Complete families of periodic solutions of differential equations.

Autor(en): **Lefschetz, Solomon**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **28 (1954)**

PDF erstellt am: **27.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-22629

Complete families of periodic solutions of differential equations

by Solomon Lefschetz, Princeton (N. J.)

To my friend Heinz Hopf on his sixtieth birthday

1. The following question has been investigated at length by Poincaré especially in connection with his research on the problem of three bodies: — Consider a real differential system

$$\frac{dy}{dt} = Y(y\,;\mu\,;t) \tag{1.1}$$

where y is an n-vector and the components of Y are holomorphic at $y = 0$, $\mu = 0$, and continuous and periodic with period 2π in T. Suppose that for $\mu = 0$ there is known a solution $\xi(t)$ with period 2π in t. Does there exist a solution $\xi(t\,;\mu)$ with period 2π in t, holomorphic in μ about $\mu = 0$ and $\to \xi(t)$ as $\mu \to 0$. Poincaré proceeds in this way: — He considers the solution $\xi(t) + x + z(x\,;t\,;\mu)$ with the initial value $\xi(0) + x$ for $t = 0$. In particular z is holomorphic in x and μ at $x = 0$, $\mu = 0$ and has period 2π in t. Furthermore $z(x\,;0\,;\mu) = 0$. Expressing then the periodicity of the solution there follows a system

$$z(x\,;2\pi\,;\mu) = 0\ . \tag{1.2}$$

If this system has a real solution $x(\mu)$ which $\to 0$ with μ, there is defined a periodic solution $\xi(t\,;\mu)$ of the desired type. Everything comes down to the determination of the real solutions of a real system (1.2) which actually depend on μ. This problem was only solved by Poincaré in specially simple cases. We propose to give a complete solution of the problem. It will rest upon a rather simple application of Kronecker's method of elimination.

2. Before proceeding let us recall a well known terminology. Let $f(u) = f(u_1, \ldots, u_p)$ be a real or complex function of the indicated variables holomorphic at the origin. We call f a *unit* if $f(u) \neq 0$, a *non-unit* otherwise. If

$$f = u_1^q + f_1(u_2, \ldots, u_p)\, u_1^{q-1} + \cdots + f_q(u_2, \ldots, u_p)\ ,$$

where the f_h are non-units, then f is referred to as a *special polynomial* in u_1. Units are written E.

3. Returning to our problem let us write (1.2) in the general and explicit form

$$X_{1h}(x_1, \ldots, x_n, \mu) = 0, \qquad h = 1, 2, \ldots, n_1 \ . \tag{3.1}$$

In point of fact here $n = n_1$. However we shall not need to take advantage of this fact and it will make our argument clearer not to assume it. We suppose that the X_{1h} are real non-units and we shall determine all the suitable families of solutions then select the real families among these.

Since we are only interested in solutions depending on μ, if X_{1h} is divisible say by μ^h we cross this factor out and continue to call the quotient X_{1h}. Thus $X_{1h}(x\,;0) \neq 0$. We may now apply a real linear transformation to the x_j and dispose of the situation so that in X_{1h} the variable x_1 appears to the power $x_1^{m_h}$ where $m_h > 0$ is the lowest degree of any term in the x_j in X_{1h}. The Weierstrass preparation theorem yields then

$$X_{1h} = X'_{1h} E(x\,;\mu)$$

where X'_{1h} is a special polynomial in x_1 (in the system $x_1, x_2, \ldots, \mu$) of degree m_h in that variable. Notice that its coefficients will all be real since their determination never involves any irrationality. To simplify matters we may therefore suppose that in (3.1) the X_{1h} are already special polynomials in x_1.

The X_{1h} may have a common factor $D_1(x_1, \ldots, x_n, \mu)$. It is readily shown to be likewise a special polynomial in x_1 and with quotients $X_{1h}/D_1 = X^*_{1h}$. Take any irreducible factor D^*_1 of D_1. Both D^*_1 and X^*_{1h} are again, up to unit-factors, special polynomials in x_1. Then $D^*_1 = 0$ represents an irreducible $(n-1)$-dimensional family of solutions depending on μ.

