On Implicit Analytic Systems

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1. On a domain containing the origin,

$$(z_1, \ldots, z_m; w_1, \ldots, w_n) = (0, \ldots, 0; 0, \ldots, 0) , \tag{1}$$

of the space of $m + n$ complex variables, let $G_1, \ldots, G_n$, where

$$G_i = G_i(z_1, \ldots, z_m; w_1, \ldots, w_n) , \tag{2}$$

be regular functions satisfying

$$G_i(0, \ldots, 0; 0, \ldots, 0) = 0 \qquad (i = 1, \ldots, n) \tag{3}$$

and

$$\det \; G_{ij}(0, \ldots, 0; 0, \ldots, 0) \neq 0, \quad \text{where} \quad G_{ij} = \partial G_i / \partial w_j . \tag{4}$$

It can be assumed that the (z, w)-domain in question is, or contains, the domain

$$|z_1| < 1, \ldots, |z_m| < 1; \quad |w_1| < 1, \ldots, |w_n| < 1 , \tag{5}$$

and that the n functions (2) are bounded on (5). In fact, the latter assumption is satisfied if, though not only if, the functions (2) have no singularities on the boundary of (5). Needless to say, the regularity of (2) on (5) means that the functions G_i are power series, in $m + n$ variables, which are convergent as $(m + n)$-fold series (and so, as is well-known, converge absolutely) on the domain (5).

The classical existence theorem of analytic implicit systems asserts that, *if r is sufficiently small*, there exists on the neighborhood

$$|z_1| < r, \ldots, |z_m| < r \tag{6}$$

of the point

$$(z_1, \ldots, z_m) = (0, \ldots, 0) \tag{7}$$

a unique set of regular functions

$$w_i = w_i(z_1, \ldots, z_m) \qquad (i = 1, \ldots, n) \tag{8}$$

satisfying the equations

$$G_i(z_1,\ldots,z_m;\ \ w_1,\ldots,w_n)=0 \qquad (i=1,\ldots,n) \tag{9}$$

and the initial conditions

$$w_i(0,\ldots,0)=0 \qquad (i=1,\ldots,n)\ . \tag{10}$$

But the usual procedures leading to the power series (8) fail to supply a reasonable estimate of the size of that "sufficiently small" $(z_1,\ldots,z_m)$-domain about (7) on which the functions (8) can be assured to be regular. Actually, the majorant methods *must* supply for the r in (6) a lower estimate which, as will be seen below, is very far from the ultimate truth.

The object of the present note is to fill in this gap, by determining the "best" value of the "radius of regularity" as an absolute constant, when (9) is given in a normal form.

It will be clear from the construction of that normal form of (9), which is well-known, and from the procedure leading to the "best" value of the corresponding absolute constant r, that the method could be adapted to the determination of the "best" values in that more general situation which is dealt with in Weierstrass' preparation theorem; a situation in which (4) is relaxed to

$$\det\ G_{ij}(z_1,\ldots,z_m;0,\ldots,0)\not\equiv 0\ . \tag{11}$$

2. Let n^2 constants, a_{ij}, be defined as follows: (a_{ij}) is the inverse matrix of the initial Jacobian matrix occurring in (4). Then the n functions

$$F_i=F_i(z_1,\ldots,z_m;\ \ w_1,\ldots,w_m),\ \text{ where }\ F_i=\sum_{j=1}^{n}a_{ij}G_j\ , \tag{12}$$

are regular on the domain (5) and, if (e_{ij}) denotes the unit matrix,

$$F_{ij}(0,\ldots,0;0,\ldots,0)=e_{ij}\ ,\qquad \text{where}\qquad F_{ij}=\partial F_i/\partial w_j\ .$$

Hence, the n functions

$$f_i=f_i(z_1,\ldots,z_m;w_1,\ldots,w_m)\ ,\qquad \text{where}\qquad f_i=w_i-F_i\ , \tag{13}$$

are regular on the domain (5) and satisfy the n^2 conditions

$$f_{ij}(0,\ldots,0;0,\ldots,0)=0,\ \text{ where }\ f_{ij}=\partial f_i/\partial w_j\ \ (i,j=1,\ldots,n). \tag{14}$$

