HOELDER continuity and initial value problems of mixed type differential equations.

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§ 1. Introduction

For the partial differential equations of elliptic type with analytic coefficients the functional behavior and in particular, the question of continuation of solutions on and near the boundary are of great interest[2]). A very important method for such study is the method of H. LEWY [12a]. It has been used for successful treatment of a variety of linear and non-linear problems, as for instances, the problems on cavitational flow [6] and of the boundary behavior of minimal surfaces [12b]. This method consists in the construction of the analytic extension of the solution into the complex domain in order to obtain information on the behavior of the solution in the real. The present work originated from the idea to extend the method to mixed type equations and to treat a typical problem such as the behavior of the solutions of the Tricomi equation when the singular line is approached from the elliptic part of the domain. Because of the analytic nature of the question it seems natural to consider for $x > 0$, the typical equation

$$x^m U_{yy} + U_{xx} = 0 \qquad \text{with } m > -1 \tag{1.1}$$

where the exponent m is a real number and not necessary an odd integer. The restriction $m > -1$ is justified by various reasons[3]), one of which is that the solution of the CAUCHY-KOWALEWSKI problem of (1.1) with analytic data requires such a restriction, see BERGMAN [1].

For the use of LEWY's method it is convenient to put (1.1) in the normalized form:

$$u_{XX} + u_{yy} + \frac{2\alpha}{X} u_X = 0\,, \tag{1.2}$$

where $2\alpha = m/(m+2)$ and $(1-2\alpha)x = X^{(1-2\alpha)}$. We may restrict ourselves to positive α with $1 - 2\alpha > 0$ by considering the BELTRAMI-STOKES equation

$$X^{2\alpha} u_X = v_y^*, \qquad X^{2\alpha} u_y = -v_X^*\,. \tag{1.3}$$

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[1]) This research was supported by the United States Air Force through the Air Force Office of Scientific Research and Development Command under Contract No. AF 49(638)-107.

[2]) For results on equations of higher order with more than 2 independent variables cf. F. JOHN [10], C. B. MORREY and L. NIRENBERG [14].

[3]) For equation (1.2) with $2\alpha \geq 1$ cf. HUBER [9a]

Henceforth we shall use (1.2) for $\alpha > 0$, $1-2\alpha > 0$, and for the same α the following equation of v*:

$$v^*_{XX} + v^*_{yy} - \frac{2\alpha}{X} v^*_X = 0 . \tag{1.2$'$}$$

Another set of equations which is equivalent to (1.3) and is expressed in terms of (u, v) instead of (u, v^*), is introduced in § 2. We shall be dealing with these equations in the main part of the paper; only toward the end, § 9, shall we return to equation (1.1). There, in § 9, all the results obtained in the preceeding sections are reformulated and summarized for the solutions $U(x, y)$.

For (u, v^*) of the BELTRAMI-STOKES equation our results are briefly as follows:

1. Characterization of the initial data of (u, v^*) on $X = 0$, theorems 2 and 2$'$, § 7, (the main theorem). We consider the solutions (u, v^*) which are twice continuously differentiable in a domain D with a boundary segment l on $X = 0$. They are grouped into classes Z'_k according to their "radial" HOELDER continuity properties[4]) in $D + l$. Similarly the analytic functions $g(W)$ of a single complex variable W in D, are also grouped into classes $\mathfrak{G}'_k$ according to their HOELDER continuity properties in $D + l$. Then there is a one to one correspondence between the elements of Z'_k and the elements of $\mathfrak{G}'_k$, such that the initial data $(u(y), v^*(y))$ and $g(y)$ of the corresponding elements are related to each other by the simple transformations of the type $I_\beta^{\pm\delta}$ (see § 6.4 and § 7.4).

2. A generalization of PRIVALOFF's theorem [16] for conjugate harmonic functions, theorems 3, 3$'$, § 8. The generalization is to the effect that if u is continuous up to l and if on l, $u(y)$ is HOELDER continuous with exponent μ, $0 < \mu < 1 - 2\alpha$, then $v^*(y)$ has the exponent $\mu + 2\alpha$, and conversely.

3. By means of the induction which is based on the formulas of § 5, the above results can be extended to other values of μ and to cover the cases of differentiable initial data and data which become HOELDER continuous after integration, theorems 5, 5$'$, § 8. Theorem 6, § 8 generalizes a theorem of KELLOGG [11] for harmonic functions.

Both the equations (1.2) and (1.3) and the corresponding equations with more than two independent variables have been extensively studied by WEINSTEIN [17], HUBER and others[5]). Of particular interest for us here is a theorem by A. HUBER [9a], on the uniqueness of solution of (1.2). This theorem is stated in § 8 as theorems 4, 4$'$, and is shown on the basis of our main

[4]) For the definition, see § 6, (6D).

[5]) Cf. HUBER [9b], [9c], also BROUSSE and PONCIN [3], OLEVSKII [15].

theorem to be equivalent to the theorems 3, 3′. The uniqueness theorem can also be formulated as a reflection law for equation (1.1), § 9.1.

The proof of the main theorem in § 7 requires a generalization (theorems 1, 1′, § 6) of a theorem by HARDY and LITTLEWOOD [7], which is concerned with HOELDER continuity properties of functions represented by integrals $I^{\pm\delta}$ of fractional order. A considerable amount of details is needed in § 3 and § 4 to set up the correspondence between (u, v) and $g(W)$ in D and to insure the convergence to the limit relations $I_{\beta}^{\pm\delta}$ of the initial data.

The behavior of the solution at the endpoints of l as well as along other parts of the boundary of D, is not discussed in this paper[6]).

By the same method one could study solutions of (1.1) and (1.2) which are entire[5]) in the half-plane with initial data integrable along the whole y-axis. However we shall not go into this question here.

§ 2. The differential equations

If (u, v^*) satisfies (1.3) and each function is of class C_2, then u and v^* satisfy (1.2) and (1.2)′ respectively. Thus the functions (U, V) given respectively by

$$U(x, y) = u\left((1-2\alpha)x^{\frac{1}{1-2\alpha}}, y\right), \; V(x, y) = v^*\left((1+2\alpha)x^{\frac{1}{1+2\alpha}}, y\right) \qquad (2.1)$$

are solutions of equations of the form (1.1) respectively with $m = p = \dfrac{4\alpha}{1-2\alpha}$, $p > 0$ and $m = q = \dfrac{-4\alpha}{1+2\alpha}$, $0 > q > -1$. In this manner solutions of (1.1) with $m > -1$ can be obtained from those of (1.3). Note also the equivalence of the following initial conditions:

$$\begin{aligned} U(0, y) &= u(0, y), \quad & U_x(0, y) &= \lim_{X\to 0}\left(\frac{X}{1-2\alpha}\right)^{2\alpha} u_X(X, y), \\ V(0, y) &= v^*(0, y), \quad & V_x(0, y) &= \lim_{X\to 0}\left(\frac{X}{1+2\alpha}\right)^{-2\alpha} v^*_X(X, y). \end{aligned} \qquad (2.2)$$

By using the polar coordinate (s, θ), it is seen that the left hand side of (1.2) can be written in two different ways as divergent expressions. One expression leads to the polar coordinate form of (1.3),

$$s^{2\alpha-1}(\sin\theta)^{2\alpha}u_\theta = -v^*_s, \quad s^{2\alpha}(\sin\theta)^{2\alpha}u_s = s^{-1}v^*_\theta, \qquad (2.3)$$

[5]) See note 5 on page 2.

[6]) For the theory of boundary value problems of mixed type equations cf. BERS [2], and FRIEDRICHS [4].

while the other leads to a system of equations for (u, v) as follows:

$$(\sin\theta)^{2\alpha} s^{-1} u_\theta = -v_s, \quad (\sin\theta)^{2\alpha}[u_s + 2\alpha s^{-1} u] = s^{-1} v_\theta . \tag{2.4}$$

It is with this system for (u, v) that we shall be mainly concerned.

