On Isothermic Coordinates

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Dedicated to Professor H. Hopf on his 60th birthday

It has recently been shown ([7], pp. 686—687) that if $S: X = X(u, v)$, where $X = (x, y, z)$, is a (small piece of a) surface of class C^n and if $n \geqq 3$, then a local parametrization $X = X(U, V)$ of S can be chosen with the properties that $X(U, V)$ is of class C^n and that the element of arc-length $ds^2 = |dX|^2$ has the conformal normal form

$$ds^2 = \gamma(dU^2 + dV^2) \qquad (\gamma = \gamma(U, V) > 0) \ . \tag{1}$$

In other words, in dealing with surfaces of class C^n, where $n \geqq 3$, there is no loss of differentiability when (1) is assumed. It remained undecided whether or not the same is true if $n = 2$, an important case which, because of a low degree of differentiability, leads to peculiar difficulties.

It will be shown in this paper that these difficulties can be overcome and that, just as *loc. cit.*, the theorem holds also if the given

$$ds^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 \tag{2}$$

is not embedded as $|dX|^2$ on a surface S of class C^2, but has coefficients of class C^1 and possesses a continuous curvature.

The theorem to be proved is as follows:

(*) *On the domain* $u^2 + v^2 < r^2$, *let the coefficients* g_{ik} *of the positive definite metric* (2) *be functions* $g_{ik}(u, v)$ *of class* C^1 *and such that* (2) *possesses a continuous curvature* $K = K(u, v)$. *Then there exist mappings*

$$u = u(U, V), \ v = v(U, V) \ , \tag{3}$$

of class C^1 *and of non-vanishing Jacobian, which transform* (2) *into the normal form* (1), *and every mapping* (3) *with these properties is of class* C^2, *so that* $\gamma(U, V)$ *is of class* C^1.

It is understood that curvature for a metric (2) with coefficients of class C^1 is meant in the sense of Weyl ([6], pp. 42—44). This can be explained in the notation of H. Cartan ([1], p. 60) as follows:

A Pfaffian form $\omega = Pdu + Qdv$ with continuous coefficients $P(u, v)$, $Q(u, v)$ on a simply connected domain D is called *regular* if there exists a continuous function $f(u, v)$ on D with the property that

$$\int_J \omega = \iint_B f(u, v) du dv \tag{4}$$

holds for every domain B bounded by a piecewise smooth Jordan curve J in D. (For example, if P and Q are of class C^1, then ω is regular and $f = Q_u - P_v$.) When ω is regular, the identity (4) will be abbreviated as

$$d\omega = f du \wedge dv \ , \tag{5}$$

and f will be called the density of the Pfaffian form ω, relative to $du \wedge dv$. If $\omega_1 = P_1 du + Q_1 dv$ and $\omega_2 = P_2 du + Q_2 dv$ are two Pfaffian forms, the symbol $\omega_1 \wedge \omega_2$ is understood to be $(P_1 Q_2 - P_2 Q_1) du dv$ (so that $du \wedge dv = du dv$).

If the coefficients in (2) are of class C^1, then (2) can be written as the sum of the squares of two Pfaffian forms,

$$ds^2 = \omega_1^2 + \omega_2^2 \ , \tag{6}$$

each having coefficients of class C^1. Furthermore, there is a unique Pfaffian form ω_{12} with continuous coefficients satisfying

$$d\omega_1 = \omega_{12} \wedge \omega_2 \ , \ d\omega_2 = \omega_1 \wedge \omega_{12} \ . \tag{7}$$

If the coefficients of (2) are of class C^2, then those of ω_1, ω_2 can be chosen of class C^2 and those of ω_{12} become of class C^1. In this case, ω_{12} is regular and Riemann's definition of the curvature $K = K(u, v)$ of (2) is

$$d\omega_{12} = -K \omega_1 \wedge \omega_2 \ . \tag{8}$$

If the coefficients of (2) are only of class C^1, then ω_{12} need not be regular. Following Weyl, (2) is said to possess a continuous curvature K if ω_{12} is regular, in which case K is defined by (8).