Consider now the system

$$X^*_{1h} = 0 \ , \qquad h = 1, 2, \ldots, n_1 \ . \tag{3.2}$$

Following Kronecker introduce two linear combinations with arbitrary parameters

$$U = \Sigma\, u_h X^*_{1h} \ , \qquad V = \Sigma\, v_h X^*_{1h} \ ,$$

and form the resultant $R(U, V)$ as to x_1. We will have

$$R = \Sigma\, W_k(u, v) X_{2k}(x_2, \ldots, x_n, \mu) \ ,$$

where the W_k are monomials in the u_h and v_k. A n. a. s. c. in order that the system (3.2) possess a solution in x_1 is that

$$X_{2k}(x_2, \ldots, x_n, \mu) = 0 \ , \qquad k = 1, 2, \ldots, n_2 \ . \tag{3.3}$$

This system is wholly analogous to (3.1) but with one variable less. We reason then with (3.2) as with (3.1), and so on, and the argument manifestly terminates.

The ultimate result may be described as follows: — There may exist for each $k<n$ and each μ sufficiently small a certain number of $n-k$ dimensional families of solutions, each represented in suitable coordinates by a system

$$X_h(x_h, \ldots, x_n, \mu) = 0 \ , \qquad h = 1, 2, \ldots, k \ , \tag{3.4}$$

where X_h is a polynomial in x_k with non-unit non-leading coefficients, and X_k is special in x_k. Furthermore X_k is irreducible as a polynomial in x_k, and the X_h, $h>k$, are irreducible in a similar sense.

One may even proceed further. Let d_h be the degree of X_h in x_h and let $d = d_1 d_2 \ldots d_k$. Choose k real constants $c_1, \ldots, c_k$ such that the d values of $c_1x_1 + \cdots + c_kx_k$ are distinct. Upon making the change of variables

$$x_h \to x_h, h<k \ ; \qquad c_1x_1 + \cdots + c_kx_k \to x_k \ ,$$

we will have in place of (3.4) a system in which the equation h $(h<k)$ will be of degree one in x_h. Hence (3.4) assumes the form

$$\left.\begin{aligned} &X_k(x_k, \ldots, x_n, \mu) = 0 \ , \\ &A_0(x_{k+1}, \ldots, x_n, \mu)x_{k-h} - A_h(x_k, \ldots, x_n, \mu) = 0 \end{aligned}\right\} \tag{3.5}$$

where X_k may in fact be taken to be a real special polynomial in x_k and the A_j are non-units and real.

4. Passing now to the problem of the reality of the solutions we must distinguish three types of solutions or points.

I. The points where the Jacobian matrix J of the left-hand sides in (3.5) is of maximum rank k at the same time as $A_0 \neq 0$. These are the *ordinary points*. If M is their set and $P \in M$ then there is a neighborhood U of P in M which is a complex analytical cell of real dimension $2(n-k+1)$. This follows at once from the fact that about P one may express the coordinates and μ as power series in $n-k+1$ of them.

II. The points where J is of rank $<k$. These are the *singular points* of M and we denote their set by S.

III. The points where $A_0 = 0$ are *exceptional points* and we denote their set by E.

Each of the three types may yield real points. If $X_k = 0$ has a real solution in x_k for $x_{k+1}, \ldots, x_n, \mu$ arbitrary real and small then the system (3.5) represents a continuous family of real periodic solutions, of dimension $n - k$ for each small real μ. This will certainly occur if X_k is of odd degree in x_k.