A system (u, v) possesses the property P_μ $0 < \mu < 1$, if it is of the class C_2 in D and in the neighborhood of $W = 0$ in every sector $\varepsilon < \theta < \pi - \varepsilon$, $u = O(s^\mu)$, $v = O(s^\mu)$.

Lemma 2.1. If a solution (u, v) of (2.4) has the property P_μ, then (su_s, sv_s) and $\left(\int_0^s \frac{u(te^{i\theta})}{t}\,dt, \int_0^s \frac{v(te^{i\theta})}{t}\,dt\right)$ are also solutions of (2.4).

§ 3. The operator L

Consider a solution $u(X, y)$ of class C_2 in D. By LEWY's method, the analytic extension of u into a function $\Omega = u(\tilde{X}, \tilde{y})$ of two complex variables $\tilde{X} = X + iX'$ and $\tilde{y} = y + iy'$ is obtained in terms of the variables Z and W: $\Omega(Z, W) \equiv u(\tilde{X}, \tilde{y})$ with $Z = -\tilde{y} + i\tilde{X}$ and $W = \tilde{y} + i\tilde{X}$.

Ω has the following properties: a) it is defined for $W \in D$ and $-\bar{Z} \in D$, b) $\Omega(Z, W) = u(X, y)$ when $W = -\bar{Z}$, and c) $\partial^2\Omega/\partial Z\,\partial W + \alpha(Z + W)^{-1} \cdot (\partial\Omega/\partial Z + \partial\Omega/\partial W) = 0$. c) is derived formally from (1.2) by replacing in it (X, y) by $(\tilde{X}, \tilde{y})$.

Instead of determining Ω for all values of Z and W, it suffices here to find the function $G(r, s, \theta) = \Omega(-re^{-i\theta}, se^{i\theta})$ of the real positive variables (r, s) for each fixed value of θ, $0 < \theta < \pi$. Then we have

$$\begin{gathered}\frac{\partial^2 G}{\partial r\,\partial s} + \alpha[se^{i\theta} - re^{-i\theta}]^{-1}\left(e^{i\theta}\frac{\partial G}{\partial r} - e^{-i\theta}\frac{dG}{\partial s}\right) = 0 , \\ G(s, s, \theta) = u(W) , \quad \left(\frac{\partial G}{\partial r} - \frac{\partial G}{\partial s}\right)_{r=s} = is^{-1}u_\theta(W) .\end{gathered} \tag{3.1}$$

The last condition expresses the analyticity of $\Omega(Z, W)$ at $W = -\bar{Z}$. (3.1) is an initial value problem. Its solution is represented by RIEMANN's formula, and the RIEMANN function in this case is known explicitly. It is given by

$$\begin{gathered}R(\sigma, \tau; r, s) = \omega^\alpha F(1 - \alpha, \alpha, 1, z) \\ \omega = (s - \eta r)(\tau - \eta\sigma)^{-1}, \quad z = \eta(\sigma - r)(\tau - s)(s - \eta r)^{-1}(\tau - \eta\sigma)^{-1},\end{gathered} \tag{3.2}$$

in which the parameter θ occurs in the form of $\eta = e^{-2i\theta}$.

Our main interest in the function $G(r, s, \theta)$ is to obtain the limit

$$g(W) = \lim_{r \to 0} G(r, s, \theta) \quad \text{for } W \neq 0,\ W \in D . \tag{3.3}$$

For, according to LEWY's theory, $g(W)$ should be an analytic function of W. To carry out the limit we need the assumption that the solution (u, v) is continuous at the origin, and without loss of generality, we may set $v(0) = 0$.

The RIEMANN formula consists of three terms. One term involves the values of u at the endpoints of the initial segment, with the origin as one of the endpoints. The other two terms are integrals over the initial segment, one integrand contains u as a linear factor, and the other, u_θ. We replace u_θ in the last integral by an expression involving v_s, using the first equation of (2.4), and then carry out the integration by parts. For abbreviation we introduce the complex valued function

$$z(W) = u(X, y) + i v(X, y) , \tag{3.4}$$

and write (3.3) in the following form

$$g(W) = L z(W) . \tag{3.5}$$

Lemma 3.1. When $u = 1$, $v = 0$, then $L(1) = 1$.

Lemma 3.2. When $g_1 = L z_1$ and $g_2 = L z_2$, then for real constants c_1 and c_2, we have $c_1 g_1 + c_2 g_2 = L(c_1 z_1 + c_2 z_2)$.

Lemma 3.3. Let $\chi = \eta(1-\lambda)/(1-\lambda\eta)$ where $\eta = e^{-2i\theta}$, $0 < \lambda < 1$. The operator L has the following representation:

$$2g(W) = P(W, \eta) + iQ(W, \eta) \tag{L}$$

with

$$\begin{aligned} P(W, \eta) &= (1-\eta)^\alpha u(W) + \int_0^1 (1-\lambda\eta)^{\alpha-1} \lambda^{\alpha-1} \varphi_0(\chi, \eta) u(\lambda W) d\lambda , \\ Q(W, \eta) &= \int_0^1 (1-\lambda\eta)^{-\alpha} \lambda^\alpha \varphi_1(\chi, \eta) \frac{dv(\lambda W)}{d\lambda} d\lambda . \end{aligned} \tag{3.6}$$

Another representation of L is

$$2g(W) = P(W, \eta) + iQ'(W, \eta) \tag{L'}$$

with Q' derived from Q, as follows,

$$Q'(W, \eta) = \varphi_2(\eta) v(W) + \int_0^1 (1-\lambda\eta)^{-\alpha-1} (\varphi_3 + \lambda\varphi_4) [v(W) - \lambda^{\alpha-1} v(\lambda W)] d\lambda . \tag{3.7}$$

The functions φ_k $k = 0, 1, 2, 3, 4$ are given explicitly in terms of the hypergeometric functions:

$$F_0(\chi) = F(\alpha, \alpha, 1, \chi),\ F_1(\chi) = F(1-\alpha, 1-\alpha, 2, \chi),\ F_2(\chi) = F(1+\alpha, \alpha, 2, \chi) .$$

We have:

$$\begin{aligned}
\varphi_0 &= \alpha^2(1+\eta)F_1(\chi) + \alpha(1-\alpha)(1-\chi)^{2\alpha}F_2(\chi) \\
\varphi_1 &= c\eta^\alpha F_0(\chi), \quad c = 4^\alpha e^{i\alpha\pi}, \\
\varphi_2 &= (1-\alpha)c\eta^\alpha \int_0^1 (1-\lambda\eta)^{-\alpha}F_0(\chi)d\lambda, \\
\varphi_3 &= c\alpha\eta^\alpha F_0(\chi), \\
\varphi_4 &= -c\eta^{1+\alpha}(1-\chi)\frac{dF_0(\chi)}{d\chi}.
\end{aligned} \tag{3.8}$$

§ 4. The operator B

4.1. The function $\Omega(Z, W)$ has the property (see LEWY [12a]),

$$\lim_{Z\to 0}\Omega(Z, W) = g(W) \text{ for } W \neq 0, \quad \lim_{W\to 0}\Omega(Z, W) = \overline{g(-\overline{Z})} \text{ for } Z \neq 0,$$

and $\Omega(-\overline{W}, W) = u(W)$ is real. When $g(W)$ is given in D, to find Ω one may proceed as follows: First, find a function $\Omega'(Z, W)$ such that Ω' has the same properties a) and c) as described in § 3 for Ω, and that instead of b), Ω' satisfies the condition b'):

$$\lim_{Z\to 0}\Omega'(Z, W) = g(W) \text{ for } W \neq 0, \lim_{W\to 0}\Omega'(Z, W) = 0 \text{ for } Z \neq 0 .$$

Secondly Ω is obtained from Ω' by $\Omega(Z, W) = \Omega'(Z, W) + \overline{\Omega'(-\overline{W}, -\overline{Z})}$. It then follows that $u(W) = 2Re\Omega'(-\overline{W}, W)$.