Remark. It follows from (*) that (1) has a continuous curvature and that the relation corresponding to (8) is

$$\int_J \gamma^{-1}(\gamma_V dU - \gamma_U dV) = \iint_B 2K\gamma dU dV \ , \tag{9}$$

where K as a function of (U, V) is $K\big(u(U, V), v(U, V)\big)$. For consequences of the relation (9), see [7].

Even without the assumption that (2) has a continuous curvature, there exist mappings (3) transforming (2) into the conformal form (1) and such that the mapping functions (3) have first order partial derivatives satisfying a Hölder condition of every index less than 1 (Lichtenstein [5]). Since the mappings transforming the conformal form (1) into another conformal form $ds^2 = \gamma_0(dU_0^2 + dV_0^2)$ are of the form $U + iV = F(Z_0)$, where F is an analytic function of $Z_0 = U_0 + iV_0$, it follows that every mapping (3) taking (2) into the form (1) has the property that the first order partial derivatives of the functions (3) satisfy a Hölder condition. This fact will not be used below.

(*) states that the additional assumption of the existence of a continuous curvature $K = K(u, v)$ implies that the assertion of Hölder continuity for the first order partial derivatives can be improved to the assertion of the existence and continuity of second order partial derivatives. It is known ([3], p. 265) that the assertion of (*) becomes false if the assumption concerning the existence of a continuous curvature is omitted.

The truth of (*) was implied by Weyl but was not proved thus far; cf. [6], pp. 49—50 and [7], p. 685, footnote.

Proof of (*). Let (3) be a mapping of class C^1 transforming (2) into (1). It is sufficient to show that γ in (1) is of class C^1. For, according to [2], p. 222, any mapping of class C^1 which transforms a metric (2) with coefficients of class C^1 into another metric with the same property must be of class C^2. (A simplified proof of this general fact, depending on the methods of this paper, will be given elsewhere. For the case when one of the metrics is of the conformal form (2), as in (*), see [7], pp. 681—682).

Introduce the complex-valued Pfaffian form

$$\varphi = \omega_1 + i\omega_2 . \tag{10}$$

Then (6) becomes

$$ds^2 = \varphi\bar{\varphi} . \tag{11}$$

It also follows from (10) that $\varphi \wedge \bar{\varphi} = -2i\omega_1 \wedge \omega_2$, while (7) and (8) become

$$d\varphi = -i\,\omega_{12} \wedge \varphi \tag{12}$$

and

$$d\omega_{12} = -\tfrac{1}{2} iK\varphi \wedge \bar{\varphi} . \tag{13}$$

The form ω_{12}, with the additional condition that it is real-valued, is uniquely determined by (12).

After the change of parameters (3), it follows from (11) that $\varphi\bar{\varphi}$ becomes the right side of (2). Hence it is readily verified that there exists a continuous (complex-valued) function $\alpha = \alpha(U, V)$ satisfying

$$\alpha\bar{\alpha} = \gamma(\neq 0) \tag{14}$$

and

$$\varphi = \alpha dw, \quad \text{where} \quad w = U + iV , \tag{15}$$

and $U = U(u, v)$, $V = V(u, v)$ is the mapping inverse to (3).

In the (u, v)-coordinates, the coefficients of φ are of class C^1, so that φ is regular. Since the definition of a regular Pfaffian form shows that regularity is preserved under transformations of class C^1, it follows that $\varphi = \alpha(U, V)(dU + idV)$ is regular. Let the continuous function $\tau = \tau(U, V)$ be defined by

$$d\varphi = \tau\alpha d\bar{w} \wedge dw \tag{16}$$

or, equivalently, by $d\varphi = 2i\tau\alpha dU \wedge dV$; so that τ is $(2i\alpha)^{-1}$ times the density of φ, relative to $dU \wedge dV$.

