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Groups and spaces of loops

by H. Samelson [1])

For Professor H. Hopf on his sixtieth birthday

1. It has become customary to call H-structure an object consisting of a space X and a multiplication μ in X with homotopy-unit, i. e. a (continuous) map μ of the cartesian product $X \times X$ into X, such that for a certain point x_0 of X the two maps l_{x_0}, r_{x_0} of X into itself, defined by $x \to \mu(x_0, x)$, resp. $x \to \mu(x, x_0)$ (left and right translation by x_0) are homotopic to the identity (cf. [8]; essentially this concept — the Γ-manifold — appeared in [5]). There are two large well known classes of H-spaces: topological groups, and spaces of loops (with fixed base point) in topological spaces (cf. [8] for the definition of the latter). It is our purpose to show that in a certain sense and to a certain extent the first category is contained in the second. We then give proofs, suggested by this situation, for two propositions. First, we give an answer to the question, raised by Eilenberg, whether the map $(x, y) \to xyx^{-1}y^{-1}$, where x and y run through the quaternions of norm 1, is homotopic to a constant map; the answer is that it is not. (We note that a different and somewhat simpler proof for the same fact has been found independently by G. W. Whitehead.) The second application concerns a special fact about Pontryagin-multiplication in Eilenberg-MacLane spaces.

2. The structures mentioned above possess a further operation, namely an inversion, i. e. a map $\sigma : X \to X$ such that the map $x \to \mu(x, \sigma(x))$ is homotopic to a constant map; for groups this is the ordinary inverse, for loop spaces this is the map obtained by reversing the loops, i. e. by replacing the parameter t by $1 - t$. (We often write xy or $x \cdot y$ instead of $\mu(x, y)$ and x^{-1} instead of $\sigma(x)$.)

If (X, μ) and (X', μ') are two H-structures, then a map f: $X \to X'$ is an H-homomorphism, if the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} \\ X' \times X' & \xrightarrow{\mu'} & X' \end{array}$$

[1]) The work on this note was supported by a grant from the National Science Foundation.

is homotopy-commutative, i. e. if the two possible maps of $X \times X$ into X' are homotopic. If the structures have an inversion, one requires that in addition the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow f & & \downarrow f \\ X' & \xrightarrow{\sigma'} & X' \end{array}$$

be homotopy-commutative. We call a map $g: Y \to Y'$ a weak homotopy equivalence, if g induces isomorphisms of all the homotopy groups of Y and Y'; as well known, g induces then also isomorphisms of all the (singular) homology groups of Y and Y' (one proves this by introducing the mapping cylinder C of g, and noticing that because of the vanishing of the relative homotopy groups of C mod Y one can construct a chain deformation from C to Y which shows that the relative homology groups vanish; this is a simple case of Hurewicz's isomorphism theorem; cf. also the theorem of J. H. C. Whitehead [12], Theorem 1). If Y and Y' are sufficiently cell-complex-like, such a map is a homotopy equivalence. The following lemma is easily proved by the same technique.

Lemma 1. If $g: Y \to Y'$ is a weak homotopy equivalence, and if $h: P \to Y$ is a map of a finite polyhedron P into Y, then h is homotopic to a constant map, if and only if $g \circ h$ is.

We recall (cf. e. g. [1]) that a principal bundle for a topological group G is a space E, on which G operates without fixed points (we write the operation as $x \cdot g$ or xg, for $x \in E$, $g \in G$), and which satisfies an additional continuity assumption: the map of $E \times G$ into $E \times E$, defined by $(x, g) \to (x, xg)$ is a homeomorphism into, or equivalently, g is a continuous function of the pair (x, xg). Denoting by B the base space of the decomposition of E into the orbits under G, and by p the projection of E onto B, we shall require also that p is a fiber map in the sense of Serre, i. e. that the covering homotopy theorem holds for finite polyhedra (cf. [8]).

A principal bundle will here be called universal if it is contractible over itself to a point. The base space of a universal bundle is called a classifying space for G. It is well known that e. g. all compact Lie groups admit universal bundles [1]; as a matter of fact, in this case the universal bundles can be constructed such that they and the corresponding classifying spaces are locally finite polyhedra.