Given $g(W)$ in D, continuous at $W = 0$, we can construct $\Omega'(Z, W)$ explicitly in two different ways. As it turns out, Ω' is in fact a function of W and $\xi = -Z/W$, $\Omega' \equiv E(W, \xi)$, such that E is analytic for W in D, and for arbitrary ξ in the complex ξ-plane which has a slit along the positive real axis. From E one derives the values of a solution u by

$$u(W) = 2ReE(W, \eta) . \tag{4.1}$$

One way of deriving $E(W, \xi)$ is to solve the differential equation of E in the variables W and ξ (because of condition (c) of Ω'),

$$\xi(1-\xi)E_{\xi\xi} - W(1-\xi)E_{W\xi} + (1-\xi-\alpha-\alpha\xi)E_\xi + \alpha W E_W = 0 ,$$

with $E(W, 0) = g(W)$ for $W \neq 0$, and $E(0, \infty) = 0$.

Another way is to find $E(W, \eta)$ by setting $E(W, \eta) = G(s, s, \theta)$ where $G(r, s, \theta)$ is the solution of the differential equation in (3.1) with assigned values along the characteristics:

$$G(0, s, \theta) = g(W), \ W \neq 0; \quad G(r, 0, \theta) = 0 . \tag{4.2}$$

This characteristic initial value problem can be solved explicitly by using RIEMANN's formula:

$$G(r, s, \theta) = \left(\frac{s}{s - r\eta}\right)^{\alpha} g(W) - \int_0^s t^{\alpha} \frac{\partial t^{-\alpha} R(0, t; r, s)}{dt} g(te^{i\theta}) dt \qquad (4.3)$$

where R is the RIEMANN function (3.2).

It is immediately seen after putting $r = s$ in (4.3) that each of the terms on the right hand side will diverge when $\theta \to 0, \pi$. Hence a regrouping of the terms in (4.3) is necessary.

First, the variable z in (3.2) is in this case $z = \frac{\eta}{1 - \eta} \frac{s - t}{t}$ which is inconvenient for the range of integration $0 \leq t \leq s$ of (4.3). We shall introduce the variable

$$\zeta = \frac{z}{z - 1} = \frac{\eta(1 - \lambda)}{\eta - \lambda}, \qquad \lambda = t/s,$$

and express the hypergeometric function which occur in R in terms of ζ. Secondly we may either (a) carry out the integration by parts in (4.3), or (b) we may replace the factor $g(te^{i\theta})$ in the integrand of (4.3) by $[g(se^{i\theta}) - g(te^{i\theta})]$, so that the first term on the right hand side of (4.3) has to be modified accordingly. In the case of (a) we find

$$E(W, \eta) = \int_0^1 R(\lambda, \eta) \lambda^{-\alpha} \frac{\partial \lambda^{\alpha} g(\lambda W)}{\partial \lambda} d\lambda, \; R(\lambda, \eta) = \lambda^{2\alpha}(\lambda - \eta)^{-\alpha} F_0(\zeta). \quad (4.4)$$

For the case of (b) we have the following

Lemma 4.1. $E(W, \eta)$ can be represented by

$$E(W, \eta) = e(\eta) g(W) + \int_0^1 K(\lambda, \eta) [g(W) - g(\lambda W)] d\lambda, \qquad (4.5)$$

where

$$K(\lambda, \eta) = -\alpha(1 - \alpha) \lambda^{2\alpha - 1} (\lambda - \eta)^{-1-\alpha} \eta F_2(\zeta)$$

and

$$e(\eta) = 1 - \frac{e^{i\alpha\pi} \Gamma(2\alpha)}{\Gamma(1 + \alpha) \Gamma(\alpha)} \eta^{\alpha} F(\alpha, 2\alpha, 1 + \alpha, \eta).$$

The function $E(W, \xi)$ is obtained simply by replacing in the above η by ξ.

Remark. The easiest way to verify $e(\eta)$ is by the following consideration. When $g(W) \equiv 1$, then $\Omega(Z, W) \equiv 1$, hence $u(W) \equiv 1$ and $Re E(W, \eta) \equiv Re e(\eta) \equiv \frac{1}{2}$. But on the other hand $e(\eta)$ must satisfy the differential equation of E, hence $(1 - \eta) e. + (1 - \eta - \alpha - \alpha\eta) e_{\eta} = 0$, which together with $Re e(\eta) = \frac{1}{2}$ determines $e(\eta)$.

Lemma 4.2. Another representation of $E(W, \eta)$ is as follows:

$$E(W,\eta)=\Phi_0(\eta)g(W)+\int_0^1(\lambda-\eta)^{-1-\alpha}\Phi_1(\zeta,\eta)[g(W)-\lambda^{2\alpha-1}g(\lambda W)]d\lambda\,, \tag{4.6}$$

where $\Phi_1 = -\alpha(1-\alpha)\eta F_2(\zeta)$ and $\Phi_0 = e(\eta) + \int_0^1 K(\lambda,\eta)(1-\lambda^{1-2\alpha})d\lambda$,

$$\Phi_0(\eta) = (1-\eta)^{1-2\alpha} + e^{-i\alpha\pi}\frac{\Gamma(2-2\alpha)}{\Gamma(2-\alpha)\Gamma(1-\alpha)}\eta^{1-\alpha}F(1,\alpha,2-\alpha,\eta)\,.$$

4.2. We proceed to derive a formula for $v(W)$. For this purpose it is convenient to assume that $g(W)$ has the property $P_\mu : g(W) = 0(|W|^\mu)$ in every sector $\varepsilon < \theta < \pi - \varepsilon$. From $u(W) = 2ReE(W,\eta)$ it follows $-u_\theta = 2Im(WE_W - 2\eta E_\eta)$, hence we have

$$v(W) = 2ImD(W,\eta) \tag{4.7}$$

with $D(W,\eta) = (\sin\theta)^{2\alpha}E(W,\eta) - 2(\sin\theta)^{2\alpha}\int_0^s E_\eta(te^{i\theta},\eta)\eta\frac{dt}{t}$.

Obviously the factor $(\sin\theta)^{2\alpha}$ on the right hand side of (4.7) is inconvenient for the limit process $\theta \to 0, \pi$. Hence we rewrite it in the form $(\sin\theta)^{2\alpha} = c^{-1}\eta^{-\alpha}(1-\eta)^{2\alpha}$, then shift the factor $(1-\eta)^{2\alpha}$ under the integral sign (integrals over λ) and finally substitute $(1-\eta)^{2\alpha}$ by $(1-\zeta)^{2\alpha}(\lambda-\eta)^{2\alpha}\lambda^{-2\alpha}$. We find

Lemma 4.3. $v(W) = 2ImD(W,\eta)$ with the following representation of

$$\begin{aligned}D(W,\eta) = {}&\Phi_s(\eta)g(W) + \int_0^1(\lambda-\eta)^{\alpha-1}[\Phi_2+\Phi_3+\Phi_4]\frac{g(\lambda W)}{\lambda}d\lambda\\ &+\alpha\int_0^1(\lambda-\eta)^{\alpha-1}[\Phi_3+\lambda\Phi_4]\frac{1}{\lambda}\int_0^{\lambda s}\frac{g(te^{i\theta})}{t}dt\,d\lambda\end{aligned} \tag{4.8}$$

where
$$\begin{aligned}\Phi_2(\zeta,\eta) &= \alpha(1-\alpha)c^{-1}\eta^{1-\alpha}(1-\zeta)^{2\alpha}F_2(\zeta)\,,\\ \Phi_3(\zeta,\eta) &= -2\alpha c^{-1}\eta^{1-\alpha}(1-\zeta)^{2\alpha}F_0(\zeta)\,,\\ \Phi_4(\zeta,\eta) &= -2c^{-1}\eta^{-\alpha}(1-\zeta)^{2\alpha}\zeta\frac{dF_0}{d\zeta}\,,\\ \Phi_5(\zeta) &= c^{-1}\eta^{-\alpha}(1-\eta)^{\alpha}\,.\end{aligned}$$

We define the operator B by

$$z(W) = Bg(W) = 2ReE(W,\eta) + 2iImD(W,\eta)\,. \tag{B}$$

As for the operator L, we have for B also $B(1) = 1$[7]) and $c_1z_1 + c_2z_2 = B(c_1g_1 + c_2g_2)$ when $z_k = Bg_k$, c_k real, $k = 1, 2$.