Since (16) can be written as

$$d\varphi = \tau d\bar{w} \wedge \varphi = (\tau d\bar{w} - \bar{\tau} dw) \wedge \varphi ,$$

where $\tau d\bar{w} - \bar{\tau} dw$ is a purely imaginary form, the remark following (13) shows that

$$-i\omega_{12} = \tau d\bar{w} - \bar{\tau} dw . \tag{17}$$

Since regularity is preserved under C^1-mappings, the assumption of (*) that (2) has a continuous curvature (that is, that ω_{12} is regular in (u, v)-coordinates) implies that (17) is a regular form (in dU, dV).

Let the density of the form (17), relative to $dU \wedge dV$, be $-2ik(U, V)$. Since the coefficients of (17) are purely imaginary, k is a real-valued continuous function and

$$d(\tau d\bar{w} - \bar{\tau} dw) = -k d\bar{w} \wedge dw . \tag{18}$$

It has been shown by H. Cartan ([1], pp. 62—63) that a Pfaffian form ω is regular on a simply connected domain D if and only if there exists a sequence of Pfaffian forms $\omega^1, \omega^2, \ldots$ which have smooth coefficients on D and which approximate ω in the sense that, as $n \to \infty$, the coefficients and the densities of ω^n tend to those of ω uniformly on every compact subset of D. [The sufficiency of this condition for the regularity of ω is clear. The necessity is proved by defining ω^n to be the form whose coefficients are the convolutions of the corresponding coefficients of ω

with a smooth, non-negative function $f^n(U, V)$ which vanishes outside the circle $U^2 + V^2 = 1/n^2$ and which satisfies $\iint f^n dU dV = 1$. It follows from Fubini's theorem that the density of ω^n is the convolution of $f^n(U, V)$ with the density of w.]

This theorem of H. Cartan implies that if $\varphi = \alpha dw$ is regular and has the density $2i\tau\alpha$, then the form βdw, where

$$\beta = \log \alpha \, , \tag{19}$$

is regular and has the density $2i\tau$. Thus

$$d(\beta dw) = \tau d\bar{w} \wedge dw \, . \tag{20}$$

Since $\beta + \bar{\beta} = \log \alpha\bar{\alpha} = \log \gamma$, by (14) and (19), the proof of (*) will be complete if it is shown that the real part of β is of class C^1. To this end, let

$$\tfrac{1}{2}\beta = a(U, V) + ib(U, V) \, , \quad \tau = g(U, V) - ih(U, V) \, , \tag{21}$$

where a, b, g, h are real-valued (continuous) functions. Then (20) means that

$$d(adU - bdV) = hdU \wedge dV \, , \quad d(bdU + adV) = gdU \wedge dV \tag{22}$$

and (18) means that

$$d(hdU + gdV) = kdU \wedge dV \, . \tag{23}$$

Thus (*) is contained in the following lemma:

Lemma. *Let $a(U, V)$, $b(U, V)$ be real-valued continuous functions on $U^2 + V^2 < R^2$ with the properties that the Pfaffian forms $adU - bdV$, $bdU + adV$ are regular and that, if $h(U, V)$, $g(U, V)$ are the respective densities relative to $dU \wedge dV$, the form $hdU + gdV$ is regular. Then a, b are of class C^1.*

Proof. If n is a sufficiently large integer, let $f^n(U, V)$ be a non-negative, smooth (say, of class C^2) function satisfying $\iint f^n(U, V) dU dV = 1$ and vanishing outside the circle $U^2 + V^2 = 1/n^2$, and let a^n, b^n, g^n, h^n, k^n, respectively, denote the convolutions of a, b, g, h, k with f^n; for example,

$$a^n(U, V) = \iint a(U + x, V + y) f^n(x, y) dx dy \, .$$

Thus a^n, b^n, g^n, h^n, k^n are smooth (say, of class C^2) functions on $U^2 + V^2 < (R - 1/n)^2$, and tend, as $n \to \infty$, to a, b, g, h, k uniformly on every compact subset of $U^2 + V^2 < R^2$.