We also recall Serre's basic observation [5]: If X is a 0-connected space, x_0 a point of X, then the space of all paths (continuous maps of the unit interval $I = [0, 1]$ into X), which end at x_0, is a fiber space over X relative to the projection $p(w) = w(0)$ for any $w: I \to X$. The whole space is contractible; the typical fiber is the space $\Lambda(X)$ of loops in X, based at x_0.

3. We can now state our result.

Theorem I. *If the group G admits a universal bundle E, with base space B and projection p, then, corresponding to the contraction of E, there exists an H-homomorphism, which is also a weak homotopy equivalence, of G into the space $\Lambda(B)$ of loops in B.*

For the proof we first establish the existence of a weak homotopy equivalence in a somewhat more general situation, and show later that in the case of Theorem I this map is an H-homomorphism.

Proposition I: Let L be a fiber space in the sense of Serre, with base M, projection q, typical fiber F; suppose L is contractible to a point. Then there exists a map f of F into $\Lambda(M)$ (space of loops in M) which is a weak homotopy equivalence; and a fiber map $\bar{h}$ of L into the space Z of paths in M, ending at some point b_0, which induces f in the fiber and the identity in the base, and which induces an isomorphism of the spectral sequences of L and Z from E_2 on.

That the homotopy groups of F and $\Lambda(M)$ are isomorphic, follows immediately from consideration of the homotopy sequences of the pairs (L, F) and $(Z, \Lambda(M))$ (with the usual identification of the relative groups of the pair and the absolute groups of the base space M, cf. [10], p. 90), since L and Z are contractible; the interest of the proposition lies in the existence of the map f.

Proof. Let k: $L \times I \to L$ be the contraction, let a_0 be the point, into which L is contracted, and set $b_0 = p(a_0)$. Denote by Y (resp. Z) the space of all paths in L (resp. M), which end at a_0 (resp. b_0); let p be the projection of Z onto M. In well known fashion (cf. [8], p. 474) k induces a map $\bar{k}$: $L \to Y$, by $\bar{k}(x)(t) = k(x, t)$ for any $x \in L$ and $t \in I$. Composition with q yields a map $\bar{h}$: $L \to Z$, defined by $\bar{h}(x)(t) = q \circ k(x, t)$. The map $\bar{h}$ is a fiber map relative to q, p and the identity of M: $p \circ \bar{h}(x) = q(x)$; indeed, $p \circ \bar{h}(x) = \bar{h}(x)(0) = q \circ k(x, 0) = q(x)$, since $k(x, 0) = x$. In particular, the fiber F_0 through a_0 is mapped into $\Lambda(M)$. Let h_* denote the associated map of the homotopy sequence of (L, F_0) into that of $(Z, \Lambda(M))$.

Since $\bar{h}$ induces the identity map of M, h_* is the identity map of $\pi_n(M)$ (for all $n \geq 0$). The spaces L and Z are contractible, and so have vanishing homotopy groups. It follows now from the five-lemma ([4], p. 16), that h_* induces an isomorphism between $\pi_n(F_0)$ and $\pi_n(\Lambda(M))$. This proves the weak equivalence of F_0 and $\Lambda(M)$, with f being the restriction of $\bar{h}$ to F_0, considered as a map into $\Lambda(M)$. The map $\bar{h}$ is therefore a fiber map of L into Z, which induces the identity of the base, and maps the homology groups of the fibers isomorphically; it is well known that $\bar{h}$ induces then an isomorphism of the spectral sequences from E_2 on.

4. We now turn to the situation of Theorem I. We identify G with the fiber of E through a_0 (the point toward which E is contracted by the contraction k) by sending the element g of G into $a_0 g$. Applying proposition I, we have the map f of G into $\Lambda(B)$, which is a weak homotopy equivalence; explicitly f is given by $f(g)(t) = p \circ k(a_0 g, t)$. We now show that f is an H-homomorphism.