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[7]) When $g(W) \equiv i$, the corresponding u is a monotonic function of θ and independent of s.

§ 5. The correspondence between $g(W)$ and $z(W)$

From now on we shall consider in D analytic functions $g(W)$ and solutions $z(W)$ of (2.4) of class C_2, such that both g and z have the property P_μ. In particular, $g(0) = z(0) = 0$. Without making any assumption on the boundary behavior of g and z except at the origin, we have for the operators L and B,

Lemma 5.1. There is a one to one correspondence between g and z with the property P_μ[8]).

That the property P_μ is preserved by the operators L and B follows essentially from the linearity of the operators. Note that the bounds required in P_μ depend on the sectors $\varepsilon < \theta < \pi - \varepsilon$, and in each sector the integrands of L and B are bounded. The one to one correspondence follows from the uniqueness of solutions of (3.1) and (4.2).

We denote the correspondence by $g \longleftrightarrow z$.

Lemma 5.2. $g \longleftrightarrow z$ implies $\int\limits_0^W \frac{g(\varrho)}{\varrho} d\varrho \longleftrightarrow \int\limits_0^s \frac{z(te^{i\theta})}{t} dt$ and $W\frac{dg}{dW} \longleftrightarrow s\frac{\partial z}{ds}$,

provided the last two functions have the property P_μ.

In the representations of L and B it is seen that integration and differentiation with respect to s can be carried out under the integral signs. Hence the statement in the lemma can be verified directly.

§ 6. A theorem on HOELDER continuity

6.1. The transformations T_β^δ and $T_\beta^{-\delta}$.

Let $\eta = e^{2i\theta}$, $0 \leq \theta \leq \pi$. For each value of θ we consider the transformasions

$$M_1(s, \eta) = T_\beta^\delta m_1(s, \eta), \qquad M_2(s, \eta) = T_\beta^{-\delta} m_2(s, \eta)$$

of the complex valued functions m_i and M_i, $i = 1, 2$, where T_β^δ and $T_\beta^{-\delta}$ $1 > \delta > 0$, $\beta \geq 0$, are given respectively by the following integrals:

$$M_1(s, \eta) = \int\limits_0^s s^\beta (t - s\eta)^{\delta-1} \Phi(s, t, \eta) t^{-\beta} m_1(t, \eta) dt, \tag{6.1}$$

$$M_2(s, \eta) = \int\limits_0^s s^\beta (t - s\eta)^{-\delta-1} \Phi(s, t, \eta) [s^{-\beta} m_2(s, \eta) - t^{-\beta} m_2(t, \eta)] dt. \tag{6.2}$$

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[8]) The method of obtaining solutions u by associating them with analytic functions of a complex variable is well known. It has been extensively used by BERGMAN in more general cases. His work on this subject [1] were concerned with expansions of solutions which have analytic data on the initial segment.

Φ is a given function whose explicit form is immaterial. It is continuous in its argument and satisfies the conditions:

$$|\Phi(s,t,\eta)| < \Phi_0 \tag{6A}$$

$$|\Phi(s,t,\eta) - \Phi(s-h,t,\eta)| < \Phi_0 h\,|s-h-t\eta|^{-1}, \quad 0 \leq t < s-h, \tag{6B}$$

where Φ_0 is a constant independent of h, s, t and η. Furthermore Φ is assumed to depend only on η and on the ratio of t and s[9]),

$$\Phi(s,t,\eta) \equiv \Psi(t/s,\eta) \,. \tag{6C}$$

Definition. A real or complex valued function $f(s,\eta)$ which is continuous in D up to l, is said to belong to the class H_μ with $0 < \mu < 1$, if

$$\begin{aligned} &\text{(a)}\quad f(0,\eta) = 0 \quad \text{and} \\ &\text{(b)}\quad |f(s,\eta) - f(s-h,\eta)| = Oh^\mu \end{aligned} \tag{6D}$$

hold where the HOELDER constant is independent of η.

Theorem 1. In T_β^δ let $m_1 \in H_{k_1}$ with $k_1 \geq 0, k_1 + \delta < 1$ and $k_1 - \beta > -1$, then $M_1 \in H_{k_1+\delta}$.

Theorem 1′. In $T_\beta^{-\delta}$ let $m_2 \in H_{k_2}$ with $1 \geq k_2 > \delta > 0$ and $k_2 - \beta > -1$, then $M_2 \in H_{k_2-\delta}$.

This is a generalization of a theorem of HARDY and LITTLEWOOD [7] on the transformations I^δ and $I^{-\delta}$ by integration and differentiation of fractional order δ:

$$M(s) = I^\delta m(s) = \frac{1}{\Gamma(\delta)} \int_0^s (s-t)^{\delta-1} m(t)\,dt$$

and

$$M(s) = I^{-\delta} m(s) = \frac{1}{\Gamma(1-\delta)}\, m(s) - \frac{1}{\Gamma(-\delta)} \int_0^s (s-t)^{-\delta-1}[m(s) - m(t)]\,dt \,.$$

The following is the scheme of a proof of theorem 1. The difference operator Δ_h, $h > 0$ is applied to the variable s of the functions M_i:

$$\Delta_h M_i(s,\eta) = \int_0^s - \int_0^{s-h} \ldots dt \tag{6.3}$$

where in the second integrand, $(s-h)$ takes the place of s while t and η are unchanged. A new variable r is then introduced by $r = s - t$, to replace t as the variable of integration.

---

[9]) (6C) is used only once, that is in the proof of lemma 6.2, b).

For abbreviation let

$$
\begin{aligned}
&s^{-\beta} m_i(s) = p_i(s) \qquad i = 1, 2, \\
&s^{\beta}(t - s\eta)^{\delta-1} = d_1(s, t, \eta), \qquad s^{\beta}(t - s\eta)^{-\delta-1} = d_2(s, t, \eta).
\end{aligned}
\tag{6.4}
$$

By rearranging the terms in (6.3) we obtain the following decompositions,

$$\Delta_h M_i(s, \eta) = J_{i1} + J_{i2} + J_{i3} + J_{i4}, \qquad i = 1, 2,$$

where

$$J_{11} = p_1(s)\{\int_0^s d_1(s, t, \eta)\Phi(s, t, \eta)dr - \int_h^s d_1(s - h, t, \eta)\Phi(s - h, t, \eta)dr\},$$

$$J_{12} = -\int_0^h d_1(s, t, \eta)\Phi(s, t, \eta)\,\Delta_r p_1(s)dr,$$

$$J_{13} = -\int_h^s \Delta_h d_1(s, t, \eta)\Phi(s - h, t, \eta)\,\Delta_r p_1(s)dr,$$

$$J_{14} = -\int_h^s d_1(s, t, \eta)\,\Delta_h\Phi(s, t, \eta)\,\Delta_r p_1(s)dr,$$

and similarly,

$$J_{21} = \int_h^s d_2(s, t, \eta)\Phi(s, t, \eta)dr\,\Delta_h p_2(s),$$

$$J_{22} = \int_0^h d_2(s, t, \eta)\Phi(s, t, \eta)\,\Delta_r p_2(s)dr,$$

$$J_{23} = \int_h^s \Delta_h d_2(s, t, \eta)\Phi(s - h, t, \eta)[\Delta_r p_2(s) - \Delta_h p_2(s)]dr,$$

$$J_{24} = \int_h^s d_2(s, t, \eta)\,\Delta_h\Phi(s, t, \eta)[\Delta_r p_2(s) - \Delta_h p_2(s)]dr.$$

The differences $\Delta_h M_i$ will be estimated, as in [7], by considering two cases 1) $h < s \leq 2h$ and 2) $s > 2h$.

Lemma 6.1. When $h < s \leq 2h$, we have

$$|M_i(s, \eta)| < Nh^{k_i \pm \delta}, \qquad |M_i(s - h, \eta)| < Nh^{k_i \pm \delta}$$

where the $+$ sign holds for $i = 1$ and $-$ sign for $i = 2$, and the constant N is independent of h, s and η.