Regarding the singular points let $Z_j(x; \mu)$, $j = 1, 2, \ldots, \nu$ be the minors of order $n - k + 1$ of J. The set S is then defined by the system

$$X_k = 0 \ , \qquad A_0 x_{k-h} - A_h = 0 \ , \qquad Z_j = 0 \ . \tag{4.1}$$

This may be subjected to the same treatment as (3.4). It will yield a finite number of families of complex dimension $< n - k + 1$ whose real points are to be found.

For the exceptional points the argument is the same save that (4.1) is replaced by

$$X_k = 0 \ , \qquad A_0 = A_1 = \cdots = A_{k-1} = 0 \ . \tag{4.2}$$

It is clear from the preceding argument that the complete determination of all real periodic solutions may be accomplished in a finite number of steps.

5. As a mild application let us determine the families of periodic solution of period 2π of the system

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2 + \mu g_1(x_1, x_2, \sin t, \cos t, \mu) \ , \\ \frac{dx_2}{dt} &= x_1 + \mu g_2(x_1, x_2, \sin t, \cos t, \mu) \end{aligned} \tag{5.1}$$

where the g_i are polynomials in the indicated variables. If we set $x = x_1 + ix_2$, $g = g_1 + ig_2$ then (5.1) assumes the form

$$\frac{dx}{dt} - ix = \mu g(x, \bar{x}, e^{it}, e^{-it}, \mu)$$

with g still a polynomial. To simplify matters we shall assume that g does not contain $\bar{x}$ so that the system to be treated is

$$\frac{dx}{dt} - ix = \mu g(x, e^{it}, e^{-it}, \mu) \tag{5.2}$$

with g a polynomial in the indicated variables. This system has recently been discussed by Friedrichs (Symposium on non-linear circuit analysis, Brooklyn Polytechnic Institute).

We are then looking for solutions $x(t, \mu)$ of period 2π of (5.2) which as $\mu \to 0$ tend to a solution of the first approximation

$$\frac{dx}{dt} - ix = 0 \ . \tag{5.3}$$

Let ξ be the initial value of x so that (5.3) has the general solution ξe^{it} and (5.2) a general solution of the form

$$x = \xi e^{it} + \mu A_1(\xi; t) + \cdots \ . \tag{5.4}$$

The substitution of (5.4) in (5.2) yields a simple recurrent system for the A_h together with $A_h(\xi, 0) = 0$. As a consequence the periodicity condition assumes the general form

$$F_0(\xi) + \mu F_1(\xi) + \cdots = 0 \tag{5.5}$$

where the F_j are polynomials and $\neq 0$. If $\xi(\mu)$ is a solution and $\xi(0) = \xi_0$, then ξ_0 must be a root of the equation

$$F_0(\xi) = 0 \ . \tag{5.6}$$

Let ξ_0 be a root of order p. We have then from the preparation theorem

$$\begin{aligned} &F_0(\xi) + \mu F_1(\xi) + \cdots \\ &= \{(\xi - \xi_0)^p + f_1(\mu)(\xi - \xi_0)^{p-1} + \cdots + f_p(\mu)\} \, E(\xi - \xi_0, \mu) \ , \end{aligned}$$

where the $f_h(\mu)$ are non-units. Hence the solution of (5.5) for $\xi(\mu)$ such that $\xi(0) = \xi_0$, reduces to that of

$$(\xi - \xi_0)^p + f_1(\mu)(\xi - \xi_0)^{p-1} + \cdots = 0 \ . \tag{5.7}$$

The required solutions may be obtained in a systematic manner by the Puiseux process. In the present case there will be s so-called circular systems each consisting of q conjugate sets

$$\xi - \xi_0 = \mu^{r/q} E(\mu^{1/q}) \, , \qquad r > 0 \ . \tag{5.8}$$

The values $\xi(\mu)$ defined by (5.8) correspond to a single periodic family $x\big(\xi(\mu), t\big)$ such that $x\big(\xi(\mu), 0\big) = \xi_0$ and $\Sigma q = p$. Thus we have obtained a complete solution of our problem.

Received March 18, 1954