Spatial behaviour of rotationally and directionally restrained beam-columns

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Spatial Behaviour of Rotationally and Directionally Restrained Beam-Columns

Comportement spatial de poutres-colonnes munies de ressorts directionnels et rotationnels

Räumliches Verhalten von auf Druck und Biegung beanspruchten, allgemein gelagerten Stützen

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Introduction

In the last few years a number of papers have been published on the inelastic stability of biaxially bent isolated steel beam-columns [1 to 6]. However, very few studies [7 to 12] consider the behaviour of restrained column. In practice columns are connected at both ends to other members of the structure, so that the response of a column is influenced by the restraint offered by the adjacent members. This restraint is simulated in the present study by isolated rotational and directional springs at the column ends.

The beam-column investigated is shown in Fig. 1. The member is initially straight and prismatic. Forces are only applied at its ends. The points where the external loads are applied are considered fixed to the profil, in otherwards, these points move with the displacements of the end section. However, the directions of the external forces are assumed to be conservative. As for the end conditions of twisting, only two cases namely warping restrained or warping permitted, are considered. The equilibrium equation for twisting is valid to the case where an external torsion is applied at the ends. However, in the present paper, the influence of the external torsion is not investigated.

When a cross-section subjected to biaxial bending and axial load partially plastifies, the shear center and the centroid of the elastic core shift, and the principal axes rotate, with respect to their initial positions. In a majority of

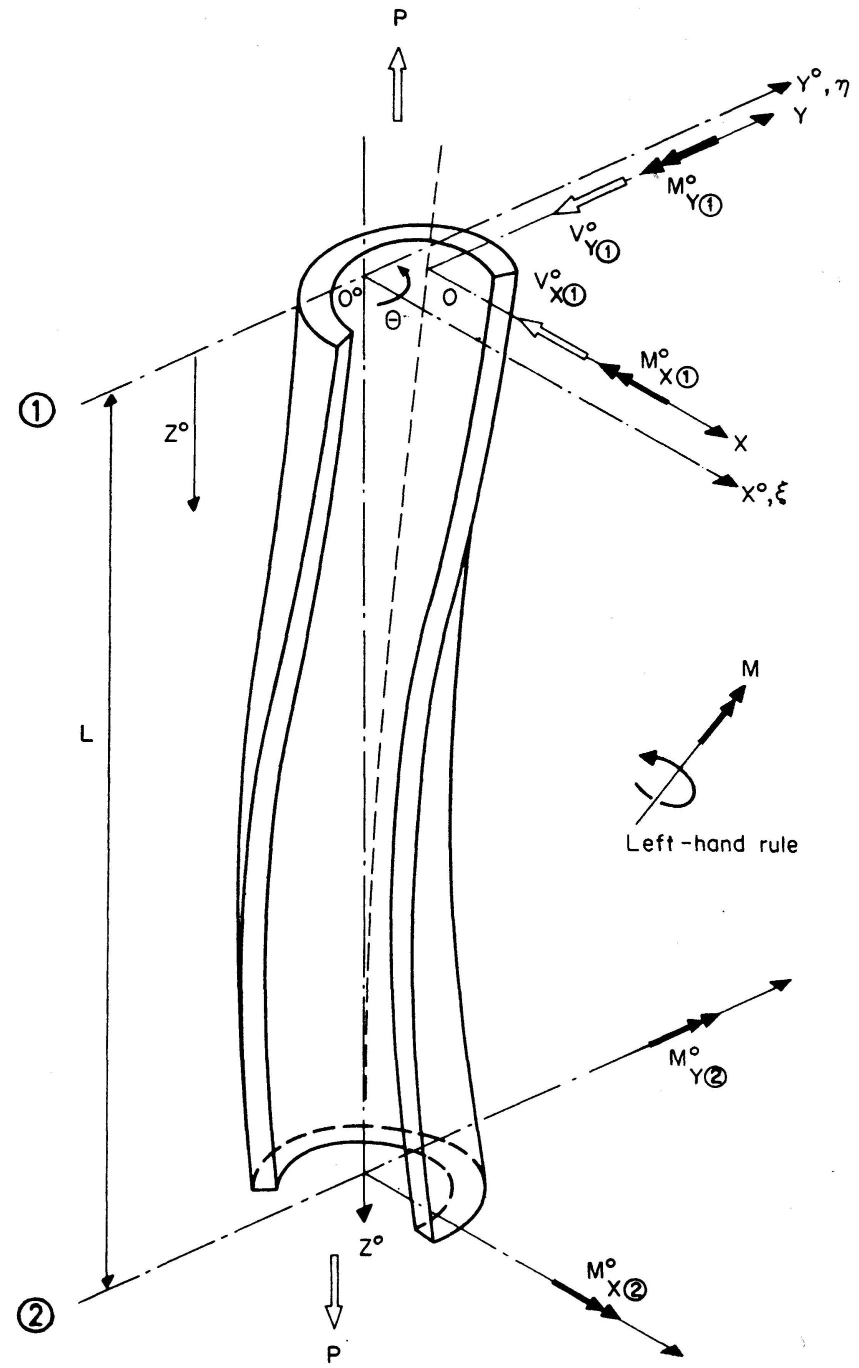


Fig. 1. Restrained beam-column, coordinate systems and external loads.

earlier studies, equilibrium is considered with respect to the principal coordinate system and the shear center. Also, the shift in the shear center and the rotation of the principal axes as yielding progresses are neglected, thus introducing an unknown degree of error. If the equilibrium equations are written with respect to an arbitrary system of coordinate axes, the resulting equations are more involved but include automatically the above mentioned variations.

The authors developed one such system of equations (three coupled fourth order differential equations) in Ref. [12], and some numerical examples were presented. However, due to the inherent difficulties in the formulation of the finite difference method, the approach could not be used when a biaxial plastic hinge forms within the length of the column, as the stiffness becomes (approaches) zero. Unfortunately, such a condition presents itself quite often in the case of restrained columns.

In the present paper the equilibrium equations are integrated twice (mathematically) to obtain three coupled second order differential equations. The system is then solved by using the finite difference method. The approach is not only more accurate and requires less computer time compared with the one using the fourth order differential equations, but circumvents the difficulties caused by the presence of plastic hinges in the latter method.

In a majority of the earlier studies the deflected shape of the column is to be assumed (like sine curve etc.). No such assumption is made in the present analysis. Also, no symmetry in the end conditions and/or end loading need exist. The influence of residual stresses and warping stresses is included in the analysis.

Theory

The restrained beam-column considered is shown in Fig. 1. It is of length L and is provided with two directional and two rotational springs at one end, and two rotational springs at the other end. In what follows, the differential equations defining the deformed configuration of this beam-column will be derived.

Assumptions

The following usual assumptions are made in the analysis.

1. The stress-strain diagram of the material is ideally elastic-plastic. Strain hardening and strain reversal are neglected.
2. The cross-section retains its original shape after deformation.
3. The deformations are small.
4. The shearing strain is negligible.
5. Yielding is governed by normal stress only.

Coordinate Systems and Sign Conventions

In deriving the equilibrium equations the following systems of coordinates are used:

The first one is a left handed orthogonal coordinate system X^0-Y^0-Z^0, fixed in space (Fig. 1). The longitudinal axis Z^0 passes through an arbitrary reference point 0 of the cross-sections of the initially straight and untwisted

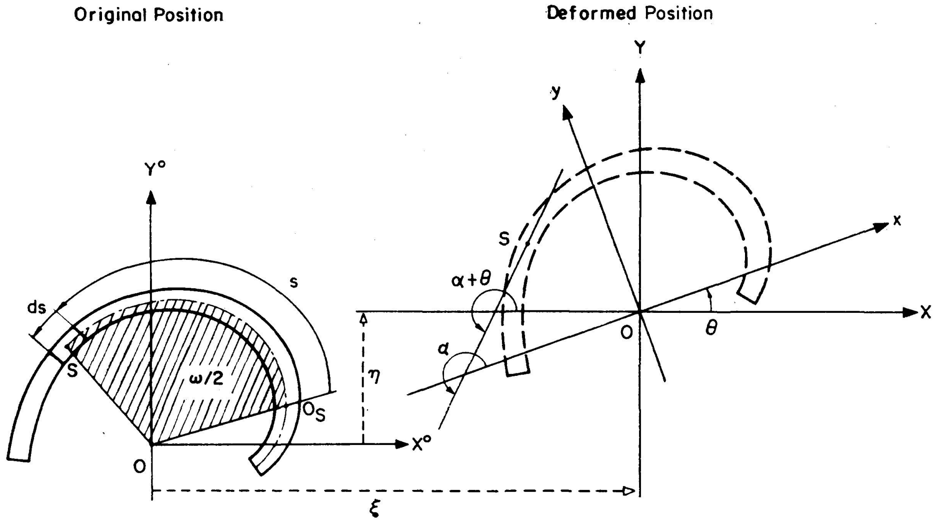


Fig. 2. Coordinate systems, displacements and twist of a cross-section.

prismatic beam-column. The origin $0°$ is taken at the end (1) of the bar. The positive sense of the Z^0 axis is directed towards the end (2).

In addition, a system of axes x-y fixed to the cross-section and deforming with it is considered at each section (Fig. 2). The origin coincides with the reference point 0 of the cross-section. The axes x, y, in the unloaded position, coincide with the axes X^0, Y^0 respectively.

The deformation of the beam-column is defined by the displacement components ξ, η of the reference point of the cross-section in the X^0 and Y^0 directions, respectively, and by the angle of twist θ of the cross-section. The angle of twist θ is taken positive about the Z^0 axis according to the left hand rule and ξ, η are positive in the positive directions of the corresponding axis.

