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Approximation by Use of Kernels Originating from Abel Transforms of Series

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1. Introduction. It is our main object to improve, by direct methods, a theorem to which the author [3] was led by consideration of Abel power series transforms of Tauberian series whose partial sums behave in a prescribed fashion with respect to given rectifiable closed curves. These curves were determined by *periodic* complex valued functions $z(t)$ for which t represented arc length and for which therefore $z(t)$ satisfied the condition

$$\int_a^b |\, dz(t)\,| = b - a \qquad \text{whenever} \qquad -\infty < a < b < \infty \ .$$

In our new Theorem 1.1 below the restriction of periodicity is removed, and the condition on the variation of $z(t)$ is relaxed. If $z(t)$ $(= x(t) + i\,y(t))$ is a function such that, as t increases over the interval $-\infty < t < \infty$, the point $z(t)$ traces in the complex plane a curve C, bounded or unbounded, which is rectifiable over each finite segment $a \leqq t \leqq b$, and if the parameter t is the product of arc length and a non-zero constant K, the $z(t)$ falls within the class of functions covered by the theorem. Hence the theorem is essentially a theorem on approximation of given locally rectifiable curves by curves determined by this curve and a specific transformation.

Theorem 1.1. *Let $z(t)$ be a complex valued function of the real variable t, defined over $-\infty < t < \infty$ and such that for some constant M*

$$\int_a^b |\, dz(t)\,| \leqq M(b - a) \qquad -\infty < a < b < \infty \ . \tag{1.2}$$

Let q and h positive parameters, and let

$$w(q, h, t) = \frac{q}{h} \int_{-\infty}^{\infty} \exp\left[-\left(q\, e^{x/h} - x/h\right)\right] z(t + x)\, dx \ . \tag{1.3}$$

Then

$$| w(q, h, t) - z(t) | \leqq A(q) hM \qquad -\infty < t < \infty \tag{1.4}$$

where $A(q)$ *is defined by*

$$A(q) = \gamma + \log q + 2 \int_q^\infty x^{-1} e^{-x} dx \ . \tag{1.5}$$

Moreover $A(q)$ *is, for each* q, *the best constant in the sense that there exists a function* $z(t)$ *and a value of* t *such that equality holds in* (1.4).

The constant γ in (1.5) is Euler's constant. The function $A(q)$ is a minimum, and hence our estimate of the left member of (1.4) is best, when $q = \log 2$. As is shown in [2] and [3], the minimum value of $A(q)$ is $A_0 = A(\log 2) = .9680448$. The theorem which we are improving did not say that $A(q)$ is the best constant in (1.4); in fact equality cannot hold in (1.4) when $z(t)$ is periodic and hence bounded. In section 3 we show that the additional hypothesis that $z(t)$ is periodic does not imply that the constant $A(q)$ in (1.4) can be replaced by a smaller constant. A technique for applying the results is illustrated in section 4. Finally, in section 5, we note that our methods can be applied to give similar results for other transformations, the application to the Gauss transformation being particularly simple.

2. Proof of Theorem 1.1. We put (1.3) in the form

$$w(q, h, t) = \int_{-\infty}^{\infty} K(x) z(t + x) dx \tag{2.1}$$

where

$$K(x) = (q/h) \exp [-(qe^{x/h} - x/h)] \tag{2.11}$$

and observe that

$$K(x) > 0 \ , \qquad \int_{-\infty}^{\infty} K(x) dx = 1 \ . \tag{2.12}$$

Hence

$$w(q, h, t) - z(t) = \int_{-\infty}^{\infty} K(x) [z(t + x) - z(t)] dx \ . \tag{2.13}$$

Our hypotheses imply that if a denotes the smaller and b the greater of the arguments t and $t + x$, then

$$| z(t + x) - z(t) | = | \int_a^b dz(t) | \leqq \int_a^b | dz(t) | \leqq M(b - a) = M | x | \tag{2.14}$$

and

$$| w(q, h, t) - z(t) | \leqq M \int_{-\infty}^{\infty} K(x) | x | dx \ . \tag{2.15}$$

We note that equality will hold in (2.14) and (2.15) if $t = 0$ and $z(t) = M | t |$.

