# On finite groups which are necessarily commutative.

Autor(en): **Szép, J.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **20 (1947)**

PDF erstellt am: **27.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-18059

# On finite groups which are necessarily commutative

By J. SZÉP, Budapest

It is known that every group of prime order is Abelian. We want to show some further cases in which the commutativity of the group is a consequence of arithmetical properties of its order. Indeed, $p_1, \ldots, p_n$ denoting different primes, we have the following

**Theorem 1.** *Every group of order* $p_1 \ldots p_n$ *is commutative provided that* $p_i \not\equiv 1 \pmod{p_k}$ *for* $i \neq k,\ i, k = 1, \ldots, n$.

With the same trouble, we prove the following more general

**Theorem 2.** *Every solvable group G of an order* $m = p_1^{\alpha_1} \ldots p_n^{\alpha_n}$ *is commutative provided its Sylow-groups are commutative and, for* $i = 1, \ldots, n$, $\alpha_i < \gamma_i$, $\gamma_i$ *denoting the least positive integer for which* $p_i^{\gamma_i} \equiv 1 \pmod{p_k}$ *for some* $k = (1, \ldots, n)$.

For $n = 1$, our assertion is trivial. Let $n \geqq 2$ and suppose, the theorem holds for $n - 1$ (instead of $n$). $G$ being solvable, it has, by a theorem of Hall[1]), a subgroup $H$ of order $m p_n^{-\alpha_n}$. By hypothesis, $H$ is commutative. Denote $P$ the subgroup of order $p_1^{\alpha_1}$ of $H$ (which is a Sylow-group of $G$), further, denote $N$ the normalizer of $P$. As $H \subseteqq N$, we have for the order $\nu$ of $N$:

$$m p_n^{-\alpha_n} \mid \nu\,, \qquad \nu \mid m\,.$$

Decomposing $G$ according to the modul $P$, $P$:

$$G = P + PA_2P + PA_3P + \cdots\,,$$

---

1) *P. Hall*, A characteristic property of soluble groups, Journal London Math. Soc. **12** (1937), pp. 198—200.

we denote the number of right-side cosets of $P$ in the terms right-hand side by $a_1(=1)$, $a_2, a_3, \ldots$ respectively. Then we have

$$mp_1^{-\alpha_1} = \sum a_i \ .$$

Each $a_i$ is a divisor of the order $p_1^{\alpha_1}$ of $P$ and the number of the terms with $a_i = 1$ equals the index $\nu p_1^{-\alpha_1}$ of $P$ in $N$[2]), and therefore

$$mp_1^{-\alpha_1} \equiv \nu p_1^{-\alpha_1} \pmod{p_1} \ .$$

Hence $\dfrac{m}{\nu} \equiv 1 \pmod{p_1}$. Here the left-hand side is a power $\leqq p_n^{\alpha_n} < p_n^{\gamma_n}$ of $p_n$, and so we have necessarily $\dfrac{m}{\nu} = 1$, $\nu = m$, $N = G$. Thus we have got that $P$ is normal in $G$, and this holds for every Sylow-group of $G$. Since any two of these Sylow-groups have relatively prime orders, they are commutable element by element[3]). Moreover, they are Abelian and so the same holds for their product $G$, as stated.

*Remark.* Owing to Dirichlet's theorem, for each $n$, there is an infinite set of numbers $p_1, \ldots, p_n$ for which theorem 1 applies.

(Eingegangen den 5. Januar 1947.)

---

[2]) *A. Speiser,* Theorie der Gruppen von endlicher Ordnung, 2nd Edition (Berlin 1937), theorems 64, 66.

[3]) *A. Speiser,* loc. cit., theorem 17.