# Motions with Maximal Displacements

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*To Paul Finsler on his sixtieth birthday*

This note deals with the surprisingly strong implications of a nearly trivial remark. We consider an abstract Finsler space, that is, a space in which geodesics with the usual geometric properties, apart from differentiability, exist. The exact requirements are found below. The remark is this: *If for a motion $\Phi$ of a Finsler space a point $z$ exists at which the displacement*, or the distance $xx\Phi$ from a point $x$ to its image $x\Phi$ under $\Phi$, *attains a maximum which is not too large, then the shortest geodesic arc from $z$ to $z\Phi^2$ passes through $z\Phi$.*

Among the facts which we deduce from this observation we mention the following: A closed group of motions of a compact space (without any differentiability conditions) is a Lie group. In a compact space without conjugate points and with an abelian fundamental group no geodesic has multiple points, and the closed geodesics in a given free homotopy class have the same length and cover the space simply.

## 1. The axioms. Proof of the remark.

The space is assumed to be a $G$-space, see [1] or [2]. The axioms for a $G$-space $R$ are:

I *$R$ is metric.* The distance of $x$ and $y$ is denoted by $xy$.

II *$R$ is finitely compact*, i. e. the Theorem of Bolzano-Weierstrass holds.

III *$R$ is convex in Menger's sense*, see [3]. If we introduce the notation $(xyz)$ to indicate that $x$, $y$, $z$ are distinct and $xy + yz = xz$, the last condition means: if $x \neq z$ then $y$ with $(xyz)$ exists. $S(p, \varrho)$ will denote the set of points $x$ satisfying $px < \varrho$.

IV *Prolongation is locally possible:* every point $p$ has a neighborhood $S(p, \varrho_p)$, $\varrho_p > 0$, such that for any two distinct points $x$, $y$ in $S(p, \varrho_p)$ a point $z$ with $(xyz)$ exists.

V *Prolongation is unique:* If $(xyz_1)$, $(xyz_2)$ and $yz_1 = yz_2$ then $z_1 = z_2$.

It follows from I, II, III that any two points $y, z$ can be connected by a *segment* $T(y, z)$, i. e. a curve $x(t)$, $\alpha \leq t \leq \beta = \alpha + yz$ such that $x(\alpha) = y$, $x(\beta) = z$ and $x(t_1)x(t_2) = |t_1 - t_2|$, see [3] or [7, p. 12]. A *geodesic* is a curve $x(t)$, $-\infty < t < \infty$, with the property that for every real $t_0$ a positive $\varepsilon(t_0)$ exists such that $x(t_1)x(t_2) = |t_1 - t_2|$ for $|t_i - t_0| \leq \varepsilon(t_0)$ $i = 1,2$. Axioms I to IV imply the *existence of geodesics:* a representation $x(t)$, $\alpha \leq t \leq \beta$, $\alpha < \beta$, of a segment can be extended to all real $t$ to represent a geodesic. This extension is *unique* if V holds.

The function $\varrho_p$ in IV may be erratic but it can be replaced by a *continuous* function: if $\varrho(p) = \sup \varrho_p$, where $\varrho_p$ satisfies IV at $p$, then $S(p, \varrho(p))$ also satisfies IV. If $\varrho(p) = \infty$, then for any two distinct points $x$, $y$ a point $z$ with $(xyz)$ exists. There fore $\varrho(q) = \infty$ for any other point $q$ and if $x(t)$ represents a geodesic, then

$$x(t_1)x(t_2) = |t_1 - t_2|$$

for any $t_1$, $t_2$. We call a geodesic with this property a *straight line*. Thus for $\varrho(p) = \infty$ all geodesics are straight lines, and the $G$-space is called *straight*. In the terminology of the calculus of variations the straight spaces are the simply connected spaces without conjugate points.

If the space is not straight, then $0 < \varrho(p) < \infty$ and

$$|\varrho(p) - \varrho(q)| \leq pq \, .$$

For if $\varrho(p) > \varrho(q)$ and $\varrho(p) > pq$ then the triangle inequality yields $S(q, \varrho(p) - pq) \subset S(p, \varrho(p))$ hence $\varrho(p) - \varrho(q) \leq pq$. The number $\varepsilon(t_0)$ occurring in the definition of a geodesic $x(t)$ may be chosen as $\varrho(x(t_0))$.

Axiom V implies that the segment $T(x, y)$ is unique if a point $z$ with $(xyz)$ exists, see [1, p. 216]. In particular $T(x, y)$ is always unique for $x, y \in S(p, \varrho(p))$.