Furthermore, by (13), (12) and (3),

$$f_i(0,\ldots,0\,;\,0,\ldots,0)=0 \qquad (i=1,\ldots,n)\ . \tag{15}$$

Finally, it is seen from (13) and (12) that, since $\det a_{ij}\neq 0$ in (12), the system (9) is equivalent to

$$w_i=f_i(z_1,\ldots,z_m\,;\,w_1,\ldots,w_n) \qquad (i=1,\ldots,n)\ . \tag{16}$$

Accordingly, (9), (3), (4) are equivalent to (16), (3), (4). This equivalent form (f) of the original system (G) is the normal form, referred to above, for which the problem of the "best absolute r" will be solved, as follows:

On the domain (5), *let*

$$f_i(z_1,\ldots,z_m\,;\,w_1,\ldots,w_n) \qquad (i=1,\ldots,n) \tag{17}$$

be n regular functions satisfying the $n+n^2$ initial conditions (15), (14) *and the n inequalities*

$$|\,f_i\,|<1 \quad \text{on the domain (5)} \quad (i=1,\ldots,n)\ . \tag{18}$$

Then the system (16) *and the initial conditions* (10) *determine about the point* (7) *of the $(z_1,\ldots,z_m)$-space n functions* (8) *which are regular on the domain*

$$|\,z_1\,|<1,\ldots,|\,z_m\,|<1\ . \tag{19}$$

In addition, the solutions (8) *satisfy the inequalities*

$$|\,w_i(z_1,\ldots,z_m)\,|<1 \quad \text{on the domain (19)} \quad (i=1,\ldots,n)\ . \tag{20}$$

3. Since the boundary of the domain (19) can be part of a natural boundary of the functions (17), the latter being required to be regular only on the domain (5), it is clear that the functions (8) need not be regular if the domain (19) is replaced by any domain (6) belonging to an $r>1$. But $r=1$ is the best r in a less trivial sense also. In fact, even if the n functions (17) are polynomials or, for that matter, linear polynomials in each of the $m+n$ variables z, w, the functions (8) can acquire singularities within the domain (6) belonging to $r=1+\varepsilon$, if $\varepsilon>0$ is arbitrarily fixed.

In order to see this, let $m = 1$, $n = 1$, and let (16), which then is a single equation of the form $w = f(z, w)$, be so chosen that $f(z, w)$ becomes z times a function of w alone:

$$w = z\, f(w) \ . \tag{21}$$

Then what correspond to (15) and (14) are satisfied, (5) reduces to

$$|\, z \,| < 1 \ , \qquad |\, w \,| < 1 \ , \tag{22}$$

and what (18) requires is that the function $f(w)$, which is supposed to be regular in the circle $|\, w \,| < 1$, be such as to satisfy

$$|\, f(w) \,| < 1 \quad \text{if} \quad |\, w \,| < 1 \ . \tag{24}$$

The theorem states that, for every such $f(w)$, the equation has a (unique) solution $w = w(z)$ which is regular in the circle $|\, z \,| < 1$ and satisfies

$$|\, w(z) \,| < 1 \quad \text{if} \quad |\, z \,| < 1 \ ; \qquad w(0) = 0 \ . \tag{25}$$

Let $f(w)$ be the linear polynomial

$$f(w) = (\varepsilon + w) / (1 + \varepsilon) \ ,$$

where ε is a positive constant. Then (24) is satisfied. On the other hand, (21) reduces to a linear equation which is seen to have the solution

$$w(z) = \varepsilon\, z / (1 + \varepsilon - z) \ .$$

Hence, the solution has a pole on the boundary of the circle $|\, z \,| < 1 + \varepsilon$. Since $r = 1 + \varepsilon$ can be chosen arbitrarily close to 1, the assertion preceding (21) follows[1].

4. The case (21) of (16) also makes it clear that the theorem cannot be proved by *any* form of the method of majorants.

In fact, the *best* majorant equation of (21) is

$$w = z\, f^*(w) \ , \qquad (|\, w \,| < 1) \ .$$

where

$$f^*(w) = \sum_{k=0}^{\infty} |\, a_k \,|\, w^k \quad \text{if} \quad f(w) = \sum_{k=0}^{\infty} a_k\, w^k \qquad (|\, w \,| < 1) \ .$$

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[1]) This example does not prove that, if $f(w)$ is regular in the circle $|\, w \,| < 1$ and satisfies (24), then the solution $w(z)$ of (21), which vanishes at $z = 0$ and is regular in the circle $|\, z \,| < 1$, can have a singularity *on* the boundary, $|\, z \,| = 1$.

