A New Curvature Theory for Surfaces in a Euclidean 4-Space.

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1.1. Introduction

In the study of local properties of surfaces in the Euclidean 4-space R_4, our attention has so far centred on the Kommerell conic (Kommerell [7], p. 553) and the curvature ellipse (Schouten-Struik [11], pp. 104—111). Let ξ, ξ' be the tangent and normal planes of a surface (A) at the point A. Then the Kommerell conic (K) of (A) at A is the locus of the point K of intersection of ξ' by the normal planes of (A) consecutive to ξ'. The curvature ellipse (G), also lying in ξ', is obtained as follows. Let J be any tangent unit vector of (A) at A, and (C) any curve on (A) tangent to J at A; then the component in ξ' of the curvature vector of (C) at A, with respect to R_4, depends only on J (Meusnier's Theorem); the locus of the end point of this component as J takes on all the directions in ξ is the curvature ellipse (G). The conics (K) and (G) are polar reciprocal of each other with respect to the unit circle in ξ'.

Analytically, the introduction of (G) into the study of surfaces in R_4 is quite natural because (G) is tied up closely with the two fundamental forms of (A) on which the Gauss-Codazzi-Ricci equations of (A) depend. Geometrically, however, the introduction of (K) is more natural. In view of the fact that the first curvature of a curve is defined to be the rate of change of the angle between two consecutive tangent lines, it is rather surprising that no systematic study has been made of the corresponding role played by the two angles (cf. § 1.4) between a pair of consecutive tangent planes of a surface in R_4. As far as the author is aware, the only known results in which these two angles play a direct or indirect part are the conjugate directions, the Kwietniewski-Kommerell-Eisenhart theorem (§ 1.3), and the "principal directions" of Struik [12] on an m-surface in a Riemannian n-space, which for a surface in R_4 are identical with the principal directions of the function λ defined later in § 1.5.

The purpose of this paper is to present a curvature theory for surfaces in R_4 based on the two angles between consecutive tangent planes of the surface. Let A, A^* be two consecutive points, at a distance ds apart, of the surface (A), and let $d\psi_1$, $d\psi_2$ be the two angles between the tangent planes ξ and ξ^* of (A) at A and A^*, and denote d/ds by d_s. Then $d_s\psi_1$, $d_s\psi_2$ are functions of the point A and the direction AA^*, and will be called the two *first curvatures* of (A) at the point A for the direction AA^*.

The main results which will be obtained are: (i) some new characteristic properties for a few well-known special types of point of the surface, such as the minimal point, the focus point, the R-point, etc. (§ 1.3), (ii) the introduction of R-directions at a point on the surface (§§ 2.3, 3.1) and (iii) the theorem (Corollary 5.1) that a given non-minimal surface in R_4 is essentially determined by its linear element and the two first curvatures.

It is to be pointed out that this new curvature theory also applies to, and all the results obtained in this paper, except those in § 5, also hold for surfaces in a Riemannian 4-space, provided that the angles between two consecutive tangent planes ξ, ξ^* at the points A, A^* of a surface are understood to mean the angles between ξ and the plane obtained by transporting ξ^* parallelly from A^* to A along the arc A^*A.

1.2. Curvature ellipse (G). A family of canonical frames for (A).

(Cf. Wong [13], §§ 2.1, 2.3.)

Following Boruvka [1], we shall use Cartan's [3, 4] method of moving frames to find the equations of (G). A frame $A - I_i$ $(i, j, k = 1, 2, 3, 4)$ in R_4 is the figure consisting of four mutually orthogonal unit vectors I_i attached to a point A. If a family of frames depends on a number of parameters, then between two consecutive frames in the family there are the relations:

$$dA = \omega_i I_i, \qquad dI_i = \omega_{ij} I_j \tag{1.1}$$

(summation over repeated indices). Here the ω's are linear differential forms in the parameters, which satisfy $\omega_{ij} + \omega_{ji} = 0$ and the equations of structure for R_4

$$d\omega_i = [\omega_k \omega_{ki}], \qquad d\omega_{ij} = [\omega_{ik} \omega_{kj}], \tag{1.2}$$

where a d before a differential form denotes exterior differentiation.

Let the point $A(u, v)$, depending on two parameters u, v, describe a surface (A), on each of whose tangent planes is assigned (in a continuous manner) a positive direction of rotation. We confine ourselves to a small enough region of the surface so that this orientation of the surface is possible. Now to each point $A(u, v)$ of (A), let us attach a frame

$A - I_i$ so that the unit vectors $A - I_1$, $A - I_2$ lie in the tangent plane and the rotation from $A - I_1$ to $A - I_2$ is positive. Then we have a two-parameter family of frames $A - I_i$ for which

$$\omega_3 = \omega_4 = 0 \ . \tag{1.3}$$

Exterior differentiation of these gives, by (1.2),

$$[\omega_1 \omega_{13}] + [\omega_2 \omega_{23}] = 0 \ , \qquad [\omega_1 \omega_{14}] + [\omega_2 \omega_{24}] = 0 \ ,$$

which are equivalent to

$$\begin{aligned} \omega_{13} &= a\,\omega_1 + b\,\omega_2 \ , & \omega_{14} &= a'\,\omega_1 + b'\,\omega_2 \ , \\ \omega_{23} &= b\,\omega_1 + c\,\omega_2 \ , & \omega_{24} &= b'\,\omega_1 + c'\,\omega_2 \ , \end{aligned} \tag{1.4}$$

where a, b, c, a', b', c' are functions of u, v.

Let (C) be any curve on (A) through the point A, and let

$$J = I_1 \cos \varphi + I_2 \sin \varphi$$

be the unit tangent vector of (C) at A. Then by (1.1), the component in the normal plane $\xi' = A - I_3 I_4$ of the curvature vector $d_s J$ of (C) at A is

$$\left(\frac{\omega_{13}}{ds} \cos \varphi + \frac{\omega_{23}}{ds} \sin \varphi\right) I_3 + \left(\frac{\omega_{14}}{ds} \cos \varphi + \frac{\omega_{24}}{ds} \sin \varphi\right) I_4 \ ,$$

which depends only on J. Hence the equations of the curvature ellipse (G) referred to the axes $A - I_3 I_4$ in the normal plane ξ' are, by (1.4),

$$\begin{aligned} x_3 &= \tfrac{1}{2}(a + c) + \tfrac{1}{2}(a - c) \cos 2\varphi + b \sin 2\varphi \ , \\ x_4 &= \tfrac{1}{2}(a' + c') + \tfrac{1}{2}(a' - c') \cos 2\varphi + b' \sin 2\varphi \ . \end{aligned} \tag{1.5}$$

It is obvious that for a surface (A) in R_4 there are infinitely many families of frames $A - I_i$ satisfying the conditions (1.3). The following theorem establishes the existence of a particular one of such families, called a *family of canonical frames,* which will be used exclusively in the rest of the paper.