The following is now an exact formulation of the remark mentioned in the introduction:

(1) *If $\Phi$ is a motion of the G-space $R$ which is not the identity $E$ and if* $zz\Phi = \sup_{x \in R} xx\Phi < \varrho(z)/2$, *then* $(zz\Phi z\Phi^2)$.[1])

*Proof.* Because $zz\Phi < \varrho(z)/2$ there is a point $u$ such that $(zz\Phi u)$ and $zz\Phi = z\Phi u$, briefly $z\Phi$ is a midpoint of $z$ and $u$. Then $z\Phi^2$ is a midpoint of $z\Phi$ and $u\Phi$ and the only one, because $\varrho(z\Phi) = \varrho(z)$. The relation

$$zz\Phi \geq uu\Phi \geq z\Phi u\Phi - z\Phi u = zu - z\Phi u = zz\Phi$$

[1]) It is also true, and has many applications(see [2]), that $0 < z z\Phi = \inf_{x \in R} x x\Phi < \varrho(z)/2$ implies $(z z\Phi z\Phi^2)$.

shows that $u$ is a midpoint of $z\Phi$ and $u\Phi$, hence $u = z\Phi^2$, which proves the assertion.

## 2. Applications to compact spaces.

In the compact case the following additional statements can be made:

(2) For any motion $\Phi \neq E$ of a compact $G$-space $R$ a point $z$ of maximal displacement $\alpha$ (i. e. $\alpha = zz\Phi = \sup_{x \in R} xx\Phi$) exists. If $k$ is the first integer for which $k\alpha \geq \varrho(z)/2$, then $zz\Phi^k = k\alpha$. If $k > 1$ then a geodesic $x(t)$ exists such that $x(i\alpha) = z\Phi^i$, $i = 0, \pm 1, \pm 2, \ldots$ and $x(t)$ represents a segment for $i\alpha \leq t \leq (i + k)\alpha$.

*Proof*. The existence of $z$ is obvious and for $k = 1$ there is nothing to prove. If $k > 1$ then $k\alpha < \varrho(z)$ and with $z_i = z\Phi^i$ it follows from (1) that $(zz_1z_2)$, hence $(z_{i-1}z_iz_{i+1})$ for all $i$. Since $\varrho(z_i) = \varrho(z)$ the segment $T(z_{i-1}, z_{i+1})$ is unique and passes through $z_i$. The existence of $x(t)$ follows, and $x(t)$ represents a segment for $i\alpha \leq t \leq (i + k)\alpha$ because $k\alpha < \varrho(z_i)$. In particular $x(0)x(k\alpha) = zz\Phi^k = k\alpha$.

We use the standard metric $\delta(\Phi, \Psi) = \sup_{x \in R} x\Phi x\Psi$ for motions $\Phi, \Psi$ of a compact space. Sinde $\varrho(x)$ is continuous and positive it has on a compact space $R$ a positive minimum $\varrho(R)$. An immediate consequence of (2) is

(3) *A non-trivial group of motions of a compact $G$-space $R$ has at least diameter $\varrho(R)/2$.*

„Non-trivial" means that the group contains at least one motion $\Phi \neq E$, and (2) implies that $\delta(E, \Phi^k) \geq \varrho(z)/2 \geq \varrho(R)/2$ for a suitable positive $k$. Well known theorems on topological groups yield the further result:

(4) **Theorem.** *A closed group of motions of a compact $G$-space $R$ is a Lie group. If the group $\Gamma$ of all motions which $R$ possesses is transitive on $R$, then $R$ is a topological manifold and* $\dim \Gamma \leq \dim R\,(\dim R + 1)/2$.

The first statement follows from [4, Theorem 53] and the second from [5, Corollary 3′, Theorem 9 and Theorem 12]. In spite of the recent result of Gleason it is an open question whether (4) extends to non-compact spaces, since no analogue to (3) is known, even when $\inf_{x \in R} \varrho(x) > 0$.

The rotations about the $z$-axis of the surface $z = (x^2 + y^2)^{-1/2}$ in $E^3$, with the length of the shortest connection on the surface as distance, show that a one-parameter group of motions of a non-compact $G$-space may not have any orbits which are geodesics. (1) and (2) imply the existence of such orbits on compact $G$-spaces:

(5) **Theorem.** *A one-parameter group of motions of a compact G-space possesses an orbit which is a geodesic.*

We assume that the one-parameter group is given in the form $\Phi(s)$ with $\Phi(s_1)\Phi(s_2) = \Phi(s_1 + s_2)$, and prove that a geodesic $x(t)$ and a positive $\alpha$ exist such that $x(t) = x(0)\Phi(\alpha t)$.