Let g and g' be any two elements of G. The assignment $t \to k(a_0 g, t)\cdot g'$, for $t \in I$, represents a path in E from $a_0\cdot g\cdot g'$ to $a_0\cdot g'$. We use this to construct a map $w_{g,g'}$ of the boundary $\dot{I}^2$ of the unit-square $I^2 = I \times I$ into E as follows:

$$w_{g,g'}(t, u) = \begin{cases} a_0 & \text{for } t = 1, 0 \leq u \leq 1 \\ a_0\cdot g\cdot g' & \text{for } t = 0, 0 \leq u \leq 1 \\ k(a_0\cdot g\cdot g', t) & \text{for } u = 0, 0 \leq t \leq 1 \\ k(a_0 g, 2t)\cdot g' & \text{for } u = 1, 0 \leq t \leq \frac{1}{2} \\ k(a_0 g', 2t-1) & \text{for } u = 1, \frac{1}{2} \leq t \leq 1 \end{cases}$$

One checks that the mapping is well defined, and that the assignment $(g, g', t, u) \to w_{g,g'}(t, u)$, for g, $g' \in G$, $(t, u) \in \dot{I}^2$, is a continuous map of $G \times G \times \dot{I}^2$ into E. We extend this to a map Φ of $G \times G \times I^2$ into E: for each (g, g'), we map the center $(\frac{1}{2}, \frac{1}{2})$ of I^2 into a_0, and we map the segment from any (t, u) in $\dot{I}^2$ to $(\frac{1}{2}, \frac{1}{2})$ in the obvious fashion on the path, described by the point $w_{g,g'}(t, u)$ under the contraction k. In $p \circ \Phi = \Psi$ we have then a map of $G \times G \times I^2$ into B. From this we get a map ψ of $G \times G \times I$ into $\Lambda(B)$ by defining $\psi(g, g', u)$ to be the path given by $\psi(g, g', u)(t) = \Psi(g, g', t, u)$; using the relation $p(a\cdot g) = p(a)$ for all $a \in E$, $g \in G$ (expressing the fact that the orbits of E under G are the fibers of p) one sees that actually $\psi(g, g', u)(0) = \psi(g, g', u)(1) = b_0$. We can consider ψ as a homotopy between ψ_0 and ψ_1, defined by $(g, g') \to \psi(g, g', 0)$, resp. $\psi(g, g', 1)$. Returning to the definition of ψ, one sees that $\psi_0(g, g')$ is identical with $f(g\cdot g')$, where f is the map $G \to \Lambda(B)$ constructed above. On the other hand $\psi_1(g, g')$ is $f(g)\cdot f(g')$, the product in $\Lambda(B)$ of $f(g)$ and $f(g')$. The last three sentences imply that the diagram

$$\begin{array}{ccc} G\times G & \longrightarrow & G \\ \big\downarrow{\scriptstyle f\times f} & & \big\downarrow{\scriptstyle f} \\ \Lambda(B)\times\Lambda(B) & \rightarrow & \Lambda(B) \end{array}$$

is homotopy-commutative, so that f is H-homomorphic with respect to multiplication.

5. Inversion can be treated similarly. For any $g \in G$ we construct a map v_g of $\dot{I}^2$ into E by

$$v_g(t, u) = \begin{cases} a_0 & \text{for } t = 1, 0 \le u \le 1 \\ a_0 g^{-1} & \text{for } t = 0, 0 \le u \le 1 \\ k(a_0 g^{-1}, t) & \text{for } u = 0, 0 \le t \le 1 \\ k(a_0 g, 1 - t)\cdot g^{-1} & \text{for } u = 1, 0 \le t \le 1 \end{cases}$$

Again this is a well defined, continuous map of $G\times\dot{I}^2$ into E, which, by means of the contraction of E, can be extended to a continuous map of $G\times I^2$ into E. The composition with p can be regarded as a map of $G\times I$ into $\Lambda(B)$ by considering t as the loop-parameter, and also as a homotopy of G into $\Lambda(B)$. The two end maps of the homotopy, for $u = 0$, resp. 1, are nothing else but the maps $g \to f(g^{-1})$, resp. $g \to f(g)^{-1}$, and so f is shown to be homotopy-homomorphic with respect to the inversion in G and $\Lambda(B)$. This finishes the proof of Theorem I.

6. Let Q denote the (multiplicative) group of quaternions of norm one (also known as SU(2), Sp(1), Spin(3)); it is homeomorphic with the 3-sphere S_3.

Theorem II. *The map* $\varkappa: Q\times Q \to Q$, *defined by* $\varkappa(x, y) = xyx^{-1}y^{-1}$, *is not homotopic to a constant.*

The theorem can be given another form which is easily seen to be equivalent.