Lemma 6.2. When $s > 2h$, we have

$$|J_{in}| < Nh^{k_i \pm \delta}, \qquad n = 1, 2, 3, 4$$

with the same qualification for the right hand side as in lemma 6.1.

In the remaining part of § 6 we give a proof of the two lemmas, from which theorems 1 and 1′ follow immediately.

6.2 Lemma 6.3. Let $1 > \delta > 0$, $r = s - t > 0$, then

$$|t - s\eta|^{\delta-1} \leq r^{\delta-1}, \qquad |t - s\eta|^{-\delta-1} \leq r^{-\delta-1} \tag{6.5}$$

hold. Furthermore, for $h < r \leq s$,

$$|\Delta_h(t - s\eta)^{\delta-1}| = O(r - h)^{\delta-1} - r^{\delta-1}, \tag{6.6}$$

$$|\Delta_h(t - s\eta)^{-\delta-1}| = Or^{-1}(r - h)^{-\delta-1}h. \tag{6.7}$$

Proof. (6.5) is obvious. For a proof of (6.6) and (6.7) note first $|a - be^{2i\theta}| \leq |a - b| + 2\sqrt{ab}\sin\theta$ for any real positive a, b and $0 \leq \theta < \pi$. Hence for $p > 0$, ($p = 1 - \delta$ or $p = 1 + \delta$),

$$|(t - s\eta)^{-p} - (t - (s - h)\eta)^{-p}| \leq \{|s - h - t\eta|^{-1} - |s - t\eta|^{-p}\}$$
$$+ 2|s - t\eta|^{-p/2}|s - h - t\eta|^{-p/2}\sin\frac{p}{2}\theta',$$

where $\theta' = \arg(t\eta - s) - \arg(t\eta - (s - h))$.

Secondly $\sin\theta' < h/|s - t\eta| \leq h/r$, hence $\sin\frac{p}{2}\theta' = Or^{-1}h$.

Finally, one verifies by differentiation that $|s - t|^{-p} - |s - t\eta|^{-p}$ is for fixed η and fixed $t < s - h$, a decreasing function of s. Hence $|s - h - t\eta|^{-p} - |s - t\eta|^{-p} \leq (r - h)^{-p} - r^{-p}$. From this follow (6.6), (6.7).

By the assumption we have $m_i(t) = Ot^{k_i}$ and $\Delta_h m_i = Oh^{k_i}$. Furthermore, $\Delta_h s^{-\beta} \leq \beta s^{-\beta}(s - h)^{-1}$ for $\beta \leq 1$, and $\Delta_h s^{-\beta} \leq \beta s^{-1}(s - h)^{-\beta}h$ for $\beta \geq 1$. For convenience we combine the cases $\beta \geq 1$ and $\beta \leq 1$ together and have the following

Lemma 6.4. Let $\beta \geq 0$, then for $i = 1, 2$,

$$\Delta_h p_i(s) = Os^{-\beta}h^{k_i} + Os^{-\beta}(s - h)^{k_i-1}h + Os^{-1}(s - h)^{k_i-\beta}h. \tag{6.8}$$

Similarly we obtain the formula for $\Delta_r p_i(s)$ by replacing h in (6.8) by r. The formula for $(\Delta_r - \Delta_h)p_i(s)$ to be denoted by (6.9), is obtained by replacing in (6.8), s by $(s - h)$ and h by $(r - h)$. Furthermore,

$$\Delta_h d_1(s, t, \eta) = Os^{\beta}[(r - h)^{\delta-1} - r^{\delta-1}] + Os^{\beta}(s - h)^{\delta-1}(s - h)^{-1}h. \tag{6.10}$$

By replacing the expression in the bracket of (6.10) by $r^{-1}(r - h)^{-\delta-1}h$ we obtain an estimate for $\Delta_h d_2(s, t, \eta)$ which we denote by (6.11).

Lemma 6.5. In the product

$$P = h^{\lambda}(s - h)^{\mu}s^{\sigma}r^{a}(s - r)^{b-1}(r - h)^{c-1}, \qquad b > 0, \quad c > 0$$

let $c+a\neq 0, c+a\neq$ any integer, and $\lambda+\mu+\sigma+a+b+c=1+\varrho$. Furthermore it is assumed that 1) $\lambda-\varrho\geq 0$ when $c+a>0$, and 2) when $c+a<0$, then $\lambda-\varrho+c+a\geq 0$. Then the integral over $h<r\leq s$ with $s\geq 2h$ can be estimated by $\int_h^s P\,dr<Nh^\varrho$ where the constant N is independent of s and h.

Proof. Consider the integral over the last three factors in P, and set $(s-r)=\tau(s-h)$, $(r-h)=(1-\tau)(s-h)$ so that the integral is over $0\leq\tau\leq 1$. This integral is equal to const. $s^a(s-h)^{b+c-1}$. $F(-a,b,c+b,(s-h)/s)$. The theorem follows from the behavior of the hypergeometric function at $h=0$, and from the fact that $\frac{1}{2}\leq(s-h)/s\leq 1$ because of $s\geq 2h$.

6.3. Proof of lemma 6.1. By assumption we have

$$h<s\leq 2h\,. \tag{6.12}$$

It suffices to show that $M_i(s,\eta)=Os^{k_i\pm\delta}$ $i=1,2$, because it follows then $M_i(s-h,\eta)=O(s-h)^{k_i\pm\delta}$ and hence $\Delta_h M_i=Oh^{k_i\pm\delta}$ due to (6.12). The estimate of $M_i(s,\eta)$ is obtained by using (6A), (6D) for $m_1(t)$ and (6.8) for $\Delta_r p_2(s)$.

Proof of lemma 6.2. By assumption we have

$$2h<s \quad\text{or}\quad 1<s/(s-h)<2\,. \tag{6.13}$$

a) Estimate of J_{12} and J_{22}. By using (6A) (6.5), (6.8) and (6.13).

b) Estimate of J_{11}. By (6C) one has $J_{11}=m_1(s)J'_{11}$ with

$$J'_{11}=\left[s^\delta-(s-h)^\delta\left(1-\frac{h}{s}\right)^\beta\right]\int_0^1(\lambda-\eta)^{\delta-1}\Psi(\lambda,\eta)\,d\lambda\,.$$

c) Estimate of J_{21}, J_{23} and J_{24}. By using (6A), (6B) for Φ, (6.5), (6.11) for d_2, and (6.8), (6.9) for p_2, one obtains products of the form same as the product P in lemma 6.5. Applications of this lemma give the required estimates.

d) Estimate of J_{13}. By proceeding in the similar way as c) one obtains six products of which three are of the form P of lemma 6.5 and the remaining three are products of $[(r-h)^{\delta-1}-r^{\delta-1}]$ respectively with r^{k_1}, $(s-r)^{k_1-1}r$ and $(s-r)^{k_1-\beta}rs^{\beta-1}$ with $\beta>1$. The integrals $\int_h^s$ over the first two products are estimated by splitting $\int_h^s=\int_h^{2h}+\int_{2h}^s$, while it is convenient to treat the last product by writing $\int_h^s=\int_h^{3/2h}+\int_{3/2h}^s$. The integrals $\int_{2h}^s$ and $\int_{3/2h}^s$ are estimated by lemma 6.5 using $1<r/(r-h)\leq 3$.

e) Estimate of J_{14}. Let the integral $\int_h^s = \int_h^{2h} + \int_{2h}^s$. In the first term we use (6A) for $\Phi(s, \eta)$ and $\Phi(s-h, \eta)$, while in the second term (6C) is used for $\Delta_h \Phi$. Lemma 6.5 is then applied to the last integral to obtain the required estimate.