Theorem 1.1. *Given any surface (A) in R_4, there exists a family of canonical frames $A - I_i$ defined over (A) such that*

$$\begin{aligned} \omega_3 = \omega_4 = 0 \ , \qquad \omega_{13} &= a\,\omega_1 \ , & \omega_{14} &= a'\,\omega_1 + b'\,\omega_2 \ , \\ \omega_{23} &= c\,\omega_2 \ , & \omega_{24} &= b'\,\omega_1 + a'\,\omega_2 \ , \end{aligned} \tag{1.6}$$

and

$$\tfrac{1}{2}(a - c) \geqslant b' \geqslant 0 \ . \tag{1.7}$$

The equations of (G), *referred to the axes* $A - I_3 I_4$, *are then*

$$x_3 = \tfrac{1}{2}(a+c) + \tfrac{1}{2}(a-c)\cos 2\varphi\ , \qquad x_4 = a' + b'\sin 2\varphi\ , \tag{1.8}$$

so that the centre of (G) *is at the point* $\{\tfrac{1}{2}(a+c), a'\}$ *and the major and minor semi-axes of* (G) *are of lengths* $\tfrac{1}{2}(a-c), b'$, *respectively.*

Proof. To arrive at (1.5), we have chosen I_1, I_2 to lie in the tangent plane ξ in an assigned order (and consequently, I_3, I_4 lie in the normal plane ξ'). We first prove the theorem in the general case when (G) is not a circle nor a line segment. As the direction J rotates from I_1 in the positive direction, φ increases from 0, and the corresponding point $G\{x_3(\varphi), x_4(\varphi)\}$ describes (G) in a direction which we take to be the *positive* direction of (G). Let C be the centre and G_1, G_2, G_1', G_2' be the four vertices of (G) in the order in which they are encountered when we describe (G) in the positive direction starting out from a vertex G_1 on the major axis of (G). Now in ξ', take $A - I_3$ parallel to CG_1, $A - I_4$ parallel to CG_2. In ξ, take $A - I_1$, $A - I_2$ to be the directions corresponding, by (1.5), to the vertices G_1, G_1', respectively. Then since the coordinates of $G_1(\varphi = 0)$ and C are $G_1(a, a')$, $C\{\tfrac{1}{2}(a+c), \tfrac{1}{2}(a'+c')\}$, we have $a' = \tfrac{1}{2}(a'+c')$, $a > \tfrac{1}{2}(a+c)$. Therefore, $a' - c' = 0$ and $a - c > 0$. Evidently, the length of the major semi-axis of (G) is $a - \tfrac{1}{2}(a+c) = \tfrac{1}{2}(a-c)$. Moreover, at $G_1(\varphi = 0)$, we have $dx_3/d\varphi = 0$, $dx_4/d\varphi > 0$. Therefore, $b = 0$, $b' > 0$.

Since now the equations of (G) are (1.8), the φ for G_2 is the smallest value of φ satisfying

$$\tfrac{1}{2}(a+c) + \tfrac{1}{2}(a-c)\cos 2\varphi = x_3 = \tfrac{1}{2}(a+c)\ , \qquad \text{i. e.} \qquad \cos 2\varphi = 0\ ,$$

Therefore, $\varphi = \dfrac{\pi}{4}$ for G_2, whose coordinates are consequently

$$G_2\{\tfrac{1}{2}(a+c), a' + b'\}\ .$$

The length of the minor semi-axis of (G) being $a' + b' - a' = b' > 0$, we have that $\tfrac{1}{2}(a-c) > b' > 0$. This finishes the proof of the theorem for the general case.

To complete the proof, we need only observe that the cases where (G) is a circle or a line segment correspond to the cases $\tfrac{1}{2}(a-c) = b'$ or $b' = 0$, respectively. *In the case where* (G) *is a circle, we shall always take* $A - I_3$ *through the centre of* (G) *so that in* (1.6) *and* (1.8) $a' = 0$.

For a given surface, the question as to how far the family of canonical frames is determined can easily be settled but is of no importance to us.

1.3. Special types of point on a surface in R_4

There are certain special types of point on a surface in R_4 which we shall meet frequently in our later work. In this section the definitions and the most important of the known properties of these special types of point are given. The relations in a, c, a', b' following each definition are the conditions for a point of the surface to be of this particular type when the surface is referred to a family of canonical frames.

(i) *Minimal point* is a point A at which the curvature ellipse (G) of (A) has its centre at A $(a + c = a' = 0)$. Eisenhart [5] proved that (A) is a minimal (in area) surface if and only if every point of (A) is a minimal point. A minimal point is said to be *general* if it is not an R-point $(b' \neq a$, see (iii) below).

(ii) *Circle point* is a point at which the (G) of (A) is a circle $(a' = 0,$ $\frac{1}{2}(a - c) = b')$. *Axial point* is a special circle point at which the circle (G) reduces to a point $(a' = b' = 0,\ a = c)$. An axial point A is said to be *special* if the (G) at A reduces to the point A $(a = 0)$. It is easy to show that a surface in R_4 is a sphere (a plane) in an R_3 if and only if every point of it is a non-special (special) axial point.

(iii) *R-point* is both a minimal point and circle point $(a' = 0,\ b' = a = -c)$. A surface is called an *R-surface* if all its points are R-points. The Kwietniewski-Kommerell-Eisenhart theorem [8, 7, 5] states that the tangent planes of (A) are all isocline to one another (cf. § 1.4) if and only if (A) is an R-surface. R-surface has also been studied by Boruvka [1] and the author (Wong [13]).