Theorem II′. *The two maps* $\theta_1, \theta_2: Q\times Q \to Q$, *defined by* $\theta_1(x, y) = xy, \theta_2(xy) = yx$, *are not homotopic to each other; Q is not homotopy-abelian.*

Proof. Let E be a universal bundle for Q; it can be constructed as locally finite polyhedron by letting Q operate in the usual manner on the spheres S_{4k-1}, the unit spheres in quaternion k-space, and joining each sphere to the next by means of the mapping cylinder of the inclusion map. The corresponding classifying space Q_∞ is essentially the infinite quaternion projective space. It follows from known properties of finite quaternion projective spaces or from the Gysin-sequence [8], that the homology groups of Q_∞ are infinite cyclic in dimensions $4n$, and zero otherwise. It is clear that the projection $p: E \to Q_\infty$ maps the sphere S_7, contained in E as described, via the „Hopf map" γ into a 4-sphere S_4 contained in Q_∞; the inclusion $i: S_4 \subset Q_\infty$ induces an isomorphism of the homotopy and homology groups of S_4 and Q_∞ in dimension 4.

7. By Theorem I we have a H-homomorphic weak homotopy equivalence f of Q into $\Lambda(Q_\infty)$. Define $d: Q \times Q \to \Lambda(Q_\infty)$ by $d(x, y) = (f(x) \cdot f(y)) \cdot (f(x)^{-1} \cdot f(y)^{-1})$. Since f is H-homomorphic, the two maps $f \circ \varkappa$ and d of $Q \times Q$ into $\Lambda(Q_\infty)$ are homotopic. By lemma 1 of § 2, $f \circ \varkappa$ is homotopic to a constant if and only if $\varkappa$ is. It is therefore sufficient for the proof of Theorem II to show that d is not homotopic to zero.

Let T denote the basic isomorphism between $\pi_n(Q_\infty)$ and $\pi_{n-1}(\Lambda(Q_\infty))$ (this is $\partial \circ p^{-1}$, cf. [7]). If α is the generator of $\pi_4(Q_\infty)$, represented by the inclusion map of S_4 in Q_∞ (cf. § 6), then $T\alpha$ is represented (up to sign) by the map f of $Q = S_3$ into $\Lambda(Q_\infty)$, since f represents a generator of $\pi_3(\Lambda(Q_\infty))$. If $[\alpha, \alpha]$ is the Whitehead product of α with itself, then $T[\alpha, \alpha]$ can be represented as follows (cf. [7]): There exists a map d', homotopic to d, of $S_3 \times S_3$ into $\Lambda(Q_\infty)$, such that the subset $S_3 \vee S_3$ (in the usual notation, cf. [7]) is carried into the point e_0 (the „constant" loop); let s denote the standard map of $(I^6, \dot{I}^6)$ ($I^6 =$ 6-cell, $\dot{I}^6$ its boundary) onto $(S_3 \times S_3, S_3 \vee S_3)$; then $T[\alpha, \alpha]$ is represented, up to sign, by the map $d' \circ s: (I^6, \dot{I}^6) \to (\Lambda(Q_\infty), e_0)$.

8. Actually one can take for d' any map homotopic to d, which maps $S_3 \vee S_3$ into e_0, as the following lemma shows. S_r denotes the r-sphere.

Lemma 2. Suppose g_0, g_1 are two maps of $S_p \times S_q (p, q \geq 1)$ into $\Lambda(X)$, the space of loops of a space X, based at x_0; suppose that $g_0(S_p \vee S_q) = g_1(S_p \vee S_q) = e_0$ (constant loop); and suppose that g_0 and g_1 are homotopic. Then there exists a homotopy $\bar{g}_t$ between g_0 and g_1, such that $\bar{g}_t(S_p \vee S_q) = e_0$ for $0 \leq t \leq 1$.