6.4. The transformations I_β^δ and $I_\beta^{-\delta}$.

These transformations are defined as generalizations of the integration I^δ and differentiation $I^{-\delta}$ of the fractional order δ:

$$I_\beta^\delta m(s) = s^\beta I^\delta (s^{-\beta} m(s)) \ , \qquad I_\beta^{-\delta} m(s) = s^\beta I^{-\delta} (s^{-\beta} m(s)) \ .$$

We have of theorem 1 the following

Corollary. If $m(t) \in H_k$, $m(0) = 0$, then

$$I_\beta^\delta m(s) \in H_{k+\delta} \,, \quad \text{for} \quad k + \delta < 1 \,, \quad k \geq 0 \,, \quad \delta > 0 \,,$$

and

$$I_\beta^{-\delta} m(s) \in H_{k-\delta} \,, \quad \text{for} \quad k - \delta > 0 \,, \quad k \leq 1 \,, \quad \delta > 0 \,.$$

It is assumed that $k - \beta > -1$.

The following simple rules of differentiation and integration may be mentioned: for $0 < \delta < 1$,

$$\begin{aligned} s \frac{d}{ds} I_\beta^\delta s^{-\delta} p(s) &= I_\beta^\delta s^{-\delta} \left(s \frac{dp}{ds} \right) , \\ \int_0^s \frac{ds}{s} I_\beta^\delta s^{-\delta} p(s) &= I_\beta^\delta s^{-\delta} \int_0^s \frac{p(t)}{t} \, dt \ . \end{aligned} \tag{6.14}$$

The formulas hold true also when δ is replaced by $-\delta$. Moreover, for $0<\delta<1$,

$$s \frac{d}{ds} I_\beta^\delta q(s) = \beta I_\beta^\delta q(s) + I_{\beta+1}^{\delta-1} s q(s) \ . \tag{6.15}$$

§ 7. The main theorem

7.1. Statement of the main theorem

Definition. By $\mathfrak{G}(\mu)$, $0 < \mu < 1$, we denote the class of analytic functions $g(W) = g_0(X, y) + i g_1(X, y)$ in D with the properties, (a) $g(0) = 0$ and (b) $g(W) \in H_\mu$ (for the definition of H_μ see § 6.1).

Specifically, $\mathfrak{G}_1(\mu)$, $\mathfrak{G}_2(\mu)$ and $\mathfrak{G}_3(\mu)$ denote those subclasses for which the ranges of μ are given respectively by $\alpha < \mu < 1 - \alpha, 1 - \alpha < \mu < 1$ and $0 < \mu < \alpha$.

Definition. The classes of twice continuously differentiable solutions $z = u + iv$ of (2.4) in D, with the property (a) $z(0) = 0$ and the following property (b) are denoted respectively by

$$Z_1(\mu)\colon \alpha < \mu < 1 - \alpha\,, \qquad \text{(b)}\ s^{-\alpha} u \in H_{\mu-\alpha} \quad \text{and} \quad s^{\alpha} v \in H_{\mu+\alpha}\,,$$

$$Z_2(\mu)\colon 1 - \alpha < \mu < 1\,, \qquad \text{(b)}\ s^{-\alpha} u \in H_{\mu-\alpha} \quad \text{and} \quad s^{\alpha} v_s \in H_{\mu+\alpha-1}\,,$$

and

$$Z_3(\mu)\colon 0 < \mu < \alpha\,, \qquad \text{(b)}\ u(W) = O(s^{\mu})\,, \quad s^{\alpha} v \in H_{\mu+\alpha} \quad \text{and}$$

$$s^{1-\alpha} \int_0^s t^{-1} u(t e^{i\theta})\, dt \in H_{\mu-\alpha+1}\,.$$

Theorem 2. The operator L, $g = L(z)$ § 3, and its inverse B, $z = B(g)$ § 4, induce a one to one correspondence between the elements of $\mathfrak{G}_k$ and those of Z_k, $k = 1, 2, 3$. The initial values of the corresponding elements are related to each other by the transformations of the type $I_{\beta}^{\pm\delta}$ (§ 6.4, with $\delta = \alpha$, $(1 - \alpha)$), which are given in the following (7.1), (7.2) and (7.3).

Let $s = |y|$ and introduce the function $f(\pm s)$ by

$$f(s) = g_0(s) + \tan \alpha\pi \;\, g_1(s)\,, \quad f(-s) = g_0(-s) - \tan \alpha\pi \;\, g_1(-s)\,.$$

Let the constants A and A_1 be $A = \Gamma(2\alpha)/\Gamma(\alpha)$, $A_1 = 4^{\alpha-1} A^{-1}$. The following are the relations between the initial values of the corresponding elements $z \longleftrightarrow g$. For $g \in \mathfrak{G}_1$, $z \in Z_1$,

$$f(\pm s) = A I_{1-2\alpha}^{\alpha} s^{-\alpha} u(\pm s)\,, \quad g_1(\pm s) = A_1 I_1^{-\alpha} s^{\alpha} v(\pm s)\,. \tag{7.1}$$

For $g \in \mathfrak{G}_2$, $z \in Z_2$, the first equation is the same as that in (7.1), but the second involves v_s instead of v,

$$f(\pm s) = A I_{1-2\alpha}^{\alpha} s^{-\alpha} u(\pm s)\,, \quad g_1(\pm s) = A_1 I_0^{1-\alpha} s^{\alpha} \frac{dv(\pm s)}{ds}\,. \tag{7.2}$$

Finally, for $g \in \mathfrak{G}_3$, $z \in Z_3$, the second equation is the same as that in (7.1), while in the first equation the function $\tilde{u}(\pm s) = \int_0^s t^{-1} u(\pm t)\, dt$ is introduced:

$$\begin{aligned} f(\pm s) &= A\, \{ I_{1-2\alpha}^{\alpha-1} s^{1-\alpha} \tilde{u}(\pm s) - \alpha I_{1-2\alpha}^{\alpha} s^{-\alpha} \tilde{u}(\pm s) \}\,, \\ g_1(\pm s) &= A_1 I_1^{-\alpha} s^{\alpha} v(\pm s)\,. \end{aligned} \tag{7.3}$$

One can express g in terms of z by simply inverting the operators $I_{\beta}^{\pm\delta}$ of (7.1) and (7.2). The inverse relation of (7.3) is given by

$$s^{1-\alpha} \tilde{u}(\pm s) = A I_{1-2\alpha}^{1-\alpha} \{ f(\pm s) + \alpha \tilde{f}(\pm s) \}\,, \; \tilde{f}(\pm s) = = \int_0^s t^{-1} f(\pm t)\, dt\,. \tag*{(7.3)$_0$}$$

7.2. An outline of the proof. At the outset one has to determine, among the many integral representations of the operators L (or B), one, which is applicable to the class Z_k (or $\mathfrak{G}_k$) under consideration. In the integral representation one replaces the variable λ of § 3 and § 4 by $\lambda = t/s$ and uses t as the variable of integration. The integrals then have the form of T_β^δ or $T_\beta^{-\delta}$ of § 6. In the place of the functions $(m_1(s, \eta), m_2(s, \eta))$ of (6.1) and (6.2) we shall substitute $(s^{-\alpha}u, s^\alpha v)$ or $(s^{-\alpha}u, s^\alpha v_s)$ or $(s^{1-\alpha}\tilde{u}, s^\alpha v)$ depending on the value of $k = 1, 2$ or 3 in the operator $L(Z_k)$. For the operator $B(\mathfrak{G}_k)$ we shall substitute $g(W)$ for m_1 or m_2 in (6.1) and (6.2). In order to apply theorem 1, it suffices to verify (by lemma 7.1) the properties (6A), (6B) and (6C) for the functions φ_i and the functions Φ_i which occur under the integral signs of (3.6) (3.7) (4.6) and (4.8). In these formulas there are additional terms not included in the integrals. The HOELDER continuity properties of those terms are easily proved (lemma 7.2). It then follows that $L(Z_k) \in \mathfrak{G}_k$ and $B(\mathfrak{G}_k) \in Z_k$.

To obtain the relations between the initial values of g and z what is needed is to carry out the limit process $\theta \to 0, \pi$ in the integral representations of L and B, under the integral sign. Since each function φ_i or Φ_i either vanishes or becomes a constant in the limit, the limit relations have form of I_β^δ or $I_\beta^{-\delta}$.