A third system of axes X-Y whose axes X and Y are always parallel to the axes X^0, Y^0 respectively, and whose origin coincides with the reference point 0 of the cross-section in its deformed position is also considered.

The position of an arbitrary point S (x, y) on the middle line of the cross-section can also be defined by its distance s from an arbitrary chosen point 0_s on the cross-section (Fig. 2). The thickness t is assumed to be a function of s.

Equilibrium Equations

The equilibrium equations with respect to the general system of coordinate axes are as follows[1]):

$$(-M_Y' + P\,\xi')' = 0, \tag{1}$$

$$(M_X' + P\,\eta')' = 0, \tag{2}$$

$$\{M_\omega + M_{SV} + K\,\theta' - \eta'\,M_Y - \xi'\,M_X - \eta\,V_{Y\,(1)} + \xi\,V_{Y\,(1)}\}' = 0 \tag{3}$$

with $$M_\omega = B', \qquad M_{SV} = G\,J\,\theta', \tag{4a}$$

[1]) A condensed version of the derivation is given in Appendix I.

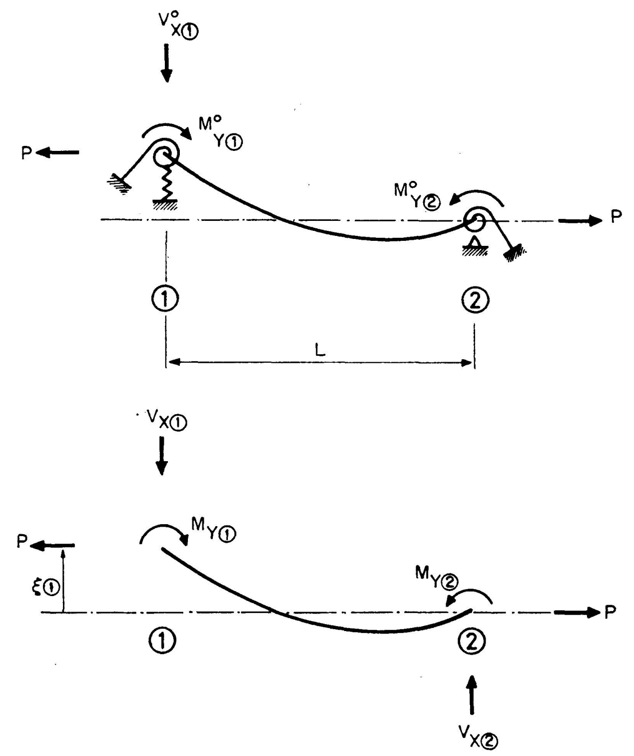
in which P is the normal force (tension positive); M_X, M_Y are the bending moments about X and Y axis respectively; M_ω is the warping torsion moment; M_{SV} is the St-Venant's torsion moment; B is the bimoment; J is the St-Venant torsion constant; G is the shear modulus; K is the modification of GJ due to normal stress; $V_{X(1)}$, $V_{Y(1)}$ are the shear forces at the end (1) in the X and Y directions respectively; σ is the normal stress and ω is the double sectorial area. The prime indicates differentiation with respect to Z^0. The following definitions are valid:

$$\begin{aligned} P &= \int_A \sigma \, dA, \\ M_X &= \int_A \sigma \, Y \, dA, \qquad M_Y = -\int_A \sigma \, X \, dA, \\ B &= \int_A \sigma \, \omega \, dA, \qquad K = \int_A \sigma \, (x^2+y^2) \, dA. \end{aligned} \tag{4b}$$

End Conditions

Fig. 3a shows the end conditions in the plane X^0–Z^0. The rotational and directional springs are attached to the point 0. M^0 is the externally applied bending moment and V^0 is the externally lateral load. Fig. 3b shows the internal bending moment M and the internal shearing force V acting at the column ends.

The rotational restraints are considered elastic-perfectly plastic as shown in Fig. 4a. The relation between the rotation φ and the reaction M^* of the



a) External forces and restraints.

b) Forces acting at the ends.

Fig. 3. Forces in the plane $X^\circ - Z^\circ$.

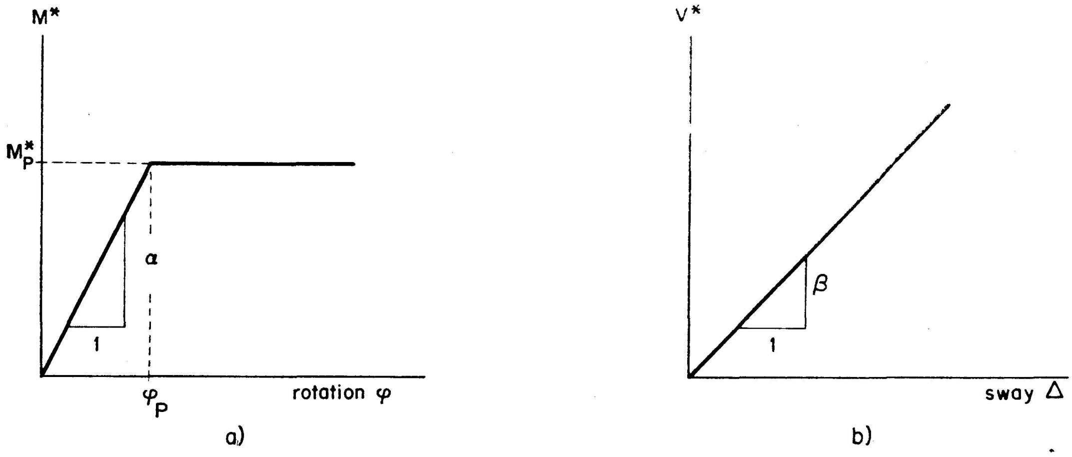


Fig. 4. Load-deformation characteristics of rotational and directional restraints.

rotational spring is linear at first, with a spring constant α. Corresponding to a value of $\varphi = \varphi_p$ and $M^* = M_p^*$, the spring plastifies and does not offer any additional restraint for values of $\varphi > \varphi_p$. Expressed mathematically

$$M^* = \alpha\varphi - \bar{\alpha}(\varphi - \varphi_p),$$

where

$$\bar{\alpha} = 0 \quad \text{for} \quad 0 < |\varphi| \leqq \varphi_p,$$
$$\bar{\alpha} = \alpha \quad \text{for} \quad |\varphi| > \varphi_p.$$

Compatibility requires that the column ends and the attached springs rotate through the same angle. Equilibrium requires that

$$M^0 = M + M^* = M + \alpha\varphi - \bar{\alpha}(\varphi - \varphi_p). \tag{5}$$

Eq. (5) can be written for each axis as follows:

$$\begin{aligned} M^0_{Y(1)} &= M_{Y(1)} - \alpha_{Y(1)}\,\xi'_{(1)} + \bar{\alpha}_{Y(1)}\,(\xi'_{(1)} - \varphi_{pY(1)}), \\ M^0_{Y(2)} &= M_{Y(2)} + \alpha_{Y(2)}\,\xi'_{(2)} - \bar{\alpha}_{Y(2)}\,(\xi'_{(2)} - \varphi_{pY(2)}), \\ M^0_{X(1)} &= M_{X(1)} + \alpha_{X(1)}\,\eta'_{(1)} - \bar{\alpha}_{X(1)}\,(\eta'_{(1)} - \varphi_{pX(1)}), \\ M^0_{X(2)} &= M_{X(2)} - \alpha_{X(2)}\,\eta'_{(2)} + \bar{\alpha}_{X(2)}\,(\eta'_{(2)} - \varphi_{pX(2)}), \end{aligned} \tag{6}$$

where the subscript (j) refers to the value at the end (j).

The directional restraint is considered perfectly elastic with a spring constant β (Fig. 4b), so that the reaction V^* from the directional restraint corresponding to a sway Δ is given by:

$$V^* = \beta\Delta$$

and equilibrium at the end requires that

$$V^0 = V + V^* = V + \beta\Delta. \tag{7}$$

Eq. (7) can be written for each axis as follows:

$$\begin{aligned} V^0_{X(1)} &= V_{X(1)} - \beta_{X(1)}\,\xi_{(1)}, \\ V^0_{Y(1)} &= V_{Y(1)} - \beta_{Y(1)}\,\eta_{(1)}. \end{aligned} \tag{8}$$

Finally, the end conditions for twisting are as follows: If the column end is prevented from warping

$$\theta' = 0. \tag{9}$$

If it is free to warp

$$B = 0. \tag{10}$$

Integration of Equilibrium Equations

If the equilibrium Eqs. (1), (2) and (3) are developed as a function of deformations, fourth order differential equations are obtained [12]. In the present paper the equilibrium equations are integrated twice mathematically under general end conditions shown in Fig. 3 and are then developed as a function of deformations, resulting in second order differential equations.