Choose $\varepsilon > 0$ such that $\delta(E, \Phi(s)) < \varrho(R)/2$ for $|s| < \varepsilon$. Let $0 < u < \varepsilon$. By (2) there are points $z$ and $z'$ of maximal displacement under $\Phi(u)$ and $\Phi(u/2)$ respectively. Then the choice of $\varepsilon$ and (1) imply

$$\begin{aligned} z'z'\Phi(u) &= 2z'z'\Phi(u/2) \geq 2zz\Phi(u/2) \\ &= zz\Phi(u/2) + z\Phi(u/2)z\Phi(u) \geq zz\Phi(u) \geq z'z'\Phi(u). \end{aligned}$$

Hence $z$ is also a point of maximal displacement for $\Phi(u/2)$ and generally for $\Phi(2^{-n}u)$. Moreover $(zz\Phi(u/2)z\Phi(u))$ and generally

$$(zz\Phi(2^{-n-1}u)z\Phi(2^{-n}u)) \; .$$

If $x(t)$ is the geodesic with $x(0) = z$ which represents for

$$0 \leq t \leq zz\Phi(u) = \beta$$

the (unique) segment $T(z, z\Phi(u))$ then (2) yields

$$x(i\,2^{-n}\beta) = z\Phi(i\,2^{-n}u)$$

for all $i$ and non-negative $n$. A trivial continuity argument shows that $x(\beta t) = z\Phi(ut)$ or $x(t) = x(0)\Phi(\alpha t)$ for all $t$, where $\alpha = u/\beta$.

## 3. Compact spaces without conjugate points and abelian fundamental groups.

For a $G$-space $R'$ which satisfies the usual differentiability hypotheses of the calculus of variations the absence of conjugate points means that the universal covering space $R$ of $R'$ is straight.

The relation of the theorem mentioned in the introduction to motions with maximal displacements comes from:

(6) **Theorem.** *If $R$ is straight and $\Phi$ is a motion of $R$ for which a point $z$ with $0 < zz\Phi = \sup_{x \in R} xx\Phi$ exists, then $xx\Phi$ is independent of $x$. The points $x\Phi^i$, $i = 0, \pm 1, \pm 2, \ldots$ lie for each $x$ on a straight line $\mathfrak{g}_x$.*

For it follows from (1) that the points $z_i = z\Phi^i$ satisfy

$$(z_{i-1}z_iz_{i+1})$$

hence lie on a straight line $\mathfrak{g}_z$. If $x$ is any other point of $R$ and $x_i = x\Phi^i$ then

$$n \cdot zz\Phi = zz_n \leq zx + \sum_{i=1}^{n} x_{i-1}x_i + x_n z_n = 2zx + n \cdot xx\Phi$$

or $xx\Phi \geq zz\Phi - 2zx/n$. Since $n$ is arbitrary $xx\Phi \geq zz\Phi$, hence $xx\Phi = zz\Phi$.

Thus every point $x$ of $R$ is a point of „maximal" displacement for $\Phi$, therefore (1) shows that the points $x_i$ lie on a line $\mathfrak{g}_x$.

Clearly for any two points $x$, $y$ either $\mathfrak{g}_x = \mathfrak{g}_y$ or $\mathfrak{g}_x \cap \mathfrak{g}_y = 0$, since $u \in \mathfrak{g}_x \cap \mathfrak{g}_y$ implies $u\Phi^i \in \mathfrak{g}_x \cap \mathfrak{g}_y$ hence $\mathfrak{g}_x = \mathfrak{g}_y$.

Let the universal covering space $R$ of the $G$-space $R'$ be straight. There is a wellknown correspondence between the classes of conjugate elements in the fundamental group $\mathfrak{F}$ of $R'$ and the classes of freely homotopic curves in $R'$, see for instance [8, § 49]. If, as in [1], $\mathfrak{F}$ is realized as the group of motions of $R$ which lie over the identity of $R'$ then the closed geodesics in a free homotopy class $K_\Phi$ determined by a motion $\Phi \neq E$ in $\mathfrak{F}$ correspond to the straight lines in $R$ which are taken into themselves by $\Phi$, the so-called axes of $\Phi$, see [2]. If $x$ lies on an axis of $\Phi$ then $xx\Phi$ is the length of the corresponding geodesic.