The HOELDER continuity property of $g(W)$ in D or at least in closed subdomains of D adjacent to l, follows from the same property of the initial values of its real or imaginary part. Hence to prove $g \in \mathfrak{G}(\mu)$, it suffices to prove the required property for $g_0(y)$, $g_1(y)$ or $f(y)$, by the corollary of theorem 1 § 6.4. On the other hand, to prove $z \in Z_k$, theorem 1 will be needed.

7.3. Proof of theorem 2.

(a) For the operator $L(Z_k)$. When $k = 1$, the integral representation (L′) is used with (3.7) and the first equation of (3.6). For $k = 2$, use (L) with (3.6). By the assumptions on Z_1 and Z_2 the limit process can be carried out under the sign, when $\theta \to 0, \pi$. The result is the limit relations (7.1) (7.2). The HOELDER continuity property of $g(W)$ follows from that of $g_1(y)$ and $f(y)$, (apply corollary § 6.4).

For $k = 3$ one substitutes $\tilde{u}$ for u and $\tilde{f}$ for f in (7.1), then differentiates with respect to s using (6.15).

(b) For the operator $B(\mathfrak{G}_k)$. For $k = 1$, use (B) with (4.6) and (4.8). When $k = 3$, use (4.8) and the following „integrated form" of (4.4): one replaces in (4.4) g by $\tilde{g} = \int\limits_0^W t^{-1} g(te^{i\theta})\,dt$ and u by $\tilde{u}$. The equations (4.4) and (4.6) are to be multiplied on both sides by the factor $s^{-\alpha}$, and the equation (4.8) by s^α.

As for $k = 2$, we need the differentiated form of (4.8): $\dfrac{\partial}{ds} D(W, \eta)$, which is obtained by differentiating under the integral sign of (4.8). Let

$\Psi_1 = \Phi_2 + \Phi_3 + \lambda\Phi_4$, $(\lambda - \eta)^{\alpha-1}\Psi_1 = Y(\lambda, \eta)$ and $\Psi_2 = \Phi_3 + \lambda\Phi_4$, then by integration by parts in the middle term, we find

$$\frac{\partial}{\partial s} D(W, \eta) = \Phi_5 \frac{\partial g(W)}{ds} + s^{-1} g(W) \int_0^1 Y(\lambda, \eta) d\lambda + \alpha \int_0^1 (\lambda - \eta)^{\alpha-1} \Psi_2 \frac{g(\lambda s e^{i\theta})}{\lambda s} d\lambda$$
$$+ \int_0^1 \lambda Y_\lambda [s^{-1} g(W) - (\lambda s)^{-1} g(\lambda s)] d\lambda \,. \tag{7.4}$$

We omit the detailed expansion of the last term of the above formula.

For the limit process one observes that

$$\Phi_2 \to 0, \; \Phi_3 \to 0 \;\text{ and }\; \Phi_4 \to -\tfrac{1}{2} e^{\mp i\alpha\pi}/(A_1 \Gamma(\alpha)), \;\text{ as }\; \theta \to 0, \pi.$$

To justify the application of theorem 1 to the integral representations of the operators $B(\mathfrak{G}_k)$, we need the following two lemmas.

Lemma 7.1. A function $\Phi(\zeta, \eta)$ satisfies the properties (6A) (6B) and (6C) if it is continuous for $0 \leq \lambda \leq 1$, $0 \leq \theta \leq \pi$ and $\left|\frac{d\Phi}{d\zeta}\right| <$ Const. $|1 - \zeta|^{\delta-1}$ with $\delta > 0$. Similarly, any positive power of λ has the properties (6A)–(6C).

Lemma 7.2. Let $g(W) \in H_\mu$, $g(0) = 0$, $0 < \mu < 1$. Then

$$s^\gamma (1 - \eta)^\gamma g(W) \in H_{\mu+\gamma} \qquad \text{for} \quad 0 < \gamma, \; \mu + \gamma < 1$$

and

$$s^\gamma (1 - \eta)^\gamma \frac{dg(W)}{dW} \in H_{\mu+\gamma-1} \qquad \text{for} \quad 0 < \mu + \gamma - 1 < 1$$

The simple proofs of the lemmas are omitted.

One verifies easily that Φ_2, Φ_3 and Φ_4 satisfy the conditions of lemma 7.1 with $\delta = 2\alpha$ or $(1 - 2\alpha)$. Lemma 7.2 is used to treat the terms $s^\alpha \Phi_5 g(W)$ in (4.8) and $s^\alpha \Phi_5 \frac{dg}{dW}$ in (7.4). We omit the discussion of the application of theorem 1 to the last integral of (7.4).

7.4. We shall formulate the main theorem in terms of other classes of solutions of (2.3) and (2.4) because they are more useful than the classes Z_k, $k = 1, 2, 3$. For this purpose let us consider again the transformations

$$M_1 = T_\beta^\delta m_1 \quad\text{and}\quad M_2 = T_\beta^{-\delta} m_2 \,. \tag{7.5}$$

If the two equations are each multiplied by a power s^γ then the resulting equations can be written in the form

$$s^\gamma M_1 = T_{\beta+\gamma}^\delta (s^\gamma m_1) \quad\text{and}\quad s^\gamma M_2 = T_{\beta+\gamma}^{-\delta} (s^\gamma m_2) \,. \tag{7.6}$$

Similarly,

$$s^\gamma I_\beta^\delta m_1 = I_{\beta+\gamma}^\delta (s^\gamma m_1) \quad\text{and}\quad s^\gamma I_\beta^{-\delta} m_2 = I_{\beta+\gamma}^{-\delta} (s^\gamma m_2) \tag{7.7}$$

hold. Now, if we assume that $s^\gamma m_i \in H_{k_i}$ instead of $m_i \in H_{k_i}$, then we may apply theorem 1 and its corollary to (7.6) and (7.7) to conclude that $s^\gamma M_i \in H_{k_i \pm \delta}$, provided that $k_i - \beta - \gamma > -1$. We shall say that (7.6) is derived from (7.5) by a shifting of the power s^γ under the integral sign.

Instead of z we shall use

$$z^* = u + iv^*$$

where (u, v^*) is a solution of (2.3). The following relations exist between v and v^*:

$$s^{2\alpha} v_s = v_s^* \text{ and } v^* = s^{2\alpha} v - 2\alpha \int_0^s t^{2\alpha-1} v(te^{i\theta})\,dt\,, \tag{7.8}$$

so that the property of $s^{2\alpha} v \in H_\mu$ is equivalent to $v^* \in H_\mu$.

We introduce the following classes of solutions with (a) $z^*(0) = 0$ and the properties (b):

$Z_1'(\mu')$: $0 < \mu' < 1 - 2\alpha$, (b) $u \in H_{\mu'}$, $v^* \in H_{\mu'+2\alpha}$,

$Z_2'(\mu')$: $1 - 2\alpha < \mu' < 1 - \alpha$, (b) $u \in H_{\mu'}$, $v_s^* \in H_{\mu'+2\alpha-1}$, $v_s^*(0) = 0$,

$Z_3'(\mu')$: $1 - \alpha < \mu' < 1$, (b) $u \in H_{\mu'}$, $v_s^* \in H_{\mu'+2\alpha-1}$, $v_s^*(0) = 0$.

For comparison with the definition of Z_k § 7.1, set $\mu' = \mu - \alpha$ for $k = 1, 2$ and $\mu' = \mu - \alpha + 1$ for $k = 3$.

Correspondingly, $\mathfrak{G}_1'(\mu')$ for $0 < \mu' < 1 - 2\alpha$, and $\mathfrak{G}_2'(\mu')$ for $1 - 2\alpha < \mu' < 1 - \alpha$ are the classes of analytic functions $g(W)$ with (a) $g(0) = 0$ and (b) $s^\alpha g(W) \in H_{\mu'+\alpha}$[10]), while $\mathfrak{G}_3'(\mu)$ consists of functions with (a) $g(W) = O\, s^{\mu'}$ and (b) $s^\alpha \dfrac{dg}{dW} \in H_{\mu'+\alpha-1}$ for $1 - \alpha < \mu' < 1$.