(iv) *Focus point* is a point A at which (G) has a focus at A $(a' = b'^2 + ac = 0)$. Calapso [2] proved that if every point of (A) is a focus point, then the conjugate lines on (A) are geodesics of (A). The converse is not always true.

(v) *Line-segment point* is a point at which (G) degenerates into a line segment $(b' = 0)$. Perepelkine [10] called such point "semi-umbilical" point. Fabricius-Bjerre [6] prove that (A) has no normal torsion if and only if every point of (A) is a line-segment point.

1.4. Angles between two consecutive planes in R_4

In this section we give some formulas and definitions which will be needed in our later work. While the theory of angles between two planes in R_4 is classic (see, for example, Manning [9]), the formulas given here are believed to be new.

Let $A - I_i$, $A^* - I_i^*$ be two consecutive frames connected by

$$dA = A^* - A = \omega_i I_i \ , \qquad dI_i = I_i^* - I_i = \omega_{ij} I_j \ , \tag{1.9}$$

where the ω's are infinitesimals, and consider the consecutive planes $\xi = A - I_1 I_2$, $\xi^* = A^* - I_1^*, I_2^*$. Then it is easy to show that the angle $d\psi$ between the direction $J(\theta) = I_1 \cos\theta + I_2 \sin\theta$ in ξ and its projection in ξ^* is given by

$$\begin{aligned}(\sin d\psi)^2 &= (\omega_{13}^2 + \omega_{14}^2)\cos^2\theta + 2(\omega_{13}\omega_{23} + \omega_{14}\omega_{24})\sin\theta\cos\theta \\ &\quad + (\omega_{23}^2 + \omega_{24}^2)\sin^2\theta \end{aligned} \tag{1.10}$$

and that the two stationary values of $(\sin d\psi)^2$ are:

$$\left.\begin{aligned}(\sin d\psi_1)^2 \\ (\sin d\psi_2)^2\end{aligned}\right\} \begin{aligned} &= \tfrac{1}{2}(\omega_{13}^2 + \omega_{14}^2 + \omega_{23}^2 + \omega_{24}^2) \pm \{\tfrac{1}{2}(\omega_{13}^2 + \omega_{14}^2 - \omega_{23}^2 - \omega_{24}^2)^2 \\ &\qquad + 4(\omega_{13}\omega_{23} + \omega_{14}\omega_{24})^2\}^{\frac{1}{2}} \ . \end{aligned} \tag{1.11}$$

The angles $d\psi_1$, $d\psi_2$, thus determined except for signs, are called the two *angles* between the planes ξ, ξ^*. The two directions $J(\theta_1)$, $J(\theta_2)$ in ξ giving these angles $d\psi_1$, $d\psi_2$ will be called the *angle directions* of ξ with respect to ξ^*. It follows at once from (1.10) that θ_1, θ_2 are the roots of the equation

$$\tan 2\theta = \frac{2(\omega_{13}\omega_{23} + \omega_{14}\omega_{24})}{\omega_{13}^2 + \omega_{14}^2 - \omega_{23}^2 - \omega_{24}^2} \ , \tag{1.12}$$

which show that the two angle directions are mutually orthogonal.

The planes ξ and ξ^* are said to be *isocline* to each other if the $d\psi$ given by (1.10) is independent of θ. (This relation between two planes is symmetrical.) Obviously, a necessary and sufficient condition for this to be the case is that

$$\omega_{13}^2 + \omega_{14}^2 = \omega_{23}^2 + \omega_{24}^2 \ , \qquad \omega_{13}\omega_{23} + \omega_{14}\omega_{24} = 0 \ , \tag{1.13}$$

which are equivalent to

$$\omega_{24} = e\,\omega_{13} \ , \qquad \omega_{14} = -e\,\omega_{23} \qquad (e = \pm 1) \ . \tag{1.13$'$}$$

The bivaluedness of e indicates that two planes can be isocline to each other in one or the other of two senses. Moreover, it follows from (1.12) and (1.13) that *two consecutive planes ξ, ξ^* are isocline to each other if and only if the angle directions of one with respect to the other are indeterminate.*

1.5. Fundamental formulas

Let us define for a surface (A) in R_4 the following three functions:

$$\begin{aligned} \lambda &= (d_s\psi_1)^2 + (d_s\psi_2)^2 , \qquad & \mu &= (d_s\psi_1)(d_s\psi_2) , \\ \nu_e &= (d_s\psi_1 - e d_s\psi_2)^2 , \qquad & (e &= \pm 1) \end{aligned} \tag{1.14}$$

which, like the two first curvatures $d_s\psi_1$, $d_s\psi_2$ of (A) defined in § 1.1, are functions of the point A and the direction AA^* of (A) at A.

If we refer (A) to a family of canonical frames, and apply the results of the preceding section by regarding the planes ξ, ξ^* as the tangent planes of (A) at the consecutive points A, A^*, then on account of (1.11), the functions λ, μ, ν_e can be expressed in terms of the ω's as follows:

$$\begin{aligned} \lambda &= (\omega_{13}^2 + \omega_{14}^2 + \omega_{23}^2 + \omega_{24}^2)/(ds)^2 , \\ \mu &= (\omega_{13}\omega_{24} - \omega_{14}\omega_{23})/(ds)^2 , \\ \nu_e &= \{(\omega_{13} - e\,\omega_{24})^2 + (\omega_{14} + e\,\omega_{23})^2\}/(ds)^2 , \qquad (e = \pm 1) , \end{aligned} \tag{1.15}$$

where $(ds)^2 = \omega_1^2 + \omega_2^2$. It is to be noted that by omitting the ambiguous sign in the expression for μ, we have partially removed the arbitrariness of the signs of $d_s\psi_1$, $d_s\psi_2$.