Integrating Eq. (1) twice results in:

$$-M_Y + P\xi = C_1 Z^0 + C_2, \tag{11}$$

where C_1 and C_2 are the constants of integration. They can be expressed in terms of the deflection $\xi_{(1)}$ and the internal end bending moments $M_{Y\,(1)}$ and $M_{Y\,(2)}$. (Note $\xi_{(2)} = 0$.) This results in the first fundamental equation:

$$-M_Y + P\xi = \frac{Z^0}{L}(M_{Y\,(1)} - M_{Y\,(2)}) - M_{Y\,(1)} + \left(1 - \frac{Z^0}{L}\right) P\,\xi_{(1)}. \tag{12}$$

In the same manner Eq. (2) becomes

$$M_X + P\eta = -\frac{Z^0}{L}(M_{X\,(1)} - M_{X\,(2)}) + M_{X\,(1)} + \left(1 - \frac{Z^0}{L}\right) P\,\eta_{(1)}. \tag{13}$$

Finally, Eq. (3) becomes

$$\begin{aligned} B + (GJ+K)\theta = {} & -\frac{Z^0}{L}(B_{(1)} - B_{(2)}) + B_{(1)} + \left(1 - \frac{Z^0}{L}\right)(GJ+K)_{(1)}\,\theta_{(1)} \\ & + \int_0^{Z^0} (\eta' M_Y + \xi' M_X + \eta\, V_{X\,(1)} - \xi\, V_{Y\,(1)})\, dZ^0 \\ & - \frac{Z^0}{L}\int_0^{L} (\eta' M_Y + \xi' M_X + \eta\, V_{X\,(1)} - \xi\, V_{Y\,(1)})\, dZ^0, \end{aligned} \tag{14}$$

where $B_{(1)}$ and $B_{(2)}$ are the internal warping moments at the end (1) and at the end (2) respectively. If warping is permitted at an end, the corresponding warping moment is equal to zero. If warping is restrained, it is not zero but some unknown value. Note that in the development of Eq. (14) the term $(GJ+K)'\theta$ is neglected. This is reasonable as J is assumed to retain its elastic value, even when the section is partially plasticized and, generally, K is only a little bit reduced in the plastic state compared with its value in the elastic state.

Relations Between Internal Stresses, Stress Resultants and Deformations

The normal strain, ϵ, at an arbitrary point S is made up of the residual strain $\epsilon_r\,(=\sigma_r/E)$ and the normal strain due to loading. Assuming that the shearing strain vanishes for the middle surface of the cross-section,

$$\epsilon = \zeta_0' - \xi'' X - \eta'' Y - \theta'' \omega + \frac{\sigma_r}{E}, \tag{15}$$

where ξ and η represent the displacements of the origin 0 of the coordinate system X-Y, while ζ_0 is the longitudinal displacements of the point 0 considered as the pole of the sectorial area (Fig. 2), and θ is the angle of twist. σ_r is the residual stress at the point S[2]. If the strain $|\epsilon|$ is inferior to the yield strain, the fiber is still elastic and

$$\sigma = E\,[\zeta_0' - \xi'' X - \eta'' Y - \theta'' \omega] + \sigma_r \tag{16a}$$

and if it is equal or superior to the yield strain

$$\sigma = \pm\,\sigma_y \quad \text{(tension positive)}, \tag{16b}$$

when a section is partially plasticized, the following geometrical characteristics based on the cross-section remaining still elastic are introduced: A_e, the area of the elastic core; S_{Xe}, S_{Ye}, static moments of inertia; I_{Xe}, I_{Ye} moments of inertia about X and Y axes; I_{XYe}, product of inertia; $S_{\omega e}$, warping moment; $I_{\omega e}$, warping moment of inertia; and $I_{\omega Xe}$, $I_{\omega Ye}$ warping product of inertia about Y and X axes.

$$\begin{array}{llll} A_e = \int_{A_e} dA, & & & \\ S_{Xe} = \int_{A_e} Y\,dA, & S_{Ye} = \int_{A_e} X\,dA, & S_{\omega e} = \int_{A_e} \omega\,dA, & \\ I_{Xe} = \int_{A_e} Y^2 dA, & I_{Ye} = \int_{A_e} X^2 dA, & I_{\omega e} = \int_{A_e} \omega^2 dA, & (17) \\ I_{\omega Ye} = \int_{A_e} \omega\,Y\,dA, & I_{\omega Xe} = \int_{A_e} \omega\,X\,dA, & I_{XYe} = \int_{A_e} X\,Y\,dA. & \end{array}$$

With the help of the relation $P = \int_A \sigma\,dA$, the term ζ_0' could be eliminated from the Eq. (16a) giving:

$$\sigma = E\left[\left(\frac{S_{Ye}}{A_e} - X\right)\xi'' + \left(\frac{S_{Xe}}{A_e} - Y\right)\eta'' + \left(\frac{S_{\omega e}}{A_e} - \omega\right)\theta''\right] + \frac{P_e}{A_e} + \sigma_r, \tag{18}$$

where P_e is the axial load resisted by the elastic part of the cross-section $(= P - \int_{A_p} (\sigma - \sigma_r)\,dA$, A_p is the area of the plasticized part of the cross-section).

The internal moments M_X, M_Y and bimoment B can be expressed as a function of deformations ξ, η and θ making use of Eqs. (17) and (18). Thus we have:

---

[2]) It is assumed that the residual stress distribution satisfies the following conditions:

$$\int_A \sigma_r\,dA = \int_A \sigma_r X\,dA = \int_A \sigma_r\,Y\,dA = \int_A \sigma_r\,\omega\,dA = 0.$$

$$\begin{aligned}
M_Y &= -\int_A \sigma X\,dA = -\int_A \sigma X\,dA + \int_A \sigma_r X\,dA = \\
&-\int_{A_e} (\sigma-\sigma_r) X\,dA - \int_{A_p} (\sigma-\sigma_r) X\,dA = \\
&-\int_{A_e} E\left\{\left(\frac{S_{Ye}}{A_e}-X\right)\xi'' + \left(\frac{S_{Xe}}{A_e}-Y\right)\eta'' + \left(\frac{S_{\omega e}}{A_e}-\omega\right)\theta''\right\} X\,dA \qquad (19\text{a})\\
&-\frac{P_e}{A_e}\int_{A_e} X\,dA - \int_{A_p} (\sigma-\sigma_r) X\,dA = \\
&E\,\tilde{I}_{Ye}\xi'' + E\,\tilde{I}_{XYe}\eta'' + E\,\tilde{I}_{\omega Xe}\theta'' - F_Y.
\end{aligned}$$

Similarly,

$$-M_X = E\,\tilde{I}_{XYe}\xi'' + E\,\tilde{I}_{Xe}\eta'' + E\,\tilde{I}_{\omega Ye}\theta'' - F_X, \qquad (19\text{b})$$

$$-B \;= E\,\tilde{I}_{\omega Xe}\xi'' + E\,\tilde{I}_{\omega Ye}\eta'' + E\,\tilde{I}_{\omega e}\theta'' - F_\theta, \qquad (19\text{c})$$

where

$$\begin{aligned}
\tilde{I}_{Ye} &= I_{Ye} - \frac{S_{Ye}^2}{A_e}, & \tilde{I}_{\omega Xe} &= I_{\omega Xe} - \frac{S_{Ye} S_{\omega e}}{A_e}, \\
\tilde{I}_{Xe} &= I_{Xe} - \frac{S_{Xe}^2}{A_e}, & \tilde{I}_{\omega Ye} &= I_{\omega Ye} - \frac{S_{Xe} S_{\omega e}}{A_e}, \qquad (20\text{a})\\
\tilde{I}_{\omega e} &= I_{\omega e} - \frac{S_{\omega e}^2}{A_e}, & \tilde{I}_{XYe} &= I_{XYe} - \frac{S_{Xe} S_{Ye}}{A_e}
\end{aligned}$$

and

$$\begin{aligned}
F_Y &= \frac{P_e S_{Ye}}{A_e} + \int_{A_p} (\sigma-\sigma_r) X\,dA, \\
F_X &= \frac{P_e S_{Xe}}{A_e} + \int_{A_p} (\sigma-\sigma_r) Y\,dA, \qquad (20\text{b})\\
F_\theta &= \frac{P_e S_{\omega e}}{A_e} + \int_{A_p} (\sigma-\sigma_r)\,\omega\,dA.
\end{aligned}$$

The integrals in Eq. (20b) represent the part of the moments M_Y, M_X and of the bimoment B, resisted by the plasticized part, A_p, of the cross-section.

Second Order Differential Equations

In this paragraph the fundamental Eqs. (12), (13) and (14) and the end conditions (6), (8), (9) and (10) are developed as a function of deformations to solve the problem using the finite difference method.

Replacing the internal moment M_Y in Eq. (12) by the value given by Eq. (19a) and replacing the internal end-bending moments $M_{Y(1)}$ and $M_{Y(2)}$ by their values obtained from Eq. (6) gives:

$$\begin{aligned}
&E\,\tilde{I}_{Ye}\xi'' + E\,\tilde{I}_{XYe}\eta'' + E\,\tilde{I}_{\omega Xe}\theta'' - P\xi + P\left(1-\frac{Z^0}{L}\right)\xi_{(1)} \\
&\quad -\alpha_{Y(1)}\left(1-\frac{Z^0}{L}\right)\xi'_{(1)} + \alpha_{Y(2)}\frac{Z^0}{L}\xi'_{(2)} = \\
&M^0_{Y(1)} - \frac{Z^0}{L}(M^0_{Y(1)} - M^0_{Y(2)}) + F_Y \qquad (21)\\
&\quad -\bar{\alpha}_{Y(1)}(\xi'_{(1)} - \varphi_{pY(1)}) + \frac{Z^0}{L}\{\bar{\alpha}_{Y(1)}(\xi'_{(1)} - \varphi_{pY(1)}) + \bar{\alpha}_{Y(2)}(\xi'_{(2)} - \varphi_{pX(2)})\}.
\end{aligned}$$

Similarly Eq. (13) becomes

$$
\begin{aligned}
& E\,\tilde{I}_{XYe}\,\xi'' + E\,\tilde{I}_{Xe}\,\eta'' + E\,\tilde{I}_{\omega Ye}\,\theta'' - P\,\eta + P\left(1-\frac{Z^0}{L}\right)\eta_{(1)} \\
& \quad - \alpha_{X(1)}\left(1-\frac{Z^0}{L}\right)\eta'_{(1)} + \alpha_{X(2)}\frac{Z^0}{L}\,\eta'_{(2)} = \\
& \quad - M^0_{X(1)} + \frac{Z^0}{L}\,(M^0_{X(1)} - M^0_{X(2)}) + F_X \\
& - \bar{\alpha}_{X(1)}\,(\eta'_{(1)} - \varphi_{pX(1)}) + \frac{Z^0}{L}\{\bar{\alpha}_{X(1)}\,(\eta'_{(1)} - \varphi_{pX(1)}) + \bar{\alpha}_{X(2)}\,(\eta'_{(2)} - \varphi_{pX(2)})\}.
\end{aligned}
\tag{22}
$$