But the assumption (24) does not even imply that

$$\underset{|w|<1}{\text{fin sup}} \; | f^*(w) | < \infty \, ,$$

and still less that

$$| f^*(w) | < 1 \quad \text{if} \quad | w | < 1 \; ;$$

all that (24) implies is that

$$| f^*(w) | \leq 1 \quad \text{if} \quad | w | \leq \tfrac{1}{3} \, ,$$

where the constant $\frac{1}{3}$ cannot be improved[2]). Hence, no application of the principle of majorants can supply for the solution $w(z)$ that circle of regularity which is assured by the theorem.

5. The method of majorants, in the sense used above, presupposes an application of the method of comparing undetermined coefficients (which leads to recursion formulae). But the latter method must not be confused with the former, since the latter can succeed when the former fails in *every* sense[3]).

It turns out, however, that *the method of undetermined coefficients cannot succeed*, in the present case, *even if it is not weakened by a subsequent application of the principle of majorants.*

In fact, if $w^{(k)}(z)$ denotes the k-th partial sum of the power series

$$w(z) = \sum_{k=1}^{\infty} c_k z^k \, ,$$

$w(z)$ being the (unknown) solution of (21), it is clear from (21) that the application of the method of undetermined coefficients leads to the recursion formula

$$w^{(k+1)}(z) = z \left[f\big(w^{(k)}(z)\big) \right]_k \, , \qquad w^{(0)}(z) \equiv 0 \, ,$$

for the coefficients $c_1, c_2, \ldots,$ where the operator $[\;]_k$ is defined as follows:

$$[f(g(z))]_k = \sum_{h=0}^{k} A_h z^h$$

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[2]) *H. Bohr,* A theorem concerning power series, Proc. London Math. Soc., ser. 2, vol. 13 (1914), pp. 1—5.

[3]) *A. Wintner,* Zur Lösung von Differentialsystemen mit unendlichvielen Veränderlichen, Mathematische Annalen, vol. 98 (1928), pp. 273—280 (more particularly p. 277).

holds by virtue of

$$f(w) = \sum_{h=0}^{\infty} a_h w^h \ , \qquad g(z) = \sum_{h=0}^{\infty} \alpha_h z^h \ ,$$

where a_h, α_h, A_h are constants.

Since all that is known about $f(w)$ in (21) is that $f(w)$ is a power series which converges in the circle $|w|<1$ and satisfies (24), it is clear from the above recursion formulae that, if they could lead to the existence of a power series $w(z)$ which is convergent in the circle $|z|<1$, they would also lead to the inequalities

$$|w^{(k)}(z)|<1 \quad \text{for} \quad |z|<1 \qquad (k = 1, 2, \ldots ; w^{(k)} = [w]_k) \ .$$

But this cannot be accomplished. In fact, if a power series

$$\sum_{h=1}^{\infty} b_h z^h$$

is regular, and represents a function which has an absolute value not exceeding 1, in the circle $|z|<1$, then its partial sums need not be uniformly bounded there. Still less need they have there an absolute value not exceeding 1. This prevents the application of the above recursion formula, even if no use is made of the "(best) majorant" of that recursion formula.

6. Correspondingly, the proof of the theorem will be based on quite another recursion formula, on that supplied by the method of successive approximations. In the particular case (21) of (16), the latter method leads to the recursion formula

$$w^{k+1}(z) = z\, f\big(w^k(z)\big) \ , \qquad w^0(z) \equiv 0 \ ,$$

which, being free of the truncating operator $[\,]_k$, does not meet the obstacle discussed above. But the *direct* estimates of the successive approximations $w^1(z), w^2(z), \ldots$, estimates which will be based on Lipschitz's method, will supply the convergence of the process only in some „sufficiently small" circle about $z = 0$, rather than in the entire circle $|z|<1$.

The missing element, supplying the latter circle, will have to be function-theoretical in nature. It will consist in an appeal to (Stieltjes' original form of) the compactness theorem of normal families.

It may be mentioned that, since the constant occurring in the assumptions (5), (18) and in the assertions (19), (20) is the same, 1, the theorem to be proved is just a manifestation of a theorem of Poincaré-Brouwer concerning fixed points and of its generalizations. But such theorems do not deal with questions of analyticity or, for that matter, with domains (which are *open* sets). This makes clear enough the nature of the part to be played by the compactness of normal families of regular functions.