If $\Phi \neq E$ possesses a point of maximal displacement then we conclude from (6) that every point $x'$ of $R'$ lies on a closed geodesic of length $xx\Phi$ in $K_\Phi$ and that two such geodesics do not intersect. It is now easy to prove:

(7) **Theorem.** *Let $R'$ be a compact $G$-space with an abelian fundamental group and a straight universal covering space $R$. Then the closed geodesics in any (non-trivial) free homotopy class of $R'$ have the same length and cover $R'$ simply. No geodesic in $R'$ has multiple points.*

For let $\Phi$ be any motion in the fundamental group $\mathfrak{F}$ of $R'$ different from the identity (such motions exist because $R$ is non-compact, hence different from $R'$). There is a compact subset $C$ of $R$ such that

$$\bigcup C\Phi_\nu = R \;,$$

where $\Phi_\nu$ traverses $\mathfrak{F}$, see [2, p. 267]. The Function $yy\Phi$ attains on $C$ a maximum at some point $z \in C$. If $x$ is an arbitrary point of $R$ then a $\Phi_\nu \in \mathfrak{F}$ exists such that $y = x\Phi_\nu \in C$. Because $\mathfrak{F}$ is abelian

$$xx\Phi = x\Phi_\nu\, x\Phi\Phi_\nu = x\Phi_\nu\, x\Phi_\nu\Phi = yy\Phi \leq zz\Phi \;,$$

so that $zz\Phi = \sup\limits_{x \in R} xx\Phi$.

The preceding discussion shows that the closed geodesics in $K_\Phi$ all have length $xx\Phi$ and cover $R'$ simply. $K_\Phi$ is, owing to the arbitrariness of $\Phi$, an arbitrary non-trivial free homotopy class in $R'$.

There can be no geodesic monogon with a proper vertex $x'$. For such a monogon would lie in some free homotopy class $K_\Phi$, not trivial because $R$ is straight. If $x$ lies over $x'$ then the points $x\Phi^i$ would not lie on a straight line. The absence of proper monogons means that the geodesics in $R'$ have no multiple points.

Theorem (7) brings a result of E. Hopf [6] to mind, namely that a two-dimensional torus $T'$ with a Riemannian metric is euclidean, if its universal covering plane $T$ is straight. In that case (7) is therefore trivial. However, when the condition that the metric be Riemannian is omitted, then $T'$ possesses a great number of essentially different metrizations for which $T$ is straight. The geodesics in $T$ need not satisfy Desargues' Theorem, but they always satisfy the parallel axiom.

## 4. A characterization of Minkowskian geometry.

The translations of the euclidean space are obvious exemples für (6). When there are enough motions satisfying (6) these motions are necessarily ordinary translations:

(8) **Theorem.** *If a straight space possesses a transitive group of motions such that for each motion $\Phi$ in $\Gamma$ a point exists whose displacement unter $\Phi$ is maximal, then $R$ is Minkowskian and $\Gamma$ the group of translations of $R$.*

We deduce from (6) that $xx\Phi$ is constant for each $\Phi$ in $\Gamma$. Hence no motion $\Phi \neq E$ in $\Gamma$ has fixed points and $\Gamma$ is simply transitive on $R$, see [7, p. 220]. The motion in $\Gamma$ that takes $a$ into $b$ may therefore be denoted by $(a \to b)$. Because of (6) the line $\mathfrak{g}(a, b)$ through $a$ and $b$, $a \neq b$, is an axis of $(a \to b)$. The proof of (8) consists of several steps the first of which is:

(a) $R$ satisfies the parallel axiom, for the terminology see [7].

To see this let $x(t)$ be any geodesic and $y$ a point not on $x(t)$. Since $x(t)$ is an axis of $\Phi = (x(0) \to x(1))$ it suffices to show that $\mathfrak{g}(y, x(t))$ tends for $t \to \infty$ or $t \to -\infty$ to the axis $\mathfrak{g}_y$ of $\Phi$ through $y$. For the statement that the line $\mathfrak{g}_y$ is an axis of the same motion $\Phi$ as $\mathfrak{g}_x$, is symmetric and transitive, hence the statement that $\mathfrak{g}_y$ is parallel to $\mathfrak{g}_x$ also has these properties.