Finally Eq. (14) becomes:

$$
\begin{aligned}
& E\,\tilde{I}_{\omega Xe}\,\xi'' + E\,\tilde{I}_{\omega Ye}\,\eta'' + E\,\tilde{I}_{\omega e}\,\theta'' - (G\,J + K)\,\theta \\
& \quad + (G\,J + K)_{(1)}\left(1-\frac{Z^0}{L}\right)\theta_{(1)} - \left(1-\frac{Z^0}{L}\right)(E\,\tilde{I}_{\omega Xe}\,\xi'' + E\,\tilde{I}_{\omega Ye}\,\eta'' + E\,\tilde{I}_{\omega e}\,\theta'')_{(1)} \\
& \quad - \frac{Z^0}{L}\,(E\,\tilde{I}_{\omega Xe}\,\xi'' + E\,\tilde{I}_{\omega Ye}\,\eta'' + E\,\tilde{I}_{\omega e}\,\theta'')_{(2)} = F_\theta - \left(1-\frac{Z^0}{L}\right)F_{\theta(1)} \\
& \quad - \frac{Z^0}{L}\,F_{\theta(2)} - \int_0^{Z^0}(\eta'\,M_Y + \xi'\,M_X + \eta\,V_{X(1)} - \xi\,V_{Y(1)})\,dZ^0 \\
& \quad + \frac{Z^0}{L}\int_0^L(\eta'\,M_Y + \xi'\,M_X + \eta\,V_{X(1)} - \xi\,V_{Y(1)})\,dZ^0.
\end{aligned}
\tag{23}
$$

In what follows, the end conditions are developed as a function of deformations. Making use of Eqs. (19a) and (19b), Eq. (6) can be written as:

$$
\begin{aligned}
& E\,\tilde{I}_{Ye}\,\xi''_{(1)} + E\,\tilde{I}_{XYe}\,\eta''_{(1)} + E\,\tilde{I}_{\omega Xe}\,\theta''_{(1)} - \alpha_{Y(1)}\,\xi'_{(1)} = \\
& \qquad M^0_{Y(1)} + F_{Y(1)} - \bar{\alpha}_{Y(1)}\,(\xi'_{(1)} - \varphi_{pY(1)}), \\
& E\,\tilde{I}_{Ye}\,\xi''_{(2)} + E\,\tilde{I}_{XYe}\,\eta''_{(2)} + E\,\tilde{I}_{\omega Xe}\,\theta''_{(2)} + \alpha_{Y(2)}\,\xi'_{(2)} = \\
& \qquad M^0_{Y(2)} + F_{Y(2)} + \bar{\alpha}_{Y(2)}\,(\xi'_{(2)} - \varphi_{pY(2)}), \\
& E\,\tilde{I}_{XYe}\,\xi''_{(1)} + E\,\tilde{I}_{Xe}\,\eta''_{(1)} + E\,\tilde{I}_{\omega Ye}\,\theta''_{(1)} - \alpha_{X(1)}\,\eta'_{(1)} = \\
& \qquad - M^0_{X(1)} + F_{X(1)} - \bar{\alpha}_{X(1)}\,(\eta'_{(1)} - \varphi_{pX(1)}), \\
& E\,\tilde{I}_{XYe}\,\xi''_{(2)} + E\,\tilde{I}_{Xe}\,\eta''_{(2)} + E\,\tilde{I}_{\omega Ye}\,\theta''_{(2)} + \alpha_{X(2)}\,\eta'_{(2)} = \\
& \qquad - M^0_{X(2)} + F_{X(2)} + \bar{\alpha}_{X(2)}\,(\eta'_{(2)} - \varphi_{pX(2)}).
\end{aligned}
\tag{24}
$$

Similarly Eq. (10) can be expressed as:

$$
E\,\tilde{I}_{\omega Xe}\,\xi''_{(j)} + E\,\tilde{I}_{\omega Ye}\,\eta''_{(j)} + E\,\tilde{I}_{\omega e}\,\theta''_{(j)} = F_{\theta(j)}, \qquad j = 1, 2. \tag{25}
$$

Equilibrium of moments acting in the plane X^0–Z^0 gives (Fig. 3b):

$$
M_{Y(1)} - M_{Y(2)} - L\,V_{X(1)} - P\,\xi_{(1)} = 0,
$$

which can be written with the help of Eqs. (6) and (8) as follows:

$$\begin{aligned}&\alpha_{Y(1)}\,\xi'_{(1)} + \alpha_{Y(2)}\,\xi'_{(2)} - (P + L\,\beta_{X(1)})\,\xi_{(1)} = \\ &\quad -M^0_{Y(1)} + M^0_{Y(2)} + L\,V^0_{X(1)} + \bar{\alpha}_{Y(1)}\,(\xi'_{(1)} - \varphi_{pY(1)}) - \bar{\alpha}_{Y(2)}\,(\xi'_{(2)} - \varphi_{pY(2)})\,.\end{aligned} \tag{26}$$

Similarly, equilibrium of the moments acting in the plane Y^0–Z^0 gives:

$$\begin{aligned}&\alpha_{X(1)}\,\eta'_{(1)} + \alpha_{X(2)}\,\eta'_{(2)} - (P + L\,\beta_{Y(1)})\,\eta_{(1)} = \\ &\quad M^0_{X(1)} - M^0_{X(2)} + L\,V^0_{Y(1)} + \bar{\alpha}_{X(1)}\,(\eta'_{(1)} - \varphi_{pX(1)}) - \bar{\alpha}_{X(2)}\,(\eta'_{(2)} - \varphi_{pX(2)})\,.\end{aligned} \tag{27}$$

Method of Solution

Eqs. (21), (22), and (23) plus the appropriate boundary conditions define the deformed shape of the beam-column under a given load. The exact solution of these three simultaneous, nonlinear, non-homogeneous coupled differential equations is not feasible. To obtain a numerical solution a finite difference technique is used.

As shown in Fig. 5, the column length, L, is divided into m equal parts of length $h = L/m$. In addition, one fictitions point beyond the column support is chosen at each end. The deformed shape of the beam-column is then defined by the three unknown components ξ, η and θ at these $m+3$ points; making a total of 3 $(m+3)$ unknowns. The three fundamental equations at each of the pivotal points 2 to $m+2$, and the six boundary conditions at each end furnish a total of 3 $(m+3)$ equations containing the 3 $(m+3)$ unknowns. So the problem can be solved.

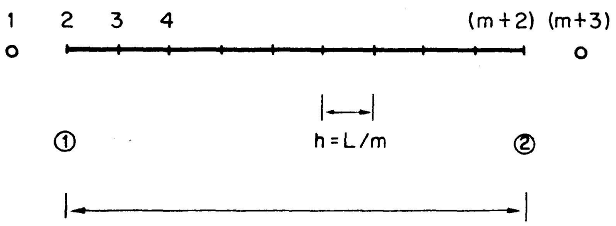


Fig. 5. Subdivision of beam-column.

Replacing the derivatives by central differences at the pivotal points in the left hand sides of the equilibrium equations and of the appropriate boundary conditions results in a system of 3 $(m+3)$ simultaneous equations. This system can be expressed in the matrix form:

$$[\mathsf{A}]\{\mathsf{U}\} = \{\mathsf{W}\}, \tag{28}$$

where

— $[\mathsf{A}]$ is a square matrix of size $3\,(m+3) \times 3\,(m+3)$, whose elements are the geometrical characteristics $E\,\tilde{I}_{Ye}$, $E\,\tilde{I}_{XYe}$, etc., of the elastic core at the pivotal points; the axial thrust P and the spring stiffnesses α.

— $\{\mathsf{U}\}$ is a vector of $3\,(m+3)$ unknown pivotal deformations

$$\{\mathsf{U}\}^T = \{\xi_1, \eta_1, \theta_1, \xi_2, \eta_2, \ldots \xi_{m+3}, \eta_{m+3}, \theta_{m+3}\}.$$

— $\{\mathsf{W}\}$ is a vector of size $3(m+3)$ which consists of the right hand sides of the equilibrium equations and of the appropriate boundary conditions. That is, the externally applied loads and the nonlinear terms resulting from twisting deformation and from plastification.

The set of simultaneous equations (28) cannot be solved directly because the vector $\{\mathsf{W}\}$ is an unknown function of $\{\mathsf{U}\}$. Furthermore, the geometric characteristics vary due to the twisting deformation and due to the yielding. So the matrice $[\mathsf{A}]$ is also an unknown function of $\{\mathsf{U}\}$. Therefore, an iteration procedure is to be followed to solve the problem.

Steps in this solution procedure are (Flow chart):

Flow Chart for Computinal Procedures

1. The input data is read and all the initial values that are necessary for computation are generated.

The beam column is divided into m intervals of equal length (in the numerical examples presented later in the report, m varied from 8 to 12).

2. A convenient load level is chosen to start the analysis. The value adopted is such that the bar is in the elastic state and elastic geometrical characteristics can be utilized for first trial calculations.

3. Trial values for deformations ζ_0, ξ, η and θ at each pivotal point are considered as follows:

For the first cycle of the first load level the deformations are assumed to be zero.

For the first cycle at any other load level, the deformations obtained from a previously converged calculation are used.