According to (22), (23) and H. Cartan's proof of his approximation theorem, outlined above,

$$a^n_U - b^n_V = g^n, \quad b^n_U + a^n_V = -h^n, \quad g^n_U - h^n_V = k^n. \tag{24}$$

These relations show that a^n satisfies Poisson's equation

$$a^n_{UU} + a^n_{VV} = k^n. \tag{25}$$

It follows at once that $a(U, V)$ is of class C^1. In fact, if $G(U, V; x, y)$ is the Green function belonging to the Laplace equation on the circle $D(\varepsilon)$: $U^2 + V^2 < (R - \varepsilon)^2$, then, in this circle, $a^n(U, V)$ is the sum of

$$\iint_{D(\varepsilon)} G(U, V;\ x, y) k^n(x, y) dx dy$$

and of the harmonic function $p^n(U, V)$ which assumes the same boundary values as $a^n(U, V)$. By the uniformity of the limit processes $a^n \to a$ and $k^n \to k$ on $U^2 + V^2 \leqq (R - \varepsilon)^2$,

$$a(U, V) = p(U, V) + \iint_{D(\varepsilon)} G(U, V;\ x, y)\, k(x, y) dx dy, \tag{26}$$

where $p(U, V)$ is the harmonic function which assumes the same boundary values as $a(U, V)$. The C^1-character of $a(U, V)$ in $D(\varepsilon)$ is implied by (26) and the continuity of k. In fact, the logarithmic potential of a continuous density is always of class C^1 (but not necessarily of class C^2).

The fact that $b(U, V)$ is of class C^1 (and has the partial derivatives $b_U = -a_V - h, b_V = a_U - g$) follows from (22), and also from (24). This completes the proof of the Lemma and of (*).

Appendix

In view of that particular case of (*) which concerns the first fundamental form of surfaces $S: X = X(u, v)$ of class C^2, it is natural to raise the question whether or not a surface $S: X = X(u, v)$ of class C^1 always has a parametrization $X = X(U, V)$ of class C^1 in which its first fundamental form has the conformal normal form (1). It turns out that the answer is in the negative.

It is known ([3], p. 262) that there exist positive definite metrics (2) with continuous coefficients for which no mapping (3) of class C^1, with a non-vanishing Jacobian, transforms (2) into (1). What is at stake is to show that there exists a metric which has this property and is, at the same time, the fundamental form of a surface $X = X(u, v)$ of class C^1.

This can be concluded by considering (at $r = 0$) suitable C^1-surfaces S of revolution,

$$S: z = z(x, y) = \int_0^r f(r) dr , \qquad r = (x^2 + y^2)^{\frac{1}{2}} \geqq 0 , \tag{27}$$

where, on some interval $0 \leqslant r \leqslant a$, the function $f(r)$ is continuous and such that

$$f(r) \gtreqless 0 \text{ according as } r \gtreqless 0 . \tag{28}$$

A suitable choice of $f(r)$ proves to be $(-\log r)^{-\frac{1}{2}}$ (if $0 < r < 1$), a choice of f made by Lavrentieff ([4], p. 420) for a similar purpose.

Consider first the parametrization of S in terms of polar coordinates (r, θ). This is not an admissible parametrization (at $r = 0$), since the resulting element of arc-length

$$ds^2 = \big(1 + f^2(r)\big) dr^2 + r^2 d\theta^2 = r^2\big(r^{-2}(1 + f^2) dr^2 + d\theta^2\big) \tag{29}$$

is not positive definite (at $r = 0$). Let $\varrho = \varrho(r)$ be the function

$$\varrho = \varrho(r) = \exp\Big(-\int_r^a r^{-1}\big(1 + f^2(r)\big)^{\frac{1}{2}} dr\Big) , \qquad (0 < r \leqq a) , \tag{30}$$

so that $d\varrho/\varrho = -r^{-1}(1 + f^2)^{\frac{1}{2}} dr$ and $ds^2 = r^2\varrho^{-2}(d\varrho^2 + \varrho^2 d\theta^2)$. If new parameters are defined by

$$u = \varrho(r)\cos\theta , \quad v = \varrho(r)\sin\theta , \tag{31}$$

then $du^2 + dv^2 = d\varrho^2 + \varrho^2 d\theta^2$, and so (29) becomes

$$ds^2 = r^2\varrho^{-2}(du^2 + dv^2) , \qquad \varrho = \varrho(r) . \tag{32}$$

This is a "conformal" form, but the factor $r^2\varrho^{-2}$ may not be continuous and positive at $r = 0$.