To indicate clearly that A^*, ξ^* depend on the length $ds = \overset{\frown}{AA^*}$, and the angle $\varphi = \measuredangle\, AI,\ AA^* = \text{Arc tan}\ \omega_2/\omega_1$, we sometimes write them as $A^*(\varphi, ds)$, $\xi^*(\varphi, ds)$. Now using (1.6) in (1.12) and (1.15) and writing $\cos\varphi$, $\sin\varphi$ for ω_1/ds, ω_2/ds, we have the following explicit formulas for $\tan 2\theta$, λ, μ, ν_e for the direction φ (at the point A):

$$\tan 2\theta(\varphi) = \frac{2\{2a'b' + (a'^2 + b'^2 + ac)\sin 2\varphi\}}{a^2 - c^2 + \{a^2 + c^2 + 2(a'^2 - b'^2)\}\cos 2\varphi} , \tag{1.16}$$

$$\lambda(\varphi) = \tfrac{1}{2}(a^2 + c^2) + a'^2 + b'^2 + \tfrac{1}{2}(a^2 - c^2)\cos 2\varphi + 2a'b'\sin 2\varphi , \tag{1.17}$$

$$\mu(\varphi) = \tfrac{1}{2}(a - c)b' + \tfrac{1}{2}(a + c)b'\cos 2\varphi + \tfrac{1}{2}(a - c)a'\sin 2\varphi , \tag{1.18}$$

$$\begin{aligned} \nu_e(\varphi) = {} & \tfrac{1}{2}(a^2 + c^2) + a'^2 + b'^2 - eb'(a - c) \\ & + (a - c - 2eb')\{\tfrac{1}{2}(a + c)\cos 2\varphi - ea'\sin 2\varphi\} . \end{aligned} \tag{1.19}$$

These formulas are fundamental as most of our results will be derived from them.

2.1. Angle directions at a point on a surface for a given direction

From (1.16) it follows that to every direction φ at a point A of a surface (A) in R_4, there correspond two mutually orthogonal directions θ, $\theta + \frac{\pi}{2}$, which are the limiting positions of the angle directions in the tangent plane ξ of (A) at A with respect to its consecutive tangent plane $\xi^*(\varphi, ds)$ as ds approaches zero, the φ being kept fixed. We shall call $\theta, \theta+\frac{\pi}{2}$ the *angle directions of* (A) *at* A *for the direction* φ. Conversely, given any pair of mutually orthogonal directions $\theta, \theta+\frac{\pi}{2}$, there exist in general two and only two directions φ_1, φ_2 such that $\theta, \theta+\frac{\pi}{2}$ are the angle directions of (A) at A for each of the directions φ_1, φ_2. We shall sometimes refer to them as the *directions of* (A) *at* A *giving rise to the angle directions* $\theta, \theta+\frac{\pi}{2}$.

We recall that the $d\psi$ given by (1.10) is the angle that a direction θ in the plane ξ makes with its projection on the consecutive plane ξ^*. For our case where $\xi, \xi^*=\xi^*(\varphi, ds)$ are two consecutive tangent planes of the surface (A), we have, on using (1.6) in (1.10) and writing $\cos\theta$, $\sin\varphi$ for ω_1/ds, ω_2/ds, that

$$\begin{aligned}(d_s\psi)^2 = {} & \{(a^2+a'^2)\cos^2\varphi+2a'b'\cos\varphi\sin\varphi+b'^2\sin^2\theta\}\cos^2\theta \\ & +2\{a'b'\cos^2\varphi+(a'^2+b'^2+ac)\cos\varphi\sin\varphi+a'b'\sin^2\varphi\}\cos\theta\sin\theta \\ & +\{b'^2\cos^2\varphi+2a'b'\cos\varphi\sin\varphi+(c^2+a'^2)\sin^2\varphi\}\sin^2\theta\ .\end{aligned}$$

The expression on the right is symmetric in φ and θ. Hence

Theorem 2.1. *At a point A of a surface (A) in R_4, the angle between a direction θ of (A) and the tangent plane $\xi^*(\varphi, ds)$ is equal to the angle between the direction φ of (A) and the tangent plane $\xi^*(\theta, ds)$.*

By definition, among all the directions τ in the tangent plane ξ, the angle directions $\theta, \theta+\frac{\pi}{2}$ of ξ with respect to the tangent plane $\xi^*(\varphi, ds)$ make stationary angles with $\xi^*(\varphi, ds)$. Therefore, using Theorem 2.1, we have that among all the tangent planes $\xi^*(\tau, ds)$, where τ is variable and ds is fixed, the two tangent planes $\xi^*(\theta, ds)$ and $\xi^*\left(\theta+\frac{\pi}{2}, ds\right)$ make stationary angles with the direction φ. Hence

Theorem 2.2. *For a surface (A) in R_4, the angle directions $\theta, \theta+\frac{\pi}{2}$ in the tangent plane ξ at any point A with respect to the tangent plane $\xi^*(\varphi, ds)$ are characterized by the property that among all the tangent planes of (A) whose points of contact are at equal small distance ds from A, the two in the directions $\theta, \theta+\frac{\pi}{2}$ make stationary angles with the direction φ.*

Corollary 2.2. *If $\xi^*(\varphi, ds)$ is isocline to ξ, then the direction φ makes the same angle with all the tangent planes of (A) whose points of contact are at equal small distance ds from A, and conversely.*

It is to be pointed out that in these theorems and in what follows, "equal (or the same)" means "equal (or the same) when the second and higher orders of the infinitesimal ds are neglected".

2.2. Directions corresponding to the vertices of (G).

Let us refer a surface (A) to a family of canonical frames (and we shall do this throughout the rest of this paper without specific mentioning). Then the directions $\varphi = 0, \frac{\pi}{2}$; $\varphi = \frac{\pi}{4}, \frac{3\pi}{4}$, which correspond, by (1.8), to the two pairs of opposite vertices of (G), will appear frequently in our later work. It is therefore interesting that we have in the following theorem a characteristic property for these directions in terms of the angle directions alone.

Theorem 2.3. *On a surface in R_4, the directions $\theta, \theta + \frac{\pi}{2}$ at a point A correspond to a pair of opposite vertices of the curvature ellipse at A if and only if these directions have the same angle bisectors as the two directions that give rise to the angle directions $\theta, \theta + \frac{\pi}{2}$.*

Proof. The condition for the directions $\theta, \theta + \frac{\pi}{2}$ to have the same angle bisectors as the directions φ_1, φ_2 is that

$$\tfrac{1}{2}(\varphi_1 + \varphi_2) = \theta + \frac{\pi}{4} .$$

If we double this and take the tangent of both sides, the result is

$$\frac{m_1 + m_2}{1 - m_1 m_2} = -\frac{1}{\tan 2\theta} , \tag{2.1}$$

where $m_1 = \tan \varphi_1$, $m_2 = \tan \varphi_2$.