For subsequent cycles at any load level, the deformations obtained from the previous cycle are used.

4. From these deformations the normal strain distribution across the cross-section at each pivotal point is computed using Eq. (15). (For details see [12].)

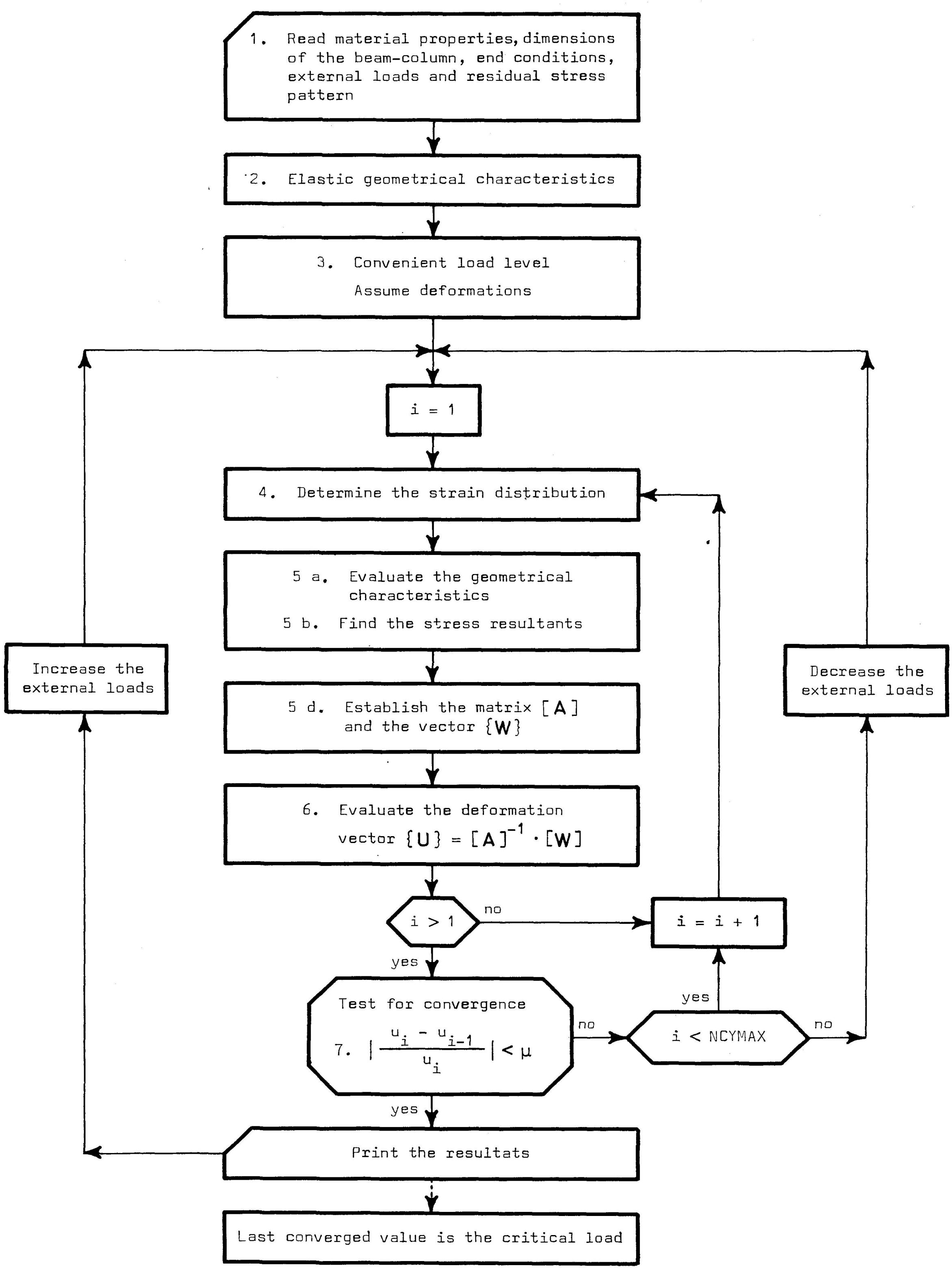
5a. From this strain distribution the geometric characteristics of the unyielded core ($A_e, S_{Xe}, I_{Xe}, \ldots, \tilde{I}_{Xe}, \tilde{I}_{Ye} \ldots$ etc.) defined by Eqs. (17) and (20a) are computed. The normal stress, σ, is then computed using Eqs. (16b) and (18).

5b. The stress resultants P, M_X, M_Y, B; the coefficient K defined by Eq. (4) and the quantities F_X, F_Y, F_θ defined by Eq. (20b) are then evaluated with the normal stress distribution computed above.

5c. The right hand sides of the equilibrium Eqs. (21), (22), (23) and of the appropriate boundary conditions are then evaluated.

5d. The quantities determined in 5a, b, c above permit to establish the elements of the matrice $[\mathsf{A}]$ and the vector $\{\mathsf{W}\}$.

Flow Chart for Computational Procedures



6. A new set of values for deformations $\{\mathsf{U}\}$ is then obtained from the resolution of the system (28).

7. The deformations calculated in two successive cycles are compared for convergence. The following rule of convergence is used:

$$\left|\frac{u_i - u_{i-1}}{u_{i-1}}\right| \leqq \mu ,$$

where u_{i-1} and u_i are the deformations at a chosen point in the $i-1$ and ith cycle, and μ represents the tolerance ratio of convergence. The value of μ in this analysis is 1%.

— If the test is satisfied, the column is still stable. So the load is increased and the calculations start from step 3.

— If the test is not satisfied and if the number of iteration cycles is less than a predefined number (NCYMAX), a new cycle of calculations with the same load starts from step 3.

— If the test is not satisfied within the predefined number of iteration cycles NCYMAX, the calculations restart from the previously converged load level with reduced load increment at step 3.

The calculations are repeated until the ratio between this reduced load increment and the last converged load becomes 1%. Thus the ultimate load is considered as lying between the last converged load and the last load for which there was no convergence.

The computational procedure described above was programmed in the FORTRAN IV language for an electronic digital computer. The CDC Cyber 7326 computer located at the Swiss Federal Institute of Technology at Lausanne was utilized for the numerical studies presented in the next paragraph.

Numerical Examples

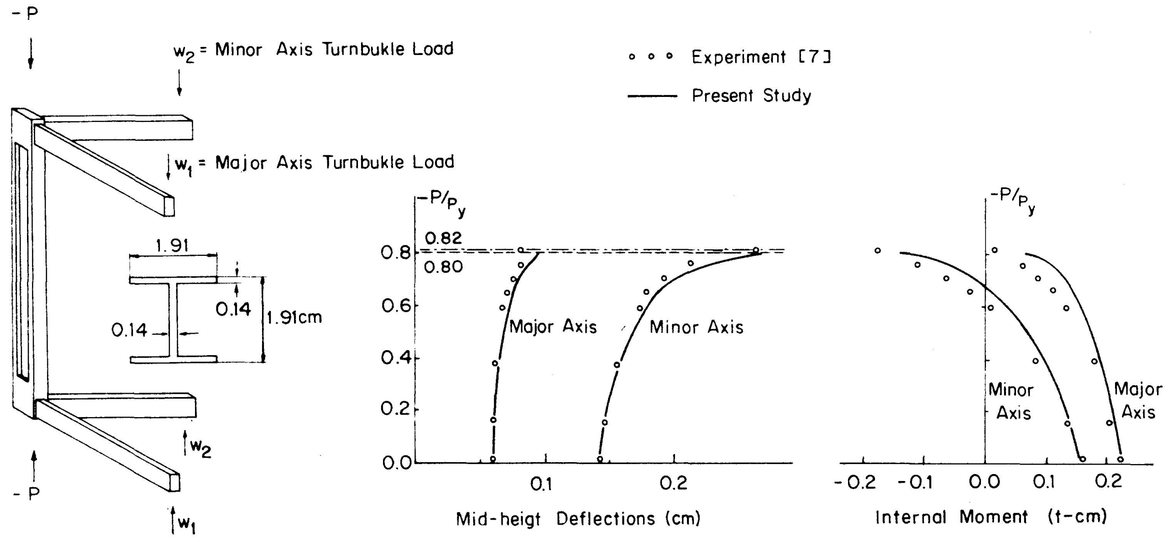
The computer program TOFLS 4 – an acronym for *TO*rsional *FL*exural *S*tability variant *4* – just described was utilized for the solution of the five problems described in this paragraph. Examples 1 and 2 were run to compare the numerical results of the proposed computational procedure with published solutions for rotationally restrained biaxially bent beam-columns [7], [8] and [9]. A biaxially bent directionally restrained beam-column is considered in Example 3. A three dimensional beam-column subassemblage was chosen as the fourth example. The last example was run to test the validity of a design method proposed by YOUNG [5].

Example 1. Rotationally Restrained Beam-Column

The first example selected is the column B 3 (Fig. 6) tested and analyzed by GENT and MILNER [7], [8]. The column was first bent by turnbuckle loads

W_1 and W_2. Then, clamping these turnbuckles rigidly, the axial load P was increased to failure. With the increase of the axial load, the column deformed increasing its joint rotations and relaxing its end moments which were controlled by the beam stiffness.