Proof. Let g_t be the given homotopy. Let (a, b) be the point common to S_p and S_q in $S_p \vee S_q$. We recall that the maps $x \to x \cdot x^{-1}$, resp. $x \cdot e_0$ are homotopic to zero, resp. to the identity, with e_0 stationary throughout the homotopy. An application of Borsuk's homotopy extension theorem shows that the map defined by

$$(x, y, t) \to g_t(x, y) \cdot g_t(a, b)^{-1}$$

is homotopic to a homotopy g', which agrees with g_0, resp. g_1 for $t = 0$, resp. 1, and sends (a, b) into e_0 for all t. We define a new homotopy g'' by

$$g''_t(x, y) = g'_t(x, b)^{-1} \cdot (g'_t(x, y) \cdot g'_t(a, y)^{-1}) \ .$$

This map in turn is homotopic to a homotopy g''', which agrees with $e_0 \cdot (g_0 \cdot e_0)$, resp. $e_0 \cdot (g_1 \cdot e_0)$ for $t = 0$, resp. 1, and sends $S_p \vee S_q \times I$ into e_0 (if f is any map into $\Lambda(X)$, then $f \cdot e_0$ means the map $x \to f(x) \cdot e_0$; similarly for $e_0 \cdot f$): On $a \times S_q \times I\, g''$ is clearly homotopic to the constant map, with $a \times S_q \times \dot{I}$ and $a \times b \times I$ staying at e_0 during the homotopy; similarly for $S_p \times b \times I$; now one applies again the homotopy extension theorem. The lemma is now proved by „removing“ the left and right factors e_0 in g_0''' and g_1''' in a similar fashion.

9. Lemma 2 of § 8 implies that the map d completely determines the element $T[\alpha, \alpha]$ (cf. § 7), and in particular that d is homotopic to zero if and only if $T[\alpha, \alpha]$ is. Since T is an isomorphism this reduces the problem to the question whether the element $[\alpha, \alpha]$ of $\pi_7(Q_\infty)$ is zero or not.

We recall some facts: $\pi_7(S_4)$ is isomorphic to the direct sum $Z + Z_{12}$ (Z = integers, Z_n = integers mod n). The Hopf map γ can be taken as a generator of Z. If i_4 is the generator of $\pi_4(S_4)$, represented by the identity map of S_4, then $[i_4, i_4]$ has Hopf invariant ± 2 (say $+2$, with suitable orientations), and $[i_4, i_4] - 2\gamma$ is a generator of the subgroup Z_{12} of $\pi_7(S_4)$, as shown by Serre [9], p. 503, and Toda [11], Theorem 4.1 [this is of course the central fact in the proof of Theorem II].

On the other hand, the first non-vanishing relative homology group of Q_∞ mod S_4 occurs in dimension 8, and is infinite cyclic, as is clear from the structure of the homology groups of Q_∞ and S_4; the same holds then in homotopy, by the theorem of Hurewicz. From the homotopy sequence of (Q_∞, S_4) it follows then that the kernel of the injection $i_* : \pi_7(S_4) \to \pi_7(Q_\infty)$ is cyclic, as image of an infinite cyclic group. But this kernel contains the Hopf map γ, since γ, as mentioned in § 6, can be factored through the contractible space E. It is clear then that the kernel of i_* : $\pi_7(S_4) \to \pi_7(Q_\infty)$ is the infinite cyclic group generated by γ, and that it therefore does not contain the element $[i_4, i_4]$; in fact the image of $[i_4, i_4]$ is an element of order 12. But the image of $[i_4, i_4]$ under i_* is of course the element $[\alpha, \alpha]$, which is therefore shown to be not zero; Theorem II is proved.

A similar question can be raised concerning the Cayley numbers: Are they homotopy-abelian, and are they homotopy-associative? Presumably the answer to both questions is no.

10. For the second application of Theorem I let $K(Z, n)$ be the Eilenberg-MacLane space for Z and n, i. e. a space X (which can be taken as a complex) for which $\pi_n(X) \approx Z$, and all other homotopy groups vanish (cf. [3], [9]). As well known, one can take the space $\Lambda(K(Z, n))$ as $K(Z, n-1)$, and so all spaces $K(Z, n)$ are H-spaces (with an inversion). This induces a multiplication in the homology group of $K(Z, n)$, the Pontryagin multiplication (cf. [2]). It is also well known that in the loop space $\Lambda(Y)$ of an H-space Y Pontryagin multiplication is anticommutative, i. e. $a * b = (-1)^{rs} b * a$, for $a \in H_r(\Lambda(Y))$, $b \in H_s(\Lambda(Y))$; $*$ denotes the Pontryagin product. [The reason for this is that in the case at hand the multiplication in $\Lambda(Y)$ is homotopy-commutative; the proof is essentially the same as the proof for the commutativity of the fundamental group of a group: For two arbitrary loops f, g in Y, based at the H-unit y_0, define a map $F_{f,g}$ of the unit square I^2 into Y by

