] and where l_α are defined as in [16]. The off diagonal component E_{12} is expressed in terms of midsurface physical quantities by

$$
\begin{aligned}
\frac{2E_{12}}{\sqrt{a}} = \hat{\gamma}_{(12)} + \cot\psi(\epsilon_{(1)} + \epsilon_{(2)}) + z\Bigg\{ &\tau + \cot\psi\,(\varkappa_{(1)} + \varkappa_{(2)}) + \frac{1}{\sin^2\psi}\left(\frac{\epsilon_{(1)}}{R_{21}} + \frac{\epsilon_{(2)}}{R_{12}}\right) \\
&- H[\hat{\gamma}_{(12)} + \cot\psi\,(\epsilon_{(1)} + \epsilon_{(2)})]\Bigg\} + z^2\Bigg\{\frac{1}{\sin^2\psi}\left(\frac{\varkappa_{(1)}}{R_{21}} + \frac{\varkappa_{(2)}}{R_{12}}\right) \\
&- H[\tau + \cot\psi\,(\varkappa_{(1)} + \varkappa_{(2)})] + \frac{1}{2\sin^4\psi}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(\frac{\epsilon_{(2)}}{R_{12}} - \frac{\epsilon_{(1)}}{R_{21}}\right) \\
&+ \frac{1}{4\sin^4\psi}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2 [\hat{\gamma}_{(12)} + \cot\psi\,(\epsilon_{(1)} + \epsilon_{(2)})]\Bigg\}.
\end{aligned} \tag{63b}
$$

Conclusions

In analogy with the definitions of the tangential strain tensor and rotation tensor of linear shell theory, derivatives of the rotation vector and the fundamental theorem of decomposition of tensor algebra are used to define unique symmetric and skew-symmetric components of a curvature change tensor. The symmetric components are shown to be Koiter's curvature change tensor of linear shell theory [2, 3], thus presenting the bending strain tensor in a new, vectorial form. The physical components of this symmetric tensor are in complete agreement with results presented elsewhere [13, 16]. The skew-symmetric components, on the other hand, are found to be expressible in terms of tangential strains thus permitting the derivation of an expression for the difference between the unsymmetric physical components of change in torsion in terms of symmetric physical strains.

The difference between the second fundamental tensor of the deformed and undeformed surface is expressed in terms of derivatives of the rotation and displacement vectors and compared with the bending strain tensor. An expression is also given for the third fundamental tensor of the deformed surface in terms of the symmetric strain measures of linear shell theory.

The mixed partial derivatives of the displacement and rotation vectors are used to obtain the compatibility equations from which the skew-symmetric components of the curvature change tensor are eliminated by means of Eq. (37).

Finally, the tangential strain tensor of a parallel surface is rewritten in several new forms and the corresponding physical components expressed in terms of symmetric physical strain measures. These results are useful in a general description of the statics and energy principles of linear shell theory in terms of physical components [27].