If $\theta, \theta + \frac{\pi}{2}$ are the angle directions for the directions φ_1, φ_2, the m_1, m_2 are the two roots for $m = \tan \varphi$ of (1.16), which is now written as

$$(\alpha' + \gamma' m^2) \tan 2\theta - (\alpha + 2\beta m + \gamma m^2) = 0 , \tag{2.2}$$

where

$$\begin{aligned} \gamma' - \alpha' &= -\{a^2 + c^2 + 2(a'^2 - b'^2)\} , \\ \gamma - \alpha &= 0 , \qquad 2\beta = 2(a'^2 + b'^2 + ac) . \end{aligned} \tag{2.3}$$

Consequently, equation (2.1) becomes

$$\frac{2\beta}{(\gamma' - \alpha')\tan 2\theta} = -\frac{1}{\tan 2\theta} . \tag{2.4}$$

To solve this equation for θ, we observe first that by (2.3) the condition $2\beta + \gamma' - \alpha' = 0$ reduces to $4b'^2 - (a - c)^2 = 0$, and is therefore the condition for the curvature ellipse (G) to be a circle.

If (G) is not a circle, equation (2.4) can only be satisfied by $\theta = 0, \frac{\pi}{2}$; or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$, which are the directions corresponding to the two pairs of opposite vertices of (G).

If (G) is a circle, equation (2.4) is satisfied by every θ. But then every point of (G) is a vertex. Hence our theorem is completely proved.

2.3. *R*-directions. Characteristic properties of the focus point, a special line-segment point, and the *R*-point.

On a surface (A) in R_4, a direction AA^* from the point A to a consecutive point A^* will be called an *R-direction* of (A) at A if the tangent plane ξ^* at A^* is isocline to the tangent plane ξ at A. Two R-directions AA_1^*, AA_2^* of (A) at A (if they exist) are said to be of the *same* or *opposite type* according as the tangent planes ξ_1^*, ξ_2^* at A_1^*, A_2^*, respectively, are isocline to ξ in the same or opposite sense.

We observe that the statements at the end of § 1.4 and in Corollary 2.2 are two characteristic properties of R-direction.

We shall now prove the following theorem.

Theorem 2.4. *For a surface (A) in R_4, a necessary and sufficient condition for all the directions of (A) at a point A to give rise to a common pair of angle directions is that A be one of the following types of point: (i) focus point, (ii) a point A at which the curvature ellipse (G) is a line segment ending at A, (iii) a point A at which (G) is a line segment subtending a right angle at A, and (iv) an R-point. At a point of the type (i), (ii) or (iii), there exist, respectively, no (real) R-direction, one R-direction or two R-directions; all other directions give rise to a common pair of determinate angle directions, which correspond to the two end points of the major axis of (G). At an R-point, every direction is an R-direction.*

Proof. At a point A of (A), all directions of (A) give rise to a common pair of angle directions if and only if the right-hand member of (1.16) has the same value for all φ. Therefore, this situation can arise only in one of the following three ways:

$$a'b' \neq 0,\ a^2 - c^2 \neq 0,\ a'^2 + b'^2 + ac = 0,\ a^2 + c^2 + 2(a'^2 - b'^2) = 0; \quad (2.6)$$

$$a^2 - c^2 = 0\ , \qquad a^2 + c^2 + 2(a'^2 - b'^2) = 0; \quad (2.7)$$

$$a'b' = 0\ , \qquad a'^2 + b'^2 + ac = 0\ . \quad (2.8)$$

To each of these sets of equations, the inequality (1.7), namely,

$$\tfrac{1}{2}(a - c) \geqslant b' \geqslant 0 \quad (2.9)$$

must be added.

Confining ourselves to real values only, simple arguments will show that (2.6) is impossible and that (2.7) is included in (2.8).

For (2.8), three cases arise according as 1) $a' = 0,\ b' \neq 0$, 2) $a' \neq 0,\ b' = 0$, 3) $a' = b' = 0$.

Case 1. $a' = 0,\ b' \neq 0$. Then $b'^2 + ac = 0$. By (2.9) we have $\frac{1}{2}(a - c) \geqslant b' > 0$. Therefore, A is a focus point, and equations (1.16) reduces to

$$\tan 2\theta = \frac{0}{(a + c)(a \cos^2 \varphi - c \sin^2 \varphi)}\ .$$

If $a + c \neq 0$, A is not an R-point. Then since $ac = -b'^2 < 0$, there are two and only two imaginary directions $a \cos^2 \varphi - c \sin^2 \varphi = 0$ giving indeterminate θ. All real directions give rise to the same determinate pair of angle directions $\theta = 0, \frac{\pi}{2}$, which correspond to the extremities of the major axis of (G).

If $a + c = 0$, A is an R-point. In this case, the angle directions for every direction are indeterminate.

Case 2. $b' = 0,\ a' \neq 0$. Then $a'^2 + ac = 0$. Since by (2.9) $a - c \geqslant 0$, we must have $ac = -a'^2 < 0$, $a > c$. Therefore, the (G) at A is a line segment subtending a right angle at A, and equation (1.16) reduces to

$$\tan 2\theta = \frac{0}{(a - c)(a \cos^2 \varphi + c \sin^2 \varphi)}\ .$$

Hence there are two and only two (real) directions $\varphi = \pm \operatorname{Arc\,tan} \sqrt{-a/c}$ giving rise to indeterminate θ. From the equations of (G)

$$x_3 = a \cos^2 \varphi + c \sin^2 \varphi, \qquad x_4 = a'\ ,$$

it follows that these two directions both correspond to the foot of the perpendicular from A to the line segment (G) and therefore are the directions corresponding to the smallest normal curvature at A. All the directions other than these two give rise to the same determinate pair of angle directions $\theta = 0, \frac{\pi}{2}$, which correspond to the two end points of (G).

Case 3. $a' = b' = 0$. Then $ac = 0$. Since $a - c \geqslant 0$ by (2.9), we have two subcases according as $a > 0$, $c = 0$ or $a = c = 0$.

If $a > 0$, $c = 0$, the (G) at A is a line segment ending at A, and equations (1.16) reduces to $\tan 2\theta = 0/a^2 \cos^2 \varphi$. Therefore, there is one and only one real direction $\varphi = \frac{\pi}{2}$ giving rise to indeterminate θ. This direction corresponds to the end point A of the line segment (G). All other directions give rise to the same pair of angle directions $\theta = 0, \frac{\pi}{2}$.