Defining $A(q)$ by the formula

$$A(q)h = \int_{-\infty}^{\infty} K(x) \mid x \mid dx \ , \tag{2.16}$$

we already have (1.4), and we see that equality holds in a special case. Using (2.11) gives

$$A(q) = \frac{q}{h^2} \int_{-\infty}^{\infty} \exp\left[-(qe^{x/h} - x/h)\right] \mid x \mid dx$$

$$= q \int_{-\infty}^{\infty} \exp\left[-(qe^{x} - x)\right] \mid x \mid dx \ , \tag{2.16}$$

and to complete the proof of Theorem 1.1, it suffices to reduce the last integral in (2.16) to the tabulated functions in (1.5). Letting I and J denote the integrals over $x>0$ and $x<0$ respectively we find, by integrating by parts and changing variables of integration, that

$$I = \int_0^{\infty} [\exp(-qe^y)(qe^y)]\, y dy = \int_0^{\infty} \exp(-qe^y) dy = \int_q^{\infty} x^{-1} e^{-x}\, dx \tag{2.17}$$

and

$$J = \int_0^{\infty} [\exp(-qe^{-y})(qe^{-y})]\, y dy$$

$$= \int_0^{\infty} [1 - \exp(-qe^{-y})] dy = \int_0^{q} x^{-1}(1 - e^{-x})\, dx \ . \tag{2.18}$$

Thus we obtain $A(q)$ in the form

$$A(q) = \int_0^{q} \frac{1 - e^{-x}}{x} dx + \int_q^{\infty} \frac{e^{-x}}{x} dx \ . \tag{2.2}$$

It is found that some straightforward calculations involving Abel's transformation lead to the formula for $A(q)$ given in (2.2) and others lead to (1.5); thus one obtains devious proofs of the formula for Euler's constant, namely

$$\gamma = \int_0^{q} \frac{1 - e^{-x}}{x} dx - \log q - \int_q^{\infty} \frac{e^{-x}}{x} dx \ , \tag{2.3}$$

which we need to pass from one to the other of the formulas for $A(q)$. Since the formula (2.3) seems to be lacking from reference books in which one would hope to find it, we give a direct proof of (2.3) starting with γ defined as the limit of γ_n where $\gamma_n = 1 + 1/2 + \cdots + 1/n - \log n$. Putting $k^{-1} = \int_0^1 t^{k-1} dt$ and then $t = e^{-x/n}$ gives

$$\gamma_n + \log n = \int_0^1 \frac{1-t^n}{1-t}\,dt = \int_0^\infty \frac{1-e^{-x}}{x}\,\frac{x/n}{e^{x/n}-1}\,dx\ . \tag{2.4}$$

Splitting the last integral at $x = q$ gives

$$\gamma_n = \int_0^q \frac{1-e^{-x}}{x}\,\frac{x/n}{e^{x/n}-1}\,dx - \int_q^\infty \frac{e^{-x}}{x}\,\frac{x/n}{e^{x/n}-1}\,dx - \log q + R_n \tag{2.5}$$

where

$$R_n = \log\frac{q}{n} + \frac{1}{n}\int_q^\infty \frac{1}{e^{x/n}-1}\,dx = \log(q/n) - \log(1-e^{-q/n}) = o(1)\ . \tag{2.51}$$

Using the Lebesgue criterion of dominated convergence for taking limits under the integral signs in (2.5), we obtain (2.3) from (2.5).

3. Periodic functions and closed curves. We now prove two theorems in which the conditions on $z(t)$ imply that $z(t)$ traverses a closed rectifiable curve in the complex plane as t increases over a period of $z(t)$ and that t represents arc length on this curve.