7. Under the assumptions and in the notations of the theorem to be proved, put

$$w_i^{k+1}(z_1,\ldots,z_m) = f_i(z_1,\ldots,z_m; w_1^k,\ldots,w_n^k) , \tag{26}$$

where $k = 0, 1, 2,\ldots,$

$$w_i^0(z_1,\ldots,z_m) = f_i(z_1,\ldots,z_m; 0,\ldots,0) \tag{27}$$

and $i = 1,\ldots,n$. It will be shown that this defines

$$w_i^0, w_i^1,\ldots, w_i^k,\ldots \qquad (i = 1,\ldots,n) \tag{28}$$

as regular functions on the domain (19), and that

$$|\, w_i^k(z_1,\ldots,z_m)\,| < 1 \text{ on the domain (19).} \tag{29}$$

First, since the n functions (17) are regular on the domain (5), the n functions (27) are regular on the domain (19). Furthermore, it is seen from (18) and (27) that (29) is true for $k = 0$. Suppose that, for a fixed k, the n functions $w_1^k,\ldots, w_n^k$ have been proved to be regular on the domain (19), and that (29) is true for this k. Then, since the functions (17) are regular on the domain (5) and satisfy (18), it follows that (26) defines n functions $w_1^{k+1},\ldots, w_n^{k+1}$ which are regular on the domain (19), and that (29) remains true if k is replaced by $k+1$.

It will be shown that the n sequences (28) are uniformly convergent on *some* domain (6), belonging to a sufficiently small r. Since (29) assures the uniform boundedness of the regular functions (28) on the domain (6) belonging to $r = 1$, it will then follow that the n sequences (28) are uniformly convergent on every closed subset of the domain (19). It will therefore follow that the n limit functions

$$w_i = \lim_{k\to\infty} w_i^k \tag{30}$$

are regular on the domain (19) and, in view of (26), satisfy (16). Finally, (20) will follow from (29).

Accordingly, the theorem will be proved if it is shown that the n sequences (28) are uniformly convergent on *some* domain (6).

8. According to (15) and (14), there belongs to every (arbitrarily small) s, where $0<s<1$, a sufficiently small $r = r(s)$, where $0<r<1$, having the following property: The n inequalities

$$| f_i(z_1,\ldots,z_m;\ w_1',\ldots,w_n') - f_i(z_1,\ldots,z_m;\ w_1'',\ldots,w_n'') | \leq s \sum_{j=1}^{n} | w_j' - w_j'' |$$

(that is, Lipschitz's conditions with a preassigned Lipschitz constant, s) are satisfied whenever

$$(z_1,\ldots,z_m;\ w_1',\ldots,w_n'), \qquad (z_1,\ldots,z_m;\ w_1'',\ldots,w_n'')$$

is a pair of points contained in the domain

$$| z_1 | < r,\ldots, | z_m | < r \ ; \quad | w_1 | < r,\ldots, | w_n | < r \ .$$

In view of (26), this implies that the n inequalities

$$| w_i^{k+1}(z_1,\ldots,z_m) - w_i^k(z_1,\ldots,z_n) | \leq s\, \lambda_k(z_1,\ldots,z_m) \ ,$$

where

$$\lambda_k(z_1,\ldots,z_m) = \sum_{j=1}^{n} | w_j^k(z_1,\ldots,z_m) - w_j^{k-1}(z_1,\ldots,z_m) | \ , \tag{31}$$

hold on the domain (6), provided that

$$| w_i^k(z_1,\ldots,z_m) | < r \text{ on the domain (6)} \qquad (i = 1,\ldots,n) \ , \tag{32}$$

and that (32) remains true when k is replaced by $k-1$. Hence, if the inequality preceding (31) is summed with respect to i, it follows that, for every k,

$$\lambda_{k+1}(z_1,\ldots,z_m) \leq n\, s\, \lambda_k(z_1,\ldots,z_m) \text{ on the domain (6)} \ , \tag{33}$$

provided that (32) is true for every k.

On the other hand, (15) and (27) show that

$$w_i^k(0,\ldots,0) = 0 \qquad (i = 1,\ldots,n) \tag{34}$$

is true if $k = 0$. It follows therefore from (15) and (26) that (34) is true for every k. But (34), (29) and Schwarz's lemma imply that (32), where $r<1$, holds for every k. Consequently, (33) is true for every k.

Finally, it is seen from (29) and (31) that

$$\lambda_1(z_1,\ldots,z_n) < n \ .$$

It follows therefore from (33) that

$$\lambda_k(z_1, \ldots, z_m) < n(ns)^{k-1} \text{ on the domain (6)} . \tag{35}$$

Choose the positive number s, which thus far was arbitrary (<1), to be less than $1/n$. Then (35) and (31) show that the n sequences (28) are uniformly convergent on the domain (6), where $r = r(s)$.

For reasons explained after and before (30), this completes the proof of the theorem.

(Received November 1, 1948.)