If $a = c = 0$, the (G) at A is the point A. Every direction gives rise to indeterminate angle directions.

Thus our theorem is proved.

3. Existence of *R*-directions

In the last section we came across some results concerning the R-directions. In this section we shall study the R-directions directly by regarding them as the *real* zero-directions of the invariant ν_e (cf. (1.19)):

$$\begin{aligned} \nu_e &= \tfrac{1}{2}(a^2 + c^2) + a'^2 + b'^2 - eb'(a - c) \\ &\quad + (a - c - 2eb')\{\tfrac{1}{2}(a + c)\cos 2\varphi - ea' \sin 2\varphi\}, \end{aligned} \tag{3.1}$$

where e is $+1$ or -1.

It is easy to show that the zero-directions of ν_e are given by

$$\left.\begin{matrix}\tan \varphi_e' \\ \tan \varphi_e''\end{matrix}\right\} = \frac{-a'\{2b' - e(a - c)\} \pm \sqrt{-1}\,\{a'^2 + (a - eb')(c + eb')\}}{a'^2 + (c + eb')^2}. \tag{3.2}$$

Therefore, each ν_e has two zero-directions, which are in general distinct and imaginary. But when the two zero-directions of ν_e coincide, they coincide in one (real) R-direction φ_e' $(= \varphi_e'')$. The condition for this is

$$a'^2 + (a - eb')(c + eb') = 0,$$

that is,

$$a'^2 - b'^2 + ac + eb'(a - c) = 0. \tag{3.3}$$

We remember that $\frac{1}{2}(a - c)$ and b' are the lengths of the semi-axes of (G). Also, as a consequence of the Gauss equation for the surface (A),

$a'^2 - b'^2 + ac$ is equal to the Gaussian curvature K of (A) (cf. Wong [13], formula (3.6) and Theorem 2.2). Therefore, a geometric interpretation of (3.3) is

$$\begin{aligned}&\text{(Power of } A \text{ with respect to the director circle of } (G) \text{ at } A)\\ &\quad = \text{(Gaussian curvature of } (A) \text{ at } A) \qquad (3.4)\\ &\quad = -\frac{2e}{\pi}\text{(Area of } (G)) \ .\end{aligned}$$

Hence

Theorem 3.1. *At a point A on a surface (A) in R_4, there exists in general no R-direction. There exists an R-direction if and only if the area of the curvature ellipse (G) at A is equal to $\frac{1}{2}\pi$ times the numerical value of the Gaussian curvature of (A) at A.*

Let us suppose that there exist at A two R-directions of the *same* type. Then for $e = +1$ or -1, (3.3) holds and furthermore, the right hand member of (3.2) is indeterminate. Therefore, we have

$$a'^2 + (a - eb')(c + eb') = a'\{2b' - e(a - c)\} = a'^2 + (c + eb')^2 = 0 \ ,$$

that is,

$$a' = c + eb' = 0 \ .$$

But then $\nu_e = \frac{1}{2}(a - eb')(1 + \cos 2\varphi)$, and it has to be zero for two directions. Therefore, $a - eb' = 0$, so that A is an R-point (and by (1.7), $e = 1$). The converse is obvious. Hence

Theorem 3.2. *If there exist at A two distinct R-directions of the same type, then A is an R-point. Conversely, at an R-point of (A), all directions of (A) are R-directions of the same type.*

Let us now suppose that there exist at A two R-directions, one of each type. Then (3.3) holds for both values of e, so that

$$a'^2 - b'^2 + ac = 0 \ , \qquad b'(a - c) = 0 \ .$$

Since $\frac{1}{2}(a - c) \geqslant b' \geqslant 0$ (cf. (1.7)), the above equations are equivalent to $b' = 0$, $a'^2 + ac = 0$. Therefore, the (G) at A is a line segment subtending a right angle at A. In this case, it follows from (3.2) that the zero direction of ν_e is given by $\tan \varphi'_e = -ea'/c$ or is indeterminate according as $a - c > 0$ or $= 0$. In the latter case, (G) is the point A.
Suppose that $a - c > 0$, then the two zero-directions φ'_e $(e = \pm 1)$ coincide if and only if $a' = 0$ or $c = 0$. If $a' = 0$, then $a = 0$, $c > 0$, and (G) is a line segment ending at A, in which case, the two zero-direc-

tions coincide in the direction $\varphi = 0$. Using $a' = b' = a = 0$ in (1.17) and (1.18), we see that for the direction $\varphi = 0$, both λ and μ are zero, i. e. $d_s\psi_1 = d_s\psi_2 = 0$. Therefore, the tangent plane of (A) at a point A^* consecutive to A in the direction $\varphi = 0$ is parallel to the tangent plane of (A) at A (to within infinitesimals of higher order than $ds = \widehat{AA^*}$). The case $c = 0$ is geometrically the same as the case $a' = 0$. Hence

Theorem 3.3. *There exist at A two and only two R-directions of different types if and only if the (G) at A is a line segment subtending a right angle at A. When two such R-directions exist, they are equally inclined to the two (mutually orthogonal) directions corresponding to the extremities of the line segment (G). In particular, these two R-directions coincide if and only if the line segment (G) has an end point at A; in this case, the two R-directions coincide in the direction AA^* corresponding to the end point A of (G), and the tangent plane at A^* is parallel to that at A (to within infinitesimals of higher order than $ds = \widehat{AA^*}$).*