Theorem 3.1. *If $q>0$ and B is a constant less than the constant $A(q)$ in (1.5), then there is a periodic function $z(t)$ such that*

$$\int_a^b |\,dz(t)\,| = b-a \qquad -\infty<a<b<\infty\ , \tag{3.11}$$

and, when $w(q,h,t)$ is the periodic function defined by (1.3),

$$|\,w(q,h,0) - z(0)\,| > Bh\ . \tag{3.12}$$

Let h and q be fixed, and choose $\varepsilon>0$ such that $A(q)h - \varepsilon > Bh$. With $K(x)$ still representing the function in (2.11) so that (2.1) holds, choose a positive number L so great that

$$\int_{-\infty}^{-L} K(x)\,|\,x\,|\,dx + \int_L^\infty K(x)\,|\,x\,|\,dx < \varepsilon/2\ . \tag{3.2}$$

Let $z(t)$ be the function of period $2L$ for which

$$z(t) = |\,t\,| \qquad -L \leqq t \leqq L\ . \tag{3.3}$$

Thus $z(t)$ is real and nonnegative, and $z(t) \leqq |\,t\,|$ for all values of t. Hence

$$w(q,h,0)-z(0)=\int_{-\infty}^{\infty}K(x)z(x)dx$$
$$\geqq\int_{-L}^{L}K(x)\,|\,x\,|\,dx-\int_{-\infty}^{-L}K(x)\,|\,x\,|\,dx-\int_{L}^{\infty}K(x)\,|\,x\,|\,dx$$
$$=\int_{-\infty}^{\infty}K(x)\,|\,x\,|\,dx-2\left[\int_{-\infty}^{-L}K(x)\,|\,x\,|\,dx+\int_{L}^{\infty}K(x)\,|\,x\,|\,dx\right]. \quad (3.4)$$

Using (2.16) and (3.2) we obtain

$$w(q,h,0)-z(0)>A(q)h-\varepsilon>Bh\ . \quad (3.41)$$

This proves Theorem (3.1).

The curve generated by the function $z(t)$ constructed in (3.3) is non-simple, but we could easily make a small deformation of $z(t)$ into a function $z^*(t)$, with the same period L, which would generate a simple closed convex curve and for which (3.5) would still hold.

In case $q=\log 2$, we can strengthen the conclusion of (3.12) by replacing it by (3.52) below.

Theorem 3.5. *If* $q=\log 2$ *and* B *is a constant less than the constant* $A_0=A(\log 2)=.9680448$, *then there is a periodic function* $z(t)$ *such that*

$$\int_a^b|\,dz(t)\,|=b-a \qquad -\infty<a<b<\infty\ , \quad (3.51)$$

and, when $w(q,h,t)$ *is the periodic function defined by* (1.3),

$$|\,w(q,h,t)-z(0)\,|>Bh \qquad -\infty<t<\infty\ . \quad (3.52)$$

Without using at present our hypothesis that $q=\log 2$, we choose $\varepsilon>0$ such that $A(q)h-\varepsilon>Bh$ and choose L so great that

$$\int_{-\infty}^{-L+2A(q)h}K(x)dx+\int_{L-2A(q)h}^{\infty}K(x)dx<\varepsilon/2\ . \quad (3.6)$$

As before, let $z(t)$ be the real nonnegative function of period $2L$ for which $z(t)=|\,t\,|$ when $-L\leqq t\leqq L$. Then $z(0)=0$ and $w(q,h,t)$ has period $2L$ so we can prove (3.52) by proving that

$$w(q,h,t)>Bh \qquad |\,t\,|\leqq L\ . \quad (3.61)$$

Using (1.4) with $M=1$, we see that if $|\,z(t)\,|>2A(q)h$, then $w(q,h,t)>A(q)h>Bh$; hence (3.61) holds when t lies in the part of the intervall $-L\leqq t\leqq L$ which lies outside the interval $-2A(q)h\leqq t\leqq 2A(q)h$. It therefore suffices to prove that

$$w(q,t,h)>Bh \qquad |\,t\,|\leqq 2A(q)h\ . \quad (3.62)$$

Assuming that $|t| \leqq 2A(q)h$, we set $a = L - 2A(q)h$ and find that

$$w(q, h, t) \geqq \left\{ \int_{-a}^{a} - \int_{-\infty}^{-a} - \int_{a}^{\infty} \right\} K(x) \, | t + x | \, dx$$

$$= \left\{ \int_{-\infty}^{\infty} - 2 \int_{-\infty}^{-a} - 2 \int_{a}^{\infty} \right\} K(x) \, || t + x | \, dx > F(t) - \varepsilon \tag{3.63}$$