Since $\varrho(r)$ is an increasing function of r for $0 \leqslant r \leqslant a$, the transformation $(x, y) \equiv (r\cos\theta, r\sin\theta) \rightarrow (u, v)$ is one-to-one. In addition, this transformation is of class C^1 with a non-vanishing Jacobian for $x^2 + y^2 \neq 0$ (and/or $u^2 + v^2 \neq 0$).

Suppose that there exists a mapping

$$U = U(x, y), \quad V = V(x, y) \tag{33}$$

of class C^1 with non-vanishing Jacobian on some circle $x^2 + y^2 < \varepsilon^2$, with the property that the element of arc-length on S has the conformal normal form

$$ds^2 = \gamma(dU^2 + dV^2) \qquad (\gamma = \gamma(U, V)) \tag{34}$$

and that, without loss of generality, $U(0,0) = V(0,0) = 0$. Then there exists on some circle $u^2 + v^2 < \delta^2$ a mapping

$$U = U(u, v), \quad V = V(u, v) \tag{35}$$

with the property that (35) is of class C^1, has non-vanishing Jacobian for $u^2 + v^2 \neq 0$ and the inverse of (35) transforms (32) into (34) for $u^2 + v^2 \neq 0$. Hence $U(u, v) + iV(u, v) = W(w)$ is a continuous (single-valued) function of $w = u + iv$ on the circle $|w| < \delta$ and is regular on the punctured circle $0 < |w| < \delta$. Consequently, $W(w)$ is regular on the circle $|w| < \delta$ and $\gamma = r^2 \varrho^{-2} |dW/dw|^{-2}$ for $|w| \neq 0$, where $\varrho = |w|$ and where $r = r(\varrho)$ is the function inverse to (30). Since $W(w)$ $(\not\equiv 0)$ is regular at $\varrho = 0$, there exist an integer $m \geqq 0$ and a constant $c > 0$ such that $\varrho |dW/dw| \sim c\varrho^{m+1}$ as $\varrho = |w| \to 0$. Hence,

$$r = r(\varrho) \sim \gamma^{\frac{1}{2}}(0, 0) c \varrho^{m+1} \quad \text{as} \quad \varrho \to 0 . \tag{36}$$

It follows that no neighborhood of the point $(x, y) = (0, 0)$ of S has a C^1-parametrization in which the element of arc-length has the form (34) if the inverse function $r = r(\varrho)$ of (30) fails to satisfy (36) for some constant $\gamma^{\frac{1}{2}}(0, 0)c > 0$ and some integer $m \geqq 0$. This is the case, for example, if

$$f(r) = (-\log r)^{-\frac{1}{2}}, \tag{37}$$

since $\log(1/\varrho) = \text{Const.} + \log r^{-1} + \frac{1}{2} \log\log r^{-1} + o(1)$ as $r \to 0$, by (30) and (37). Hence $C\varrho \sim r/(-\log r)^{\frac{1}{2}}$ as $r \to 0$, where C is a positive constant. Consequently, $r \sim C\varrho(-\log \varrho)^{\frac{1}{2}}$ as $\varrho \to 0$, and so (36) cannot hold.

On the C^1-surface S of Lavrentieff (*loc. cit.*), defined by (27) and (37), the element of arc-length in terms of its Cartesian parameters (x, y) is

$$ds^2 = (1 - x^2/R)dx^2 - 2(xy/R)dx\,dy + (1 - y^2/R)dy^2, \text{ where } R = r^2 \log r. \tag{38}$$

It is curious that the example given in [3], pp. 269—279 of a (non-embedded) continuous ds^2 which cannot be conformalized is very close to (38), namely,

$$ds^2 = dx^2 + (1 + x/2R)^2 dy^2 ,$$

where $R = r^2 \log r$, as in (38).

It is worth mentioning that the surface S defined by (27) and (37) is strictly convex, since (37) is an increasing function of r.

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