4.1. Directions for which λ or μ has the same value

From (1.17), namely,

$$\lambda(\varphi) = \tfrac{1}{2}(a^2 + c^2) + a'^2 + b'^2 + \tfrac{1}{2}(a^2 - c^2)\cos 2\varphi + 2a'b'\sin 2\varphi, \tag{4.1}$$

it follows easily that $\lambda(\varphi_1) = \lambda(\varphi_2)$ if and only if the directions φ_1, φ_2 satisfy the equation

$$\tan(\varphi_1 + \varphi_2) = \frac{4a'b'}{a^2 - c^2}\,. \tag{4.2}$$

Therefore, in particular, the directions giving stationary values to $\lambda(\varphi)$, i. e. the (Struik) principal directions of $\lambda(\varphi)$, are given by

$$\tan 2\varphi = \frac{4a'b'}{a^2 - c^2}\,. \tag{4.3}$$

For $\mu(\varphi)$ the equations corresponding to (4.1)—(4.3) are

$$\mu(\varphi) = \tfrac{1}{2}(a - c)b' + \tfrac{1}{2}(a + c)b'\cos 2\varphi + \tfrac{1}{2}(a - c)\,a'\sin 2\varphi\,, \tag{4.4}$$

$$\tan(\varphi_1 + \varphi_2) = \frac{a'(a - c)}{b'(a + c)}\,, \tag{4.5}$$

$$\tan 2\varphi = \frac{a'(a - c)}{b'(a + c)}\,. \tag{4.6}$$

A consequence of the preceding equations is

Theorem 4.1. *At a point on a surface in R_4, the two principal directions of $\lambda(\varphi)$ $\{or\ \mu(\varphi)\}$ are mutually orthogonal. And $\lambda(\varphi_1) = \lambda(\varphi_2)$ $\{or\ \mu(\varphi_1) = \mu(\varphi_2)\}$ if and only if the directions φ_1, φ_2 are equally inclined to the principal directions of $\lambda(\varphi)$ $\{or\ \mu(\varphi)\}$.*

From (1.16) it is easy to show that the pair of directions giving rise to the angle directions $\theta, \theta + \frac{\pi}{2}$ are mutually orthogonal if and only if $\tan 2\theta = 4a'b'/(a^2 - c^2)$. Hence, comparing this with (4.3), we have

Theorem 4.2. *At a point on a surface in R_4, there are in general two and only mutually orthogonal directions giving rise to the same pair of angle directions. The angle directions for this pair of mutually orthogonal directions are the principal directions of $\lambda(\varphi)$.*

4.2. Characteristic properties of minimal point and axial point

The following facts, which are evident from the definitions (1.14) of $\lambda(\varphi)$ and $\mu(\varphi)$, will be found useful.

If $\lambda(\varphi_1) = \lambda(\varphi_2)$ and $\mu(\varphi_1) = \mu(\varphi_2)$, then the values of $d_s\psi_1$, $d_s\psi_2$ for the direction φ_1 are equal to those for the direction φ_2, and conversely. In particular, if $\lambda(\varphi)$, $\mu(\varphi)$ are both independent of φ, so also are $d_s\psi_1$, $d_s\psi_2$, and conversely.

Now it follows from (4.2) and (4.5) that

$$\frac{4a'b'}{a^2 - c^2} = \frac{a'(a - c)}{b'(a + c)} \tag{4.7}$$

is the condition for the existence of a pair (and therefore, of infinitely many pairs) of directions φ_1, φ_2, such that $\lambda(\varphi_1) = \lambda(\varphi_2)$ and $\mu(\varphi_1) = \mu(\varphi_2)$ hold at the same time.

If neither side of equation (4.7) is indeterminate, the equation can be satisfied only in one of the following three ways: $a' = 0$, or $a + c = 0$, or $(a - c)^2 = 4b'^2$. If $a' = 0$, the major axis of (G) passes through A, and $\varphi_1 + \varphi_2 = 0$. If $a + c = 0$, the minor axis of (G) passes through A, and $\varphi_1 + \varphi_2 = \frac{\pi}{2}$. If $(a - c)^2 = 4b'^2$, (G) is a circle, in which case we may suppose $a' = 0$ (cf. end of § 1.2). Therefore, this case is included in the first case.

If both sides of (4.7) are indeterminate, we have

$$a + c = a' = 0\ , \tag{4.8}$$

or

$$a - c = b' = 0\ , \tag{4.9}$$

which are respectively the conditions for the point A to be minimal or axial.

Since in all cases, an axis of (G) passes through A, we have

Theorem 4.3. *In order that there may exist at a point A of a surface (A) in R_4 a pair of directions φ_1, φ_2 such that the angles between the tangent plane ξ of (A) at A and the consecutive tangent plane $\xi^*(\varphi_1, ds)$ are equal to those between ξ and the consecutive tangent plane $\xi^*(\varphi_2, ds)$, it is necessary that an axis of the (G) at A passes through A. Conversely, if an axis of the (G) at A passes through A, then any pair of directions equally inclined to the two mutually orthogonal directions corresponding to the extremities of this axis of (G) has the above-mentioned property.*

It follows from (4.2), (4.5) that $\lambda(\varphi)$, $\mu(\varphi)$ are both independent of φ if and only if (4.8) or (4.9) is satisfied. This can also be proved directly from (4.1) and (4.4). Hence we have the following characteristic property of the minimal and the axial points:

Theorem 4.4. *A necessary and sufficient condition for a point A of a surface (A) in R_4 to be minimal or axial is that the two first curvatures $d_s\psi_1$, $d_s\psi_2$ of (A) at A are both independent of the direction at A, i. e. that the tangent plane ξ of (A) at A makes the same angles with all the consecutive tangent planes $\xi^*(\varphi, ds)$ whose points of contact are at equal small distances ds from A. At a minimal point, the two first curvatures of (A) are numerically equal to the lengths of the semi-axes of (G). At an axial point, one of the two first curvatures is zero, and the other is numerically equal to the distance of the point (G) from A.*

The last part of this theorem follows from (4.1) and (4.4), which give, for a minimal point,

$$\lambda(\varphi) = a^2 + b'^2 , \qquad \mu(\varphi) = ab' ,$$

and for an axial point,

$$\lambda(\varphi) = a^2 , \qquad \mu(\varphi) = 0 .$$

The following theorem may be considered as a companion for Theorem 4.4:

Theorem 4.5. *At a point A of a surface (A) in R_4, $d_s\psi_1 + d_s\psi_2$ has the same value for all the directions of (A) at A if and only if A is a minimal or an axial point. At a point A of (A), $d_s\psi_1 - d_s\psi_2$ has the same value for all the directions of (A) at A if and only if A is a minimal point or a circle* *point (the latter including the axial point as special case). At a circle point, $d_s\psi_1 - d_s\psi_2$ is numerically equal to the distance from A to the centre of (G).*

Proof. From (3.1) and (1.7), it follows that ν_{-1} is independent of φ if and only if (4.8) or (4.9) holds; and that ν_{+1} is independent of φ if and only if (4.8) or $a - c = 2b'$ holds. In the last case, where (G) is a circle, we may suppose $a' = 0$ so that $\nu_{+1} = \{\frac{1}{2}(a + c)\}^2$. This proves the theorem.