where

$$F(t) = \int_{-\infty}^{\infty} K(x) \, | t + x | \, dx \ . \tag{3.64}$$

Seeking the minimum of $F(t)$, we find that $F''(t) = 2K(-t) > 0$ and hence that $F(t) \geqq F(T)$ where T is the unique solution of $F'(T) = 0$, that is, of

$$\int_{-\infty}^{-T} K(x) dx = \int_{-T}^{\infty} K(x) dx \tag{3.65}$$

or, because of (2.12), of

$$\int_{-T}^{\infty} K(x) dx = \tfrac{1}{2} \ . \tag{3.66}$$

Thus $\exp(-qe^{-T/h}) = 1/2$ and

$$T = h(\log q - \operatorname{loglog} 2) \ . \tag{3.67}$$

Noting that $T = 0$ when (and only when) $q = \log 2$, we now use our hypothesis that $q = \log 2$ to see with the aid of (2.16) that $F(t) \geqq F(0) = A(q)h$ and hence that

$$w(q, h, t) > A(q)h - \varepsilon > Bh \ . \tag{3.68}$$

Thus Theorem 3.5 is proved. It is of interest to observe that the fundamental number T for which (3.65) holds, and which is 0 only when $q = \log 2$, is equal to another fundamental number, $h \log q^{-1}$, for which $F(x)$ assumes its maximum value $1/(eh)$, when and only when $q = (\log 2)^{1/2}$.

4. Tauberian constants. To illustrate briefly a way in which the preceding results may be useful in proving theorems involving Tauberian constants, we use them to simplify proof of a theorem of the author [1], [2]. The theorem is concerned with the closeness with which the partial sums s_n of a Tauberian series Σu_n can be approximated by the Abel power series transform $\sigma(r) = \Sigma_{k=0}^{\infty} r^k u_k$ by making best possible choices of r.

Theorem 4.1. *The constant* $A_0 = A(\log 2) = .9680448$ *is the least constant* B^* *having the following property* P. *If* Σu_n *is a series for which*

$\limsup |nu_n| = h$, *then there is a sequence* r_n (*which may depend upon the terms of the series* Σu_n) *such that* $0 < r_n < 1$, $\lim r_n = 1$, *and*

$$\limsup_{n\to\infty} \left| \sum_{k=0}^{\infty} r_n^k u_k - s_n \right| \leqq B^* h \ . \tag{4.11}$$

By a method that has been and is being generalized in various directions by several authors, it was proved in [1], [2], and [4] in a straightforward way that if $q = \log 2$, and $t_n = e^{-q/n}$, then (4.11) holds when $B^* = A_0$. Hence A_0 has property P. To prove that A_0 is the *least* constant having property P, we must in some way produce, for each $B < A_0$, a series Σu_n such that $\limsup |nu_n| \leqq h$ and $\limsup |\sigma(r_n) - s_n| \geqq Bh$ for every sequence r_n such that $0 < r_n < 1$ and $\lim r_n = 1$. This was done in [1] by defining a very complicated real series Σu_n such that $\limsup |nu_n| = 1$, $\limsup s_n = L$, and $\sigma(r) \leqq L - B$, $0 < r < 1$. We now illustrate a greatly improved technique for the construction of such series.

Let $z(t)$ and $w(q, h, t)$ be functions of the type specified in Theorems 3.1 and 3.5. Let C and C_h be the oriented curves traversed in the complex plane by $z(t)$ and $w(q, h, t)$ as t increases. The author [3] showed that if Σu_n is a series such that $\lim |nu_n| = h$ and its partial sums s_n all lie on C and progress steadily along C in the positive direction, then the curve C_h (which is independent of q) is precisely the set of limit points of the Abel transform $\sigma(r) = \Sigma_{k=0}^{\infty} r^k u_k$.

For the function $z(t)$ constructed in the proof of Theorem 3.5, C is the real line segment $0 \leqq x \leqq L$, traversed to and fro. It follows from (3.68) and considerations of symmetry that $Bh < w(q, h, t) < L - Bh$ so that C_h is included in the interval $Bh \leqq x \leqq L - Bh$. This gives the following theorem.