The H-section column was of length 18 in. (45.7 cm) and the relevant cross-sectional data are shown in Fig. 6a. The beams were made of 0.375 in. $\times$ 0.85 in. (0.95 cm $\times$ 2.16 cm) rectangular section and are 12.1 in. (30.7 cm) long. The column material was considered elastic-perfectly plastic with a yield stress level of 35 ksi (2.44 t/cm^2). So, the squash load P_y ($= \sigma_y A$) is 4.10 kips (1.84 t). Residual stresses and torsion are not included in the analysis, as it is the case with the calculations of [8].



a) Problem definition b) c)

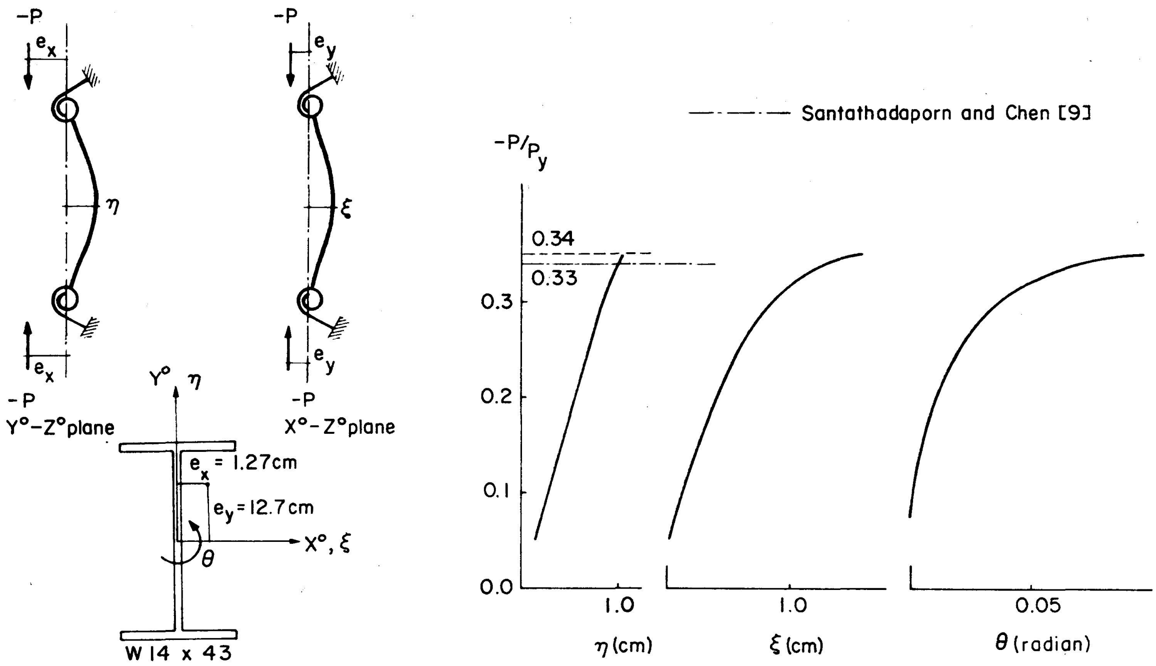
Fig. 6. Example 1: Rotationally restrained beam-column [7].

The variations of major and minor axis end moments with axial loads are shown in Fig. 6c, while Fig. 6b shows a plot of load versus the midheight major and minor axis displacements.

The correlation between computed and experimental results is good. It was also found that if torsion was included the collapse load was reduced by less than 2%.

Example 2. Rotationally Restrained Beam-Column

The second example considered is the symetric and symetrically loaded restrained beam-column (Fig. 7) studied by SANTATHADAPORN and CHEN [9]. In the analysis a WF 14 $\times$ 43 ($\approx$ IPE 380) column of length 220 in. (559 cm) was subjected to a doubly eccentric axial load ($e_x = 5.0$ in. or 12.7 cm, $e_y = 0.5$ in. or 1.3 cm). The stiffness α of the rotational restraints about X^0 and Y^0 axes was 2700 kip.in/radian (3000 t-cm/radian). The yield stress of the material was



a) Problem definition b) Load versus mid-height deformations

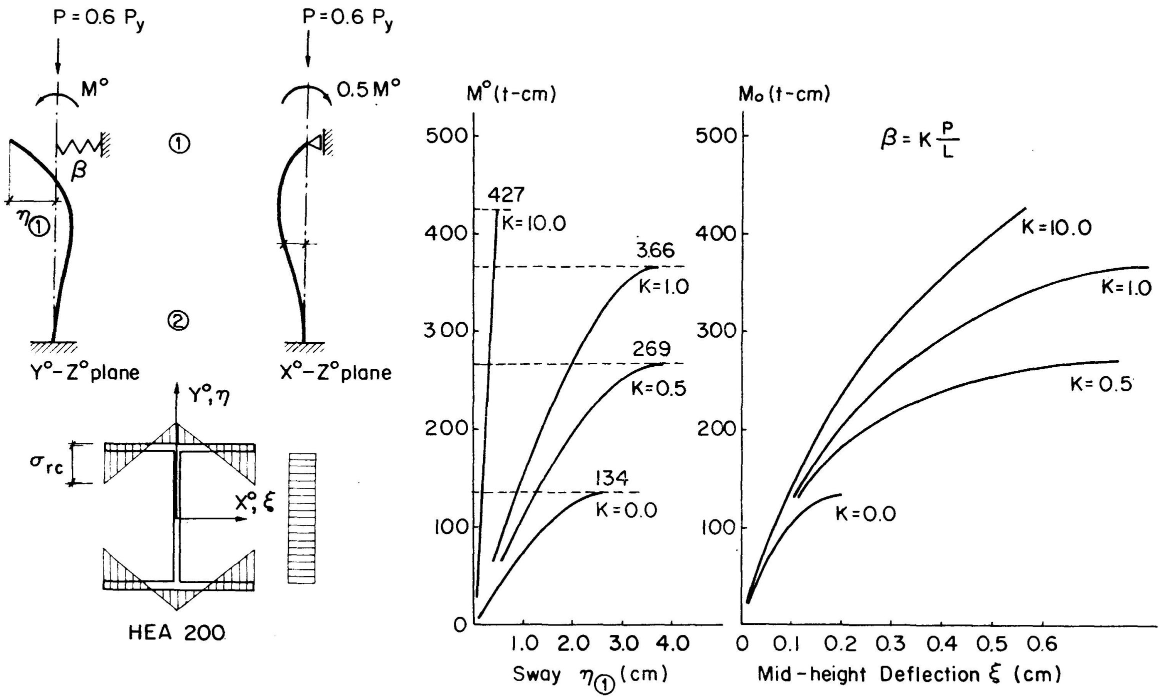
Fig. 7. Example 2: Rotationally restrained beam-column [9].

36 ksi (2.51 t/cm²). The sway and twist at the ends as well as warping of column ends were prevented. Residual stresses were not considered.

The mid-height deformations calculated by the proposed method are given in Fig. 7b. The ultimate load calculated ($=0.34$ of the squash load) compares favourably with the value (0.33) given by SANTATHADAPORN and CHEN [9].

Example 3. Directionally Restrained Beam-Column

The influence of the variation in the stiffness of directional restraint on the stability of a column was examined in the third example. The rolled steel HEA 200 Column of length 330 cm was fixed at its base (end (2)). It was provided at its upper end (end (1)) with a directional spring in the plane Y^0–Z^0 and was simply supported in the plane X^0–Z^0 (Fig. 8). Four calculations were carried out with different values for the spring constant β ($=0.0$; $0.5\frac{P}{L}$; $1.0\frac{P}{L}$ and $10.0\frac{P}{L}$). The case $\beta=0$ corresponds to a free end and the value $\beta=\infty$ corresponds to a simply supported end, so the values $\beta=0.5\frac{P}{L}$, $1.0\frac{P}{L}$ and $10.0\frac{P}{L}$ represent varying degrees of stiffness between these two extremes. In all the cases the axial load P was kept constant at the value $0.6\,P_y$. External moments, M^0 in the plane Y^0–Z^0 and $0.5\,M^0$ in the plane X^0–Z^0, were applied at the end (1) and increased proportionally till the column fails. The analyses were based on an elastic-fully plastic stress-strain diagram (with the yield stress $\sigma_y=2.4$ t/cm²) and a linearly varying symetric residual stress pattern



a) Problem definition b) Load versus displacements

Fig. 8. Example 3: Directionally restrained beam-column.

(with the maximum compressive residual stress of $0.3\,\sigma_y$ occuring at the flange tips). Warping was restrained at the ends.

Fig. 8b shows the load versus computed sways $\eta_{(1)}$ at the end (1) and mid-height deflections ξ curves for the four cases. The curves clearly indicate that the smaller the value of β, the lower the critical moment becomes. In the case of $\beta = 10.0\frac{P}{L}$ the curve terminates abruptly at a load level $M^0 = 427$ t-cm due to the appearance of a biaxial plastic hinge at the end (1). The problem in this case is therefore one of resistance and not of stability. Hence, increase of the spring stiffness beyond the value $\beta = 10.0\frac{P}{L}$ does not increase the capacity of the beam-column.

To the best of the authors' knowledge, these results appear to have been the first for predicting the load-deformation curves for biaxially loaded columns permitted to sway.