Another characteristic property of a minimal or an axial point is given in the following theorem.

Theorem 4.6. *At a point A on a surface (A) in R_4, (i) the reflection of any direction φ of (A) about the two mutually orthogonal directions corresponding to the extremities of the major axis of (G) is an angle direction of (A) for the direction φ if and only if A is a general minimal point; and (ii) any direction φ of (A) is itself an angle direction of (A) for the direction φ if and only if A is a non-special axial point.*

Proof. For a general minimal point, or for a non-special axial point, we have, respectively, (4.8) and $b' \neq a$, or (4.9) and $a \neq 0$. Therefore, equation (1.16) becomes $\tan 2\theta = \mp \tan 2\varphi$, which proves the necessity of the conditions. The sufficiency of the conditions can be proved by demanding that the right member of (1.16) be equal to $\mp \tan 2\varphi$ identically in φ.

5. Isometric correspondence of surfaces in R_4 preserving angles between consecutive tangent planes

Let the surfaces (A), $(\bar{A})$, described by the points $A(u,v)$, $\bar{A}(\bar{u},\bar{v})$, where u, v; $\bar{u}, \bar{v}$ are parameters, be in such a correspondence. Then $\bar{u}$, $\bar{v}$ are some functions of u, v. We choose the orientations in the tangent planes of (A), $(\bar{A})$ so that they agree with those induced by the isometric correspondence, and then refer (A), $(\bar{A})$ to families of canonical frames. Take any pair of corresponding points A_0, $\bar{A}_0$, and displace $(\bar{A})$ so that $\bar{A}_0$ coincides with A_0, and the oriented plane $\bar{A}_0 - \bar{I}_1\bar{I}_2$ with the oriented plane $A_0 - I_1 I_2$. Then the normal planes $\bar{A}_0 - \bar{I}_3\bar{I}_4$, $A_0 - I_3 I_4$ coincide. Rotate $\bar{A}_0 - \bar{I}_3$ in the plane $\bar{A}_0 - \bar{I}_3\bar{I}_4$ about $\bar{A}_0 = A_0$ until $\bar{A}_0 - \bar{I}_3$ coincides with $A_0 - I_3$. Then $\bar{A}_0 - \bar{I}_4$ either coincides with or is opposite to $A_0 - I_4$. It suffices to consider only the former case as the latter can be reduced to it by a reflection about the 3-plane $A_0 - I_1 I_2 I_3$.

Since (A), $(\overline{A})$ are in isometric correspondence in which the orientations $A - I_1 I_2$, $\overline{A} - \overline{I}_1 \overline{I}_2$ in the corresponding tangent planes correspond, we have

$$\begin{aligned} \overline{\omega}_1 &= \omega_1 \cos t - \omega_2 \sin t \ , \\ \overline{\omega}_2 &= \omega_1 \sin t + \omega_2 \cos t \ , \end{aligned} \tag{5.1}$$

$$\overline{b}'^2 - \overline{a}'^2 - \overline{a}\,\overline{c} = \overline{K} = K = b'^2 - a'^2 - a c \ , \tag{5.2}$$

where t is some function of u, v, and K is the Gaussian curvature of (A) (cf. Wong [13], formula (3.6)).

From (1.14) and (1.15) it follows that the condition for the correspondence to preserve angles between consecutive tangent planes is that

$$\begin{aligned} \overline{\omega}_{13}^2 + \overline{\omega}_{14}^2 + \overline{\omega}_{23}^2 + \overline{\omega}_{24}^2 &= \omega_{13}^2 + \omega_{14}^2 + \omega_{23}^2 + \omega_{24}^2 \ , \\ \overline{\omega}_{13}\,\overline{\omega}_{24} - \overline{\omega}_{14}\,\overline{\omega}_{23} &= e(\omega_{13}\omega_{24} - \omega_{14}\omega_{23}) \quad (e = \pm 1) \ , \end{aligned} \tag{5.3}$$

be satisfied at all corresponding points and for all corresponding directions at the corresponding points. Since (A), $(\overline{A})$ are referred to families of canonical frames, we have

$$\begin{aligned} \omega_{13} &= a\,\omega_1 \ , &\quad \omega_{14} &= a'\,\omega_1 + b'\,\omega_2 \ , \\ \omega_{23} &= c\,\omega_2 \ , &\quad \omega_{24} &= b'\,\omega_1 + a'\,\omega_2 \ , \end{aligned} \tag{5.4}$$

$$\tfrac{1}{2}(a - c) \geqslant b' \geqslant 0 \ , \tag{5.5}$$

and similar barred equations $(5.\overline{4})$, $(5.\overline{5})$ for $(\overline{A})$.

A rather lengthy but elementary discussion of the system of equations (5.1) — (5.5), $(5.\overline{4})$ and $(5.\overline{5})$, which will be omitted here, will show that either the surfaces (A), $(\overline{A})$ are congruent, or we have

$$\overline{b}' = b', \quad \overline{a} - \overline{c} = a - c, \quad \overline{a}' = a' = 0, \quad \overline{a} + \overline{c} = a + c = 0, \quad \sin 2t \neq 0,$$

which characterize a pair of minimal surfaces with equal curvature ellipses at corresponding points. Hence

Theorem 5.1. *In order that there may exist, between two surfaces (A), $(\overline{A})$ in R_4, an isometric correspondence preserving angles between consecutive tangent planes, it is necessary and sufficient that either (A), $(\overline{A})$ are obtainable from each other by a displacement or a displacement together with reflection about a 3-plane, or (A), $(\overline{A})$ are minimal surfaces with equal curvature ellipses at corresponding points.*

Minimal surfaces in R_4 in isometric correspondence and with equal curvature ellipses at corresponding points have been considered before but from different points of view. We refer the reader to Eisenhart [5] and Theorem 4.6 of Wong [13].

An important consequence of Theorem 5.1 is the following

Corollary 5.1. *A given non-minimal surface in R_4 is completely determined, except for a displacement or displacement with reflection about a 3-plane, by its linear element and the two first curvatures.*

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(Received 14^{th} August 1951.)