$$F_{f,g}(t, u) = f(t) \cdot g(u) \ .$$

By considering the two parts of $\dot{I}^2$ from $(0, 0)$ to $(1, 1)$ and introducing an obvious reparametrization, one gets a homotopy Φ_t of $\Lambda(Y) \times \Lambda(Y)$ into $\Lambda(Y)$, such that $\Phi_0(f, g) = f \cdot y_0 \circ y_0 \cdot g$ and $\Phi_1(f, g) = y_0 \cdot g \circ f \cdot y_0$ (here $\circ$ is the product in $\Lambda(Y)$). Left and right translations by y_0 being homotopic to the identity, one gets finally a homotopy between the two maps, defined by $(f, g) \to f \circ g$, resp. $g \circ f$. We assume here that y_0 is idempotent and stationary under the homotopies.]

11. Let now $n = 2k - 1$ be odd, and let z be the generator of $H_n(K(Z, n); Z)$. Anticommutativity implies that $2z * z = 0$. Our purpose is to show that actually $z * z = 0$.

Proof: Let $U(k)$ denote the unitary group in k variables. It is known that the homology groups of $U(k)$ are torsion free, and that the cohomology ring is a Grassmann algebra, generated by n primitive elements $a_1, \ldots, a_k$, with $\dim a_i = 2i - 1$ (cf. [1] for the concepts and facts involved). It follows from this that the Pontryagin ring of $U(n)$ also is a Grassmann algebra generated by elements $z_1, \ldots, z_k$, with $\dim z_i = 2i - 1$, which are dual to the a_i in the sense that $KI(a_i, z_j) = \delta_{ij}$ (if $\dim a_j = \dim z_j$) (cf. [6]). In particular, we have $z_i * z_i = 0$.

Now let $E_{U(k)}$ and $B_{U(k)}$ be the universal and the classifying space of $U(k)$. According to Borel [1] the elements a_i are transgressive in $E_{U(k)}$, in fact they are a basis for the subgroup of transgressive elements of $H^*(U(k))$, and $H^*(B_{U(k)})$ is a polynomial ring over Z in variables $y_1, \ldots, y_k$, with $\dim y_i = 2i$, and where y_i is obtained from a_i by transgression. According to Theorem I, $U(k)$ and the space $\Lambda = \Lambda(B_{U(k)})$ have isomorphic cohomology and Pontryagin rings; we denote by a_i' and z_i' the elements corresponding to a_i and z_i; the a_i' are generators of the group of primitive elements of $H^*(\Lambda)$, and also of the group of transgressive elements as one sees from proposition I, § 3.

13. We use the cocycle y_k to construct a mapping F of $B_{U(k)}$ into $K(Z, 2k)$, such that the basic class u of $H^{2k}(K(Z, 2k))$ maps into y_k under F^*; it is one of the basic properties of the $K(Z, n)$ that such a map exists and is unique up to homotopy. F induces a map of the space of paths in $B_{U(k)}$ into the space of paths in $K(Z, 2k)$. This map is a fiber map; it induces a map f of the loop space Λ into the loop space $\Lambda(K(Z, 2k)) = K(Z, 2k-1)$, and induces a map of the spectral sequences. Let v be the generator of $H^{2k-1}(K(Z, 2k-1))$; it is primitive and transgressive, and its transgression element is u. Under f^* it maps into a primitive and transgressive element, which therefore is a multiple of a_k'. But since u maps into y_k under F^*, it is clear that $f^*(v)$ must be a_k' itself. It follows from the invariance of Kronecker index that $f_*(z_k') = z$ (= the generator of $H_{2k-1}(K(Z, 2k-1))$. The map f, being induced by F, is multiplicative, and f_* is a homomorphism with respect to Pontryagin multiplication. We have therefore

$$z * z = f_*(z_k' * z_k') = f_*(0) = 0 \ ,$$

q. e. d.

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