Example 4. Three Dimensional Beam-Column Subassemblage

A spatial frame with rolled shape cross-sections was chosen. Instead of analysing the whole frame, only the column (1)–(2) was examined, replacing the influence of the rest of the frame by rotational restraints at the end (1). The column (1)–(2) was bent biaxially by the beam loads (3 and 2 tons respectively) and in addition, was subjected to an axial load of 60 tons (Fig. 9a). These external loads were increased proportionally, with λ indicating the

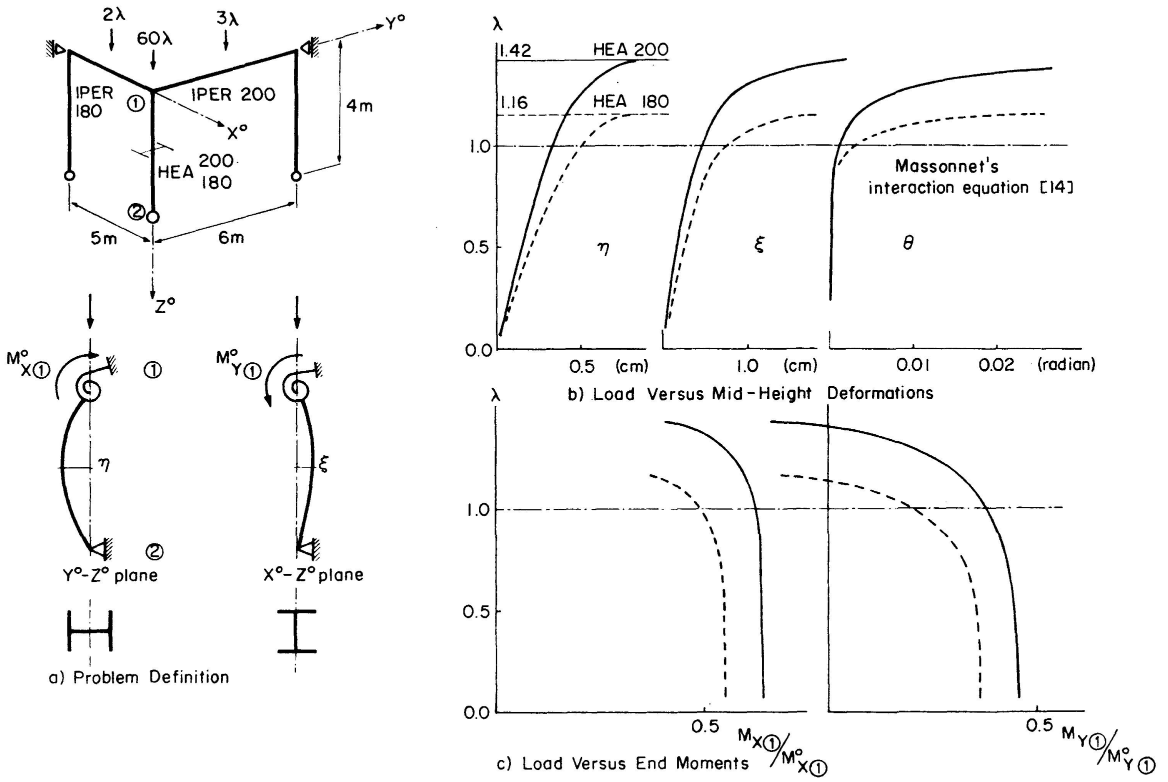


Fig. 9. Example 4: Three dimensional beam-column subassamblage.

constant of proportionality. The rotational restraints at the end (1) were estimated from a first order elastic analysis of frame ($\alpha_{Y(1)} = 22500$ t-cm/radian and $\alpha_{X(1)} = 30000$ t-cm/radian). The end (2) was simply supported. Warping was restrained at both ends. A HEA 200 rolled steel section was chosen for the column (with an yield stress $\sigma_y = 2.4\,\mathrm{t/cm^2}$). The residual stress distribution considered is the same as that given in Example 3.

The ultimate strength calculated is compared with the one estimated by MASSONNET's interaction equation of ref. (13). The mid-height deformations computed by the present study are indicated by solid line in Fig. 9b. The critical value of λ is 1.42, while the interaction equation gives 1.0. If a HEA 180 section is chosen for the column (1)–(2), the critical value of λ decreases to 1.16, which is still greater than that given by the interaction equation. The corresponding mid-height deformations are shown by dotted lines in Fig. 9b. It was verified that till the critical value of λ is reached no plastic hinges appear in the beams of the frame.

As was shown earlier in Example 1, in the case of restrained beam-columns the internal moments are relaxed with the increase of joint rotations. Especially this relaxation is considerable near the critical value (Fig. 9c). The interaction equation takes account of the effect of the rotational restraints on the effective buckling length, but can not take into account the relaxation of end moments. It is known that the interaction equation gives a good approximation for isolated simply supported beam-columns. For restrained beam-columns, however, the interaction equation is in general conservative.

Example 5. Verification of a Design Method [5]

The design proposal developed by YOUNG [5] was examined by considering three examples selected from Fig. 16 of ref. [5]. The beam-column was made of $12 \times 12 \times 65$ lb UC section and was 6.00 m long. The end thrust was $0.3\,P_y$ and a minor axis end-moment of $0.4\,M_{py}$ was applied at one end. These forces were kept constant during the loading by end-moments in the major axis. The yield stress was $2.48\,t/cm^2$. The beam-column was simply supported and warping was permitted at the ends. The calculations were carried out with different ratios of the major axis end moments: $\beta_x = 1.0$ (single curvature), 0.0 and -1.0 (double curvature).

The calculated ultimate major axis bending moments are compared in Table 1 with the values obtained from Fig. 16 of ref. [5]. M_{px} and M_{py} are the plastic moments of the section for bending about the major and minor axis respectively.

Table 1

β_x	YOUNG	Present Study	Present Study / YOUNG
1.0	$0.38\ M_{px}$	$0.36\ M_{px}$	0.95
0.0	$0.58\ M_{px}$	$0.54\ M_{px}$	0.93
-1.0	$0.61\ M_{px}$	$0.59\ M_{px}$	0.97

It is found that for examples treated in this report, only three cycles of iteration were required for values of load levels less than 90% of the calculated ultimate load. Beyond, four to seven cycles were necessary. Times of computing were of the order of 20 seconds for the CDC Computer at the Swiss Federal Institute of Technology, Lausanne.

Conclusions

A numerical procedure for analyzing biaxially bent inelastic beam-columns of arbitrary, thin walled open cross-section has been presented. The influence of material yielding, residuel stresses, warping stresses, end warping restraint, end rotational (bending) and directional restraints have been included.

The technique has been illustrated by solving some simple problems for which alternative results are available. It is found that good correlation exists between the present and existing solutions. Also, the capacity predictions compare favourably with the available experimental results. Finally, to show the feasibility of the method, an example of a biaxially bent, directionally restrained beam-column is presented.

Recently various design methods for biaxially loaded steel beam-columns have been proposed by different authors in different countries [5, 6, 14]. However, no comparison has been made to verify the relative advantages and disadvantages of these design rules. It is believed that a systematic use of the proposed numerical method – being general – should permit such a study to be undertaken.

Acknowledgements

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Keywords

Beam-columns, *biaxial loads*, computers, deformations, *inelastic analysis*, restrained columns, *stability*, strength, structural engineering, thin walled structures, *torsion*.

Appendix I

Let (X, Y) and (x, y) be the coordinates of an arbitrary point S on the middle line of a cross-section in the system X-Y and in the system x-y respectively. Then

$$X = x\cos\theta - y\sin\theta, \qquad Y = y\cos\theta + x\sin\theta. \tag{29}$$

Also, let s be the arc length along the middle line from the origin 0_s to the point S. The following relations can be written, with the help of Fig. 2:

$$\frac{dX}{ds} = \cos(\alpha+\theta), \qquad \frac{dY}{ds} = \sin(\alpha+\theta), \tag{30}$$

where α is the angle between the x axis and the tangent to the middle line of the cross-section at the point S (x, y).

The warping function, ω, which is twice the sectorial area (shown shaded in Fig. 2) is also a function of the position of S:

$$\frac{d\omega}{ds} = X\sin(\alpha+\theta) - Y\cos(\alpha+\theta). \tag{31}$$

The deformations ξ_S, η_S of the arbitrary point S (x, y) on the cross-section are related to the deformations ξ, η of the origin 0 as follows [12]:

$$\xi_S = \xi - y \sin\theta - x(1-\cos\theta), \qquad \eta_S = \eta + x\sin\theta - y(1-\cos\theta). \tag{32}$$

Deriving these equations once and making use of the relations (29) gives:

$$\xi_S' = \xi' - Y\theta', \qquad \eta_S' = \eta' + X\theta'. \tag{33}$$

Consider a section at a distance Z^0 from end (1). In the equilibrium position, internal forces σdA $(=\sigma t ds)$ normal to the element and τdA $(=\tau t ds)$ tangential to the surface develop (Fig. 10a).

By integrating the projection of these internal forces on the X axis across the section and noting that loads are applied at the column ends only, results in the relation (Fig. 10b).

$$\left[\int_A \{\tau\cos(\alpha+\theta) + \sigma\xi_S'\} t\, ds\right]' = 0. \tag{34}$$

With the help of the relation (30) and (33), Eq. (34) becomes:

$$\left[\int_s \tau \frac{dX}{ds} t\, ds - \int_A \sigma Y \theta' dA + \xi' \int_A \sigma dA\right]' = 0.$$

Applying the partial integration to the first term and using the equilibrium equation of the stresses, namely:

$$\frac{\partial(\tau t)}{\partial s} + t\frac{\partial \sigma}{\partial Z} = 0,$$

the above equation becomes:

$$\left[\int_A \frac{\partial\sigma}{\partial Z} X\, dA - \int_A \sigma Y\theta' dA + \xi'\int_A \sigma\, dA\right]' = 0.$$

Noting that X' is equal to $-Y\theta'$ and using the relation (4), Eq. (34) can be written as:

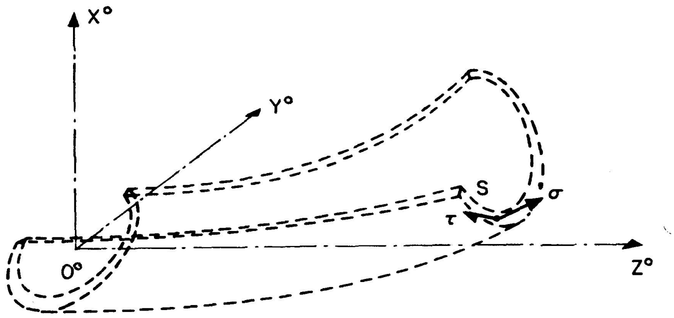
$$\left[\int_A \left(\frac{\partial\sigma}{\partial Z}X + \sigma\frac{\partial X}{\partial Z}\right) dA + \xi'\int_A \sigma\, dA\right]' = \left[\int_A \frac{\partial(\sigma X)}{\partial Z} dA + \xi'\int_A \sigma\, dA\right]' = \\ (-M_Y' + \xi' P)' = 0, \tag{35}$$

which is the first equilibrium equation.