Let $y(t)$ represent the axis of $\Phi$ through $y$ with $y(0) = y$ and $y\Phi = y(1)$. The limit sphere $\Lambda$ $(y, \mathfrak{r})$ through $y$ to $\mathfrak{r}$ (see [1, p. 240] or

[7, p. 98]), where $\mathfrak{r}$ is the ray $t \geq 0$ of $x(t)$, intersects $x(t)$ in a point $x(t_0)$ and $x(t_0)\Phi = x(t_0 + 1)$. Moreover $\Lambda(y\Phi, \mathfrak{r}) = \Lambda(y, \mathfrak{r})\Phi = \Lambda(x(t_0 + 1), \mathfrak{r})$. The asymptote $\mathfrak{a}$ to $\mathfrak{r}$ through $y$ intersects $\Lambda(y\Phi, \mathfrak{r})$ in the unique foot $f$ of $y$ on $\Lambda(y\Phi, \mathfrak{r})$. But, see [1, p. 242],

$$1 = x(t_0)x(t_0 + 1) = yf \leq yy\Phi = 1 ,$$

hence $y\Phi$ is also a foot of $y$ on $\Lambda(y\Phi, \mathfrak{r})$, so that $y\Phi = f$ and $\mathfrak{a} = \mathfrak{g}_y$, which proves (a).

We show next

(b) If $y(t)$, $t \geq 0$ represents a ray $\mathfrak{s}$ and $\mathfrak{g}$ is a straight line through $y = y(0)$ not containing $\mathfrak{s}$ then $y(t)\mathfrak{g} \to \infty$ for $t \to \infty$.

For an indirect proof assume the existence of a sequence $t_n$ with $x(t_n)\mathfrak{g} < M$. If $f_n$ is a foot of $x(t_n)$ on $\mathfrak{g}$, then $f_n \neq y$ for large $n$, and $q_n = x(t_n)(f_n \to y)$ has $y$ as foot on $\mathfrak{g}$. Because $q_n y = x(t_n)f_n < M$ there is a subsequence $\{\nu\}$ of $\{n\}$ for which $q_\nu$ tends to a point $q$. ($qy > 0$ because of [1, Theorem (11.14)]).

The line $\mathfrak{g}(x(t_\nu), q_\nu)$ is an axis of $(f_\nu \to y)$, hence parallel to $\mathfrak{g}$. It tends therefore to the parallel $\mathfrak{g}'$ to $\mathfrak{g}$ through $q$. On the other hand, the line $\mathfrak{g}(q_\nu, x(t_\nu))$ tends also to the parallel through $q$ to the line $\mathfrak{h}$ carrying $\mathfrak{s}$ (see the definition of co-ray in [1]). Since parallelism is symmetric, it would follow that $\mathfrak{g}$ and $\mathfrak{h}$ are parallel to $\mathfrak{g}'$, which is impossible because $\mathfrak{g}$ and $\mathfrak{h}$ intersect.

(c) $x_1\mathfrak{g}_2$ and $x_2\mathfrak{g}_1$ are bounded for $x_i \in \mathfrak{g}_i$ if and only if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are parallel.

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are parallel then the fact that they are axes of the same motion in $\Phi$ shows that $x_1\mathfrak{g}_2$ and $x_2\mathfrak{g}_1$ are bounded. The converse follows from (b), for a proof see [2, p. 278].

(d) $\Gamma$ is abelian.

If $\Phi$ and $\Psi$ are two non-trivial motions in $\Gamma$, select an arbitrary point $z$. If the axes of $\Phi$ and $\Psi$ through $z$ coincide, it is easily seen that $\Phi$ and $\Psi$ commute (this case can also be deduced by a limit process from the general case). We assume therefore that $z$, $p = z\Phi$ and $q = z\Psi$ are not collinear. Put $\mathfrak{g}(z, p) = \mathfrak{g}$, $\mathfrak{g}(z, q) = \mathfrak{h}$ and $\mathfrak{h}' = \mathfrak{h}\Phi$. Then $y' = y\Phi \in \mathfrak{h}'$ for $y \in \mathfrak{h}$. The relation $yy' = zp$ shows that $y'\mathfrak{h}$ and $y\mathfrak{h}'$ are bounded, by (c) the lines $\mathfrak{h}$ and $\mathfrak{h}'$ are parallel. Therefore $\mathfrak{h}'$ is an axis of $\Psi$, so that $p\Psi$ is a point $u$ of $\mathfrak{h}'$ with $zq = pu$. On the other hand $zq = z\Phi q\Phi = pq\Phi$, hence $q\Phi = u$. Therefore $\Phi = (q \to u)$, $\Psi = (p \to u)$ and

$$\Phi\Psi = (z \to p)(p \to u) = (z \to u) = (z \to q)(q \to u) = \Psi\Phi .$$

It now follows readily from a wellknown result of Pontrjagin, see [4, p. 170], that the space is a finite dimensional Minkowski space. A simple proof which does not use the theory of topological groups is found n [7, pp. 229—231].

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