Theorem 4.2. *If* $h > 0$ *and* $B < A_0 = A(\log 2) = .9680448$, *then there exist a number* L *and a real series* Σu_n *such that* $\lim |nu_n| = h$, $0 \leqq s_n \leqq L$,

$$0 = \liminf_{n\to\infty} s_n < \limsup_{n\to\infty} s_n = L \ , \tag{4.21}$$

and

$$Bh \leqq \liminf_{r\to 1} \sigma(r) \leqq \limsup_{r\to 1} \sigma(r) \leqq L - Bh \ . \tag{4.22}$$

While the statement of Theorem 4.2 did not state the fact, we know that L must be large if B is near A_0, and that our series Σu_n which satisfy $\limsup |nu_n| \leqq h$ and are able to achieve (4.21) and (4.22) are

those for which the partial sums s_n run to and fro from 0 to L as rapidly as the Tauberian condition permits. The preceding work and some heuristic considerations indicate that when B is near A_0, it is not easy for a sequence s_n, bounded or unbounded but such that $\limsup |nu_n| \leqq h$, to achieve a situation in which the sequence s_n has a limit point ζ as far distant as Bh from the nearest limit point of $\sigma(r)$; the sequences described above seem to be the simplest ones that are able to do it. It is doubtless impossible for a bounded sequence s_n, for which $\limsup |nu_n| = h$, to have its elements maneuver so artfully in the complex plane that they possess a limit point as far distant as $A_0 h$ from the nearest limit point of the Abel transform $\sigma(r)$.

It is hoped that the results and methods of [3] and this paper will prove to be useful in obtaining answers to such questions as the following. How should the partial sums s_n, of a series Σu_n for which $\limsup |nu_n| \leqq h$, maneuver in the complex plane so that the Abel transform $\sigma(r)$ has a limit point ζ as far as possible from the nearest limit point of the sequence s_n? Questions of this nature originate in a paper of Hadwiger [5] where preliminary examples and inequalities are given.

5. Other kernels. In conclusion, we remark that the proof of Theorem 1.1 shows very clearly that an inequality similar to (1.4) can be obtained for each kernel $K(x)$, with or without parameters, for which

$$\int_{-\infty}^{\infty} |K(x)||x|\,dx < \infty\ , \qquad \int_{-\infty}^{\infty} K(x)dx = 1\ . \tag{5.1}$$

For the case of the Gauss kernel, we find by a trivial calculation that if $a > 0$, if $z(t)$ satisfies the hypothesis of Theorem 1.1, and if

$$w(a, t) = a\pi^{-1/2} \int_{-\infty}^{\infty} e^{-a^2x^2} z(t + x)dx\ , \tag{5.2}$$

then

$$|w(a, t) - z(t)| \leqq (\pi^{-1/2}/a)\, M \qquad -\infty < t < \infty\ . \tag{5.3}$$

Moreover the method of proof of Theorem 1.1 shows that the coefficient of M in (5.3) is the best possible. The conditions (5.1) play fundamental roles in parts of Wiener's theory of Tauberian theorems, and our kernel $K(x)$ of (2.11), with the parameters q and h both equal to 1, appears in Wiener's original book [6, p. 105] on the subject where Abel transforms are treated.

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[5] *Hadwiger, Hugo*, Über ein Distanztheorem bei A-Limitierung, Comment. Math. Helv. **16** (1944), pp. 209—214.

[6] *Wiener, Norbert*, The Fourier integral. Cambridge 1933, 201 pp.

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The paper [3] listed above has appeared in Duke math. J. 19 (1952) 131—138. The question appearing in the last paragraph of section 5 above remains unanswered. The corresponding question, in which the Abel transform $\sigma(r)$ is replaced by the arithmetic mean transform

$$M_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1},$$

has been completely answered in the two following papers.

[7] *Agnew, R. P.*, Arithmetic means of some Tauberian series and determination of a lower bound for a fundamental Tauberian constant. Appearing in Proc. London Math. Soc.

[8] *Agnew, R. P.*, Arithmetic means and the Tauberian constant. 474541. Appearing in Acta Mathematica.