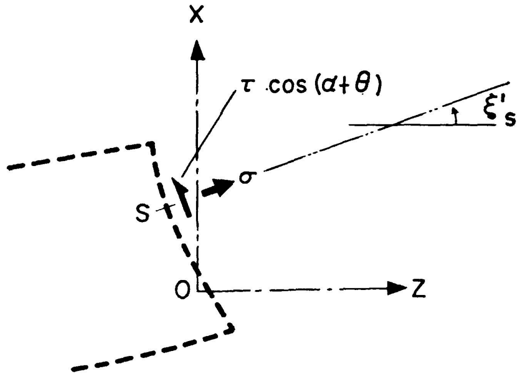
Similarly, considering the total projection of the internal forces on Y axis, a second equilibrium equation is obtained (Fig. 10c):

$$(M_X' + \eta' P)' = 0. \tag{36}$$

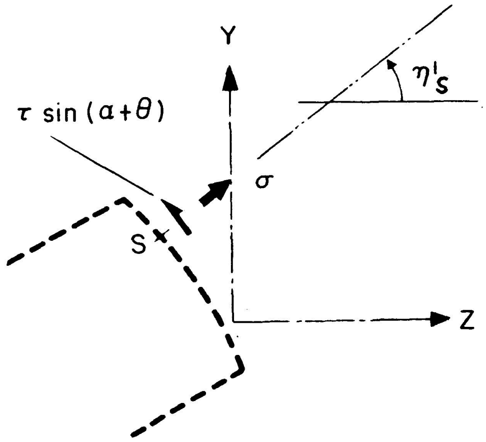
The third equilibrium equation is obtained by writing the relation between the internal twisting moment and the deformations.



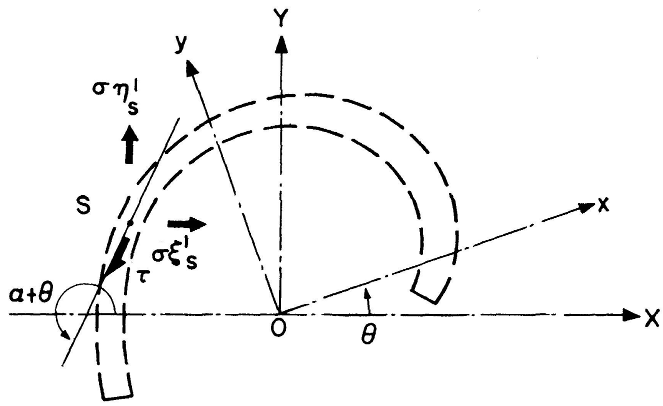
a) Normal and shear stresses acting on an element



b) Projection on X-Z plane



c) Projection on Y-Z plane



d) Shear stresses and projection of normal stresses in the section Z^0

Fig. 10. Forces acting on an elementary area.

A member of thin walled open cross-section resists twisting deformation through both classical St-Venant (uniform) torsion, M_{SV}, and non-uniform torsion which results from warping of the cross-section.

The torsional resistance due to shearing and normal stresses is (Fig. 10d):

$$\int_A \tau t\{X \sin(\alpha+\theta) - Y \cos(\alpha+\theta)\} ds + \int_A \sigma t\{\eta'_S X - \xi'_S Y\} ds. \tag{37}$$

The first term can be simplified with the help of relation (31) as:

$$\int_A \tau t \, d\omega = M_\omega .$$

So, using the relations (4) and (33), expression (37) can be written as:

$$M_\omega + K\theta' - \eta' M_Y - \xi' M_X .$$

The total twisting moment is in equilibrium with the twisting due to the end shears and with the external torsion. So[3]):

$$M_{SV} + M_\omega + K\theta' - \eta' M_Y - \xi' M_X = V_{X(1)}\eta - V_{Y(1)}\xi - H$$

the above relation after deriving once becomes:

$$[M_{SV} + M_\omega + K\theta' - \eta' M_Y - \xi' M_X - V_{X(1)}\eta + V_{Y(1)}\xi]' = 0. \tag{38}$$

Eqs. (35), (36) and (38) are the three equilibrium equations defining the deformed position of the beam-column.

Appendix II - Notations

The symbols adopted for use in this report are listed below.

A	total cross-sectional area.
A_e	cross-sectional area of the elastic core on which the absolute values of the normal strains are less than yield strain.
A_p	cross-sectional area of the plasticized zones $(=A - A_e)$.
B	bi-moment $(=\int_A \sigma\omega \, dA)$.
E	modulus of elasticity.
e_x, e_y	eccentricity of loading in x and y direction.
F_X, F_Y, F_θ	quantities defined by the Eq. (20b).
G	shear modulus.
h	distance between two successive pivotal points $(=L/m)$.
I_{Xe}, I_{Ye}	moments of inertia of the elastic about X and Y axes.

[3]) Here H is the constant part of the twisting moment. It has, as one of its components, the twisting moments $V_{X(1)}\eta_{(1)}$, $V_{Y(1)}\xi_{(1)}$ due to the end displacements. Also, with the yielding, the end shear reactions and shear forces of the end cross-section are not generally on the same lines, so that these forces provoke additional twisting. H is the sum of these two components and of any externally applied torsional moment, M_T.

I_{XYe}	product of inertia of the elastic core.
$I_{\omega Xe}, I_{\omega Ye}$	warping products of inertia of the elastic core about the Y and X axes.
$\tilde{I}_{Xe}, \tilde{I}_{Ye}, \tilde{I}_{XYe}$, $\tilde{I}_{\omega Xe}, \tilde{I}_{\omega Ye}$	quantities defined by the relations (20a).
J	St-Venant's torsional constant.
(j)	subscript used to define the end ($j=1$ refers to the end (1) and $j=2$ refers to the end (2)).
K	term defined by $\int_A \sigma(x^2+y^2)\,dA$.
L	length of the beam-column.
M_X, M_Y	internal bending moments about X and Y axes.
$M^0_{X(j)}, M^0_{Y(j)}$	external moments about X and Y axes at the end (j).
$M_{X(j)}, M_{Y(j)}$	internal bending moments about X and Y axes at the end (j).
M_{SV}	St-Venant torsional moment.
M_ω	internal warping torsional moment.
M^*	reaction on the beam-column from the rotational restraint.
m	number of subdivisions along the length of beam-column.
P	axial force, positive in tension.
P_e	part of the axial force resisted by the elastic core.
P_y	squash load $(=\sigma_y A)$.
S_{Xe}, S_{Ye}	statical moments of the elastic core about X and Y axes.
$S_{\omega e}$	warping statical moment of the elastic core.
$V^0_{X(1)}, V^0_{Y(1)}$	external shearing force in X and Y direction at the end (1).
$V_{X(1)}, V_{Y(1)}$	internal shearing force in X and Y direction at the end (1).
V^*	reaction on the beam-column from the directional restraint.
$\alpha_{X(j)}, \alpha_{Y(j)}$	stiffness factors of rotational restraints at the end (j).
α	angle defined in Fig. 2.
$\beta_{X(1)}, \beta_{Y(1)}$	stiffness factors of directional restraints at the end (1).
ϵ	normal strain.
ζ_0	longitudinal displacement.
η, ξ	displacement of the origin 0 in Y^0 or X^0 directions.
σ	normal stress.
σ_r	residual stress.
τ	average shear stress.
θ	angle of twist; and
ω	double sectorial area.

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Summary

The behaviour of biaxially bent steel beam columns of thin walled open cross-section, restrained at ends by rotational and directional springs, is presented. Residual stresses and warping stresses are included in the analysis. The second order equilibrium equations are established in the deformed position of the bar with respect to an arbitrary system of coordinate axes. This way the shift in the shear center and the shift and rotation of the principal axes of partially plastified sections are taken care of automatically. Several numerical examples are presented.

Résumé

Cet article présente le comportement des poutres-colonnes en acier à section ouverte et à parois minces soumises à la flexion biaxiale, munies aux extrémités de ressorts rotationnels et/ou directionnels. Cette étude tient compte des contraintes résiduelles ainsi que de celles de gauchissement. Un système d'axes de coordonnées est arbitrairement choisi et les équations d'équilibre de second ordre sont établies par rapport à la position déformée de la barre. De cette manière, le changement de position du centre de cisaillement, la translation et la rotation des axes principaux de la section partiellement plastifiée sont automatiquement introduits. Des exemples numériques sont étudiés.

Zusammenfassung

Die vorliegende Studie behandelt das Tragverhalten von auf Druck und zweiachsige Biegung beanspruchten Stahlstützen mit offenen, dünnwandigen Querschnitten. Der allgemeine Fall verformbarer Lagerung wird betrachtet. Der Einfluss von Eigenspannungen und Wölbspannungen ist berücksichtigt. Die Stab- und Querschnittsachsen werden willkürlich gewählt und die Gleichgewichtsbedingungen zweiter Ordnung am verformten System formuliert. Dadurch können die Änderung der Lage des Schubmittelpunktes sowie Verschiebung und Verdrehung der Hauptachsen des teilweise plastifizierten Querschnittes automatisch erfasst werden. Numerische Beispiele werden studiert.