Notations

$\bar{a}_\alpha, \bar{a}_3; \bar{a}^\alpha, \bar{a}^3$	covariant and contravariant base vectors of midsurface
$a_{\alpha\beta}, a^{\alpha\beta}$	components of first fundamental (metric) tensor
a	determinant of the matrix of $a_{\alpha\beta}$
$A_\alpha = \sqrt{a_{\alpha\alpha}}$	magnitudes of $\bar{a}_\alpha$
$b_{\alpha\beta}, b^\alpha_\beta$	components of second fundamental tensor
$c_{\alpha\beta}$	components of third fundamental tensor
C	arbitrary curve on a surface
$e_{\alpha\beta}, e^{\alpha\beta}$	components of permutation symbol defined by $e_{12}=e^{12}=-e_{21}=-e^{21}=1; \quad e_{11}=e^{11}=e_{22}=e^{22}=0$
$E_{(\alpha)}$	extensional physical strain components of z surface
$E_{\alpha\beta}$	components of symmetric tangential strain tensor of z surface
$g_{\alpha\beta}, g^{\alpha\beta}$	components of metric of z surface
$\bar{g}_\alpha, \bar{g}^\alpha$	covariant and contravariant base vectors of z surface
h	constant shell thickness
H	mean curvature of midsurface
$H_\alpha = \sqrt{g_{\alpha\alpha}}$	Lame's surface parameters; magnitudes of $\bar{g}_\alpha$
K	Gaussian curvature of midsurface
l_α	parameters associated with metric of z surface
$\bar{n}$	unit surface normal vector
P	generic point on a surface
Q	generic point on parallel surface
$\bar{r}$	position vector of a point P on the midsurface of a shell
$\bar{R}$	position vector of generic point Q on parallel surface
R_α	radii of normal curvature for parametric lines of midsurface
$R_{\alpha\beta}$	radii of torsion (twist) for the parametric lines
ds	infinitesimal distance along a curve on the midsurface
$\bar{t}_\alpha, \bar{t}^\alpha$	covariant and contravariant unit tangent vectors on the midsurface
$\bar{T}_\alpha, \bar{T}^\alpha$	covariant and contravariant unit tangent vectors on the z surface
$\bar{u}$	displacement vector of a point on the midsurface
$\bar{U}$	displacement vector of a point on the z surface
$u^\alpha, u^3, u_\alpha, u_3$	contravariant and covariant components of midsurface displacement vector, $\bar{u}$

$u^{(\alpha)}$, w	physical components of displacement vector $\bar{u}$
$\bar{v}$	arbitrary vector in space
v^{α}, v^3, v_{α}, v_3	contravariant and covariant components of $\bar{v}$
$v^{(\alpha)}$, $v^{(3)}$	physical components of $\bar{v}$
x^{α}, $x^3=z$	contravariant curvilinear "normal" coordinate system
y^i	fixed right-handed orthogonal Cartesian coordinates in Euclidean 3-space
$z=x^3$	coordinate in surface normal direction
α, β	indices
$\gamma_{(\alpha)}$	unsymmetric physical shear strain components
$\gamma_{\beta\alpha}$	unsymmetric tangential strain tensor of midsurface
$\hat{\gamma}_{(12)}=\gamma_{(1)}+\gamma_{(2)}$	total linear shear strain between the x^1 and x^2 lines on a surface
$\hat{\gamma}_{\alpha\beta}=\epsilon_{\alpha\beta}$	symmetric tangential strain tensor of midsurface
$\tilde{\gamma}_{\alpha\beta}=\omega_{\alpha\beta}$	skew-symmetric components of tangential strain tensor of midsurface
$\hat{\gamma}^z_{\alpha\beta}=E_{\alpha\beta}$	components of symmetric tangential strain tensor of z surface
$\gamma^z_{\beta\alpha}$	unsymmetric tangential strain tensor of z surface
$\Gamma^{\lambda}_{\alpha\beta}$, $\Gamma^{3}_{\alpha\beta}$, $\Gamma^{\alpha}_{\beta 3}$	Christoffel symbols of second kind for a surface
δ^{α}_{β}	Kroenecker delta
$\epsilon_{(\alpha)}$	linear physical extensional strain components for midsurface
$\epsilon_{\alpha\beta}$, $\epsilon_{\alpha 3}$	components of linear tangential strain tensor for midsurface
$\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$	components of skew-symmetric permutation tensor in 2 space
ε^{ijk}	components of skew-symmetric permutation tensor in 3 space
ζ_{α}	parameters associated with bending strains
$\eta_{\alpha\beta}$, $\eta^{\alpha}_{.\beta}$	covariant and mixed components of unsymmetric curvature change tensor
$\hat{\eta}_{\alpha\beta}=-\rho_{\alpha\beta}$	symmetric components of curvature change tensor
$\tilde{\eta}_{\alpha\beta}$	skew-symmetric components of curvature change tansor
κ_{α}	normal curvatures of parametric curves at a point P on a surface
$\hat{\kappa}_{\alpha\beta}=b^*_{\alpha\beta}-b_{\alpha\beta}$	modified curvature change tensor of linear shell theory
$(\kappa_g)_{\alpha}$	tangential curvatures for the parametric lines of a surface
$\varkappa_{(\alpha)}$	linear physical bending strain components
μ^{α}_{β}	components of a tensor relating midsurface and z surface quantities
μ	determinant of μ^{α}_{β}
$\rho_{\alpha\beta}=-\hat{\eta}_{\alpha\beta}$	Koiter's curvature change tensor of linear shell theory
$(\tau_g)_{\alpha}=\tau_{\alpha}$	torsion of parametric lines on a surface
$\boldsymbol{\tau}_{(\alpha)}$	unsymmetric physical bending strain components
$\boldsymbol{\tau}=(\boldsymbol{\tau}_{(1)}+\boldsymbol{\tau}_{(2)})$	symmetric physical bending strain component
$\varphi_{\alpha}=-\Omega_{\alpha}$	components of rotation vector
ψ	angle between curvilinear coordinates on a surface
ω^i	components of rotation vector in 3 space

ω^{α}, ω^{3}	contravariant components of rotation
ω_{ij}	components of rotation tensor in 3 space
$\omega_{\alpha\beta}=\tilde{\gamma}_{\alpha\beta}$	components of skew-symmetric rotation tensor
$\underline{\Omega}$	rotation vector
Ω_{α}, $\Omega_{n}=\Omega_{3}=\Omega$	components of rotation vector
$\Omega_{(\alpha)}$, Ω_{3}	physical components of rotation vector
Ω_{α}^{z}	components of rotation of z surface

Acknowledgement

The financial assistance provided by the National Research Council of Canada and the Defence Research Board of Canada in the form of research grants to the author, N.R.C. Grant No. A-2736 and D.R.B. Grant No. 6301-,13 is gratefully acknowledged.

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Summary

Using derivatives of the rotation vector of linear thin shell theory, the curvature change tensor is defined in a new, vectorial form. The unique symmetric components of this tensor and the corresponding physical components are compared with previous results whereas the skew-symmetric components are shown to be expressible in terms of the components of the symmetric tangential strain tensor. The difference between the unsymmetric physical components of change in torsion is derived in terms of symmetric physical strain components. The compatibility equations of linear shell theory are obtained by equating the partial mixed-derivatives of the displacement and rotation vectors and are compared with results of others. Finally, the tangential strains of the parallel surface are expressed in a new form.

Résumé

Moyennant l'utilisation des dérivées du vecteur rotationnel on définit un tenseur de variation de courbure sous une nouvelle forme vectorielle. La seule partie symétrique de ce tenseur correspond au tenseur de variation de courbure de la théorie linéaire des coques qui est reconnu partout aujourd'hui; la partie de symétrie oblique est représentée par les termes du tenseur symétrique de cisaillement. La différence entre les composantes physiques non-symétriques de la variation de torsion est représentée à l'aide des termes de tension physique symétrique. On obtient les équations de compatibilité en posant égales les dérivées composées partielles du vecteur de translation et de rotation. Finalement les tensions de cisaillement entre les surfaces parallèles de la coque sont exprimées sous une nouvelle forme.

Zusammenfassung

Unter Verwendung von Ableitungen des Drehungsvektors wird ein Krümmungsänderungs-Tensor in einer neuen vektoriellen Form definiert. Dessen einziger symmetrischer Teil erweist sich als der jetzt allgemein anerkannte Krümmungsänderungs-Tensor der linearen Schalentheorie; der schiefsymmetrische Teil wird durch Terme des symmetrischen Schubspannungstensors ausgedrückt. Die Differenz zwischen den nichtsymmetrischen physikalischen Komponenten der Torsionsänderung wird mittels Termen der symmetrischen physikalischen Spannung dargestellt. Durch Gleichsetzen der gemischten partiellen Ableitungen des Verschiebungs- und des Drehungsvektors erhält man die Kompatibilitäts-Gleichungen. Schliesslich werden die Schubspannungen des Schalenraumes in neuer Form ausgedrückt.