Theorem 2′. The operator L and its inverse B induce a one to one correspondence between elements of $\mathfrak{G}_k'$ and those of Z_k', $k = 1, 2, 3$. The following are the relations between the initial values of the corresponding elements:

$$s^\alpha f(\pm s) = A I_{1-\alpha}^{\alpha} u(\pm s)\,, \quad s^\alpha g_1(\pm s) = A_1 I_{1+\alpha}^{-\alpha} s^{2\alpha} v(\pm s), \tag{7.1$'$}$$

$$s^\alpha f(\pm s) = A I_{1-\alpha}^{\alpha} u(\pm s)\,, \quad s^\alpha g_1(\pm s) = A_1 I_\alpha^{1-\alpha} v_s^*\,, \tag{7.2$'$}$$

$$s^\alpha \frac{df(\pm s)}{ds} = A\,\{I_{-\alpha}^{\alpha-1} u(\pm s) - \alpha I_{-\alpha}^{\alpha} s^{-1} u(\pm s)\}, \quad s^\alpha \frac{dg_1(\pm s)}{ds} = A_1 I_\alpha^{-\alpha} v_s^*\,. \tag{7.3$'$}$$

Proof. Use the same integral representations of the operators L and B as in theorem 2, and apply to them the process of shifting the power s^α for

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[10]) $s^\alpha g(W) \in H_\mu$ is equivalent to $W^\alpha g(W) \in H_\mu$.

$k = 1, 2, 3$. In the last case of $k = 3$, the shifting is done after the equations have been differentiated with respect to s.

Remark. In formulating the main theorem we have restricted ourselves to three classes of functions with HOELDER continuity properties. By making use of the lemma 5.2 one can define corresponding classes with n-times differentiable initial values, such that the highest derivative are HOELDER continuous, or classes of functions whose initial values become HOELDER continuous after repeated integrations. Lemma 5.2 together with (6.14), applied to (7.1)–(7.3), give the relations of the corresponding initial values.

§ 8. HOELDER continuity of the conjugate functions

8.1. Theorem 3. Let $u(W)$ be a solution of (1.2) of class C_2 in D and continuous up to l. Let the initial values $u(y)$ have the properties (a) $u(0) = 0$ and (b) $u(y)$ is HOELDER continuous with exponent μ, $0 < \mu < 1 - 2\alpha$. Then the solution $z^* = u + iv^*$ belongs to the class $Z_1'(\mu)$, where v^* is the conjugate of u in (2.3) with $v^*(0) = 0$.

Remark. The existence of the initial value $v^*(0)$ for any conjugate v^* of u has yet to be proved. Under the assumptions of theorem 3, we shall prove first a few lemmas on solutions $z = u + iv$ of the equation (2.4) which serves as a basis for an inductive reasoning due to lemma 5.2.

Lemma 8.1. $u(W) = Os^{\mu}$.

Proof. For the range of μ, $0 < \mu < 1 - 2\alpha$, it is possible to construct a special solution $\varphi(W)$ of (1.2) with the following properties:
1) $\varphi(W) = s^{\mu}p(\tau)$, $\tau = \frac{1}{2}(1 + \cos\theta)$, $0 \leq \theta \leq \pi$, 2) $p(\tau) \geq$ a constant $p_0 > 0$ for $0 < \tau < 1$. One finds for $p(\tau)$ the following differential equation

$$\tau(1 - \tau)p'' - (1 - 2\alpha)(\tau - \tfrac{1}{2})p' + \mu(\mu + 2\alpha)p = 0,$$

with the independent solutions $p_1 = F(-\mu, \ \mu + 2\alpha, \ \frac{1}{2} + \alpha, \ \tau)$ and $p_2 = \tau^{\frac{1}{2}-\alpha}F(-\mu + \frac{1}{2} - \alpha, \ \mu + \frac{1}{2} + \alpha, \ \frac{3}{2} - \alpha, \ \tau)$. The last hypergeometric function takes on only positive values for $0 \leq \tau \leq 1$. Hence we can obtain a φ by setting $\varphi = p_1 + Cp_2$ with a suitable positive constant C. By further multiplying φ with a suitable constant if necessary, we can make $|u(W)| < \varphi(W)$ to hold on the circumference of a half-circle around the origin and on its diameter. By the maximum principle it then follows that $|u(W)| < \varphi(W) = Os^{\mu}$ also holds inside the circle.

Lemma 8.2. In the domain D let v be a conjugate solution of u in (2.4).

Then there exists a constant v_0 such that

$$\frac{1}{s}\int_0^s v(te^{i\theta})dt - v_0 = O s^{\mu} \tag{8.1}$$

and v_0 is independent of θ, $0 < \theta < \pi$.

Proof. 1) By applying the integration $\int_\varepsilon^s \int_{\theta_1}^{\theta_2} d\theta dt$ on both sides of the first equation of (2.4) and by carrying out the integration with respect to θ on the left, and with respect to t on the right hand side, one finds easily the existence of the limit of $\int_{\theta_1}^{\theta_2} v(\varepsilon e^{i\theta})d\theta$ as $\varepsilon \to 0$. Denote the limit by $J(\theta_1, \theta_2)$, it is an additive interval function of (θ_1, θ_2). Lemma 8.1 applied to the integrated equation gives

$$\int_{\theta_1}^{\theta_2} v(se^{i\theta})d\theta - J(\theta_1, \theta_2) = O s^{\mu}. \tag{8.2}$$

2) Multiply the second equation by s and integrate it as follows:

$$\int_{\theta_1}^{\theta_2}\int_{\theta_1}^{\theta}\int_\varepsilon^s \sin\theta^{2\alpha}(tu_t + 2\alpha u)\, dt = \int_\varepsilon^s \int_{\theta_1}^{\theta_2} v(te^{i\theta})d\theta dt - (\theta_2 - \theta_1)\int_\varepsilon^s v(te^{i\theta_1})dt. \tag{8.3}$$

One carries out the integration with respect to t at the left hand side and then replaces ε by 0, it is seen then that the left hand side is $O s^{1+\mu}$. Divide the whole equation by s and finally set $s \to 0$. It then follows from (8.2),

$$\lim_{s\to 0}(\theta_2 - \theta_1)\frac{1}{s}\int_0^s v(te^{i\theta_1})dt = J(\theta_1, \theta_2). \tag{8.4}$$

By interchanging θ_1 with θ_2 in (8.4) one concludes that the integral of (8.1) is independent on θ. The lemma is proved.

By writing v for $v - v_0$ one finds from (8.1) by integration by parts,

Lemma 8.1. $\displaystyle\int_0^s \frac{1}{t}\, v(te^{i\theta})dt = O s^{\mu}$.

We introduce the following functions $\tilde{z} = \tilde{u} + i\tilde{v}$ with

$$\tilde{u} = \int_0^s \frac{u(te^{i\theta})}{t}\, dt, \qquad \tilde{v} = \int_0^s \frac{v(te^{i\theta})}{t}\, dt,$$

and $\approx{z} = \approx{u} + i\approx{v}$ with

$$\approx{u} = \int_0^s \frac{\tilde{u}(te^{i\theta})}{t}\, dt, \qquad \approx{v} = \int_0^s \frac{\tilde{v}(te^{i\theta})}{t}\, dt.$$

Lemma 8.2. Both $\tilde{z}$ and $\tilde{\tilde{z}}$ are solutions of (2.4) with continuous boundary values. Moreover $\tilde{\tilde{z}} \in Z_1'(\mu)$.

Proof. The continuity of the initial values of $\tilde{v}$ follows from integrating the second equation of (2.4) over $\delta \leq \theta \leq \theta_1$ and $0 \leq t \leq s$ and then letting $\delta \to 0$. Obviously $\tilde{\tilde{u}}_s$ and $\tilde{\tilde{v}}_s$ are continuous in D and up to l except at the origin. $\tilde{z}$ and $\tilde{\tilde{z}}$ are solutions by lemma 2.1.

Proof of theorem 3. By theorem 2′ there corresponds to $\tilde{\tilde{z}}$ an analytic function $g \in \mathfrak{G}_1'(\mu)$, such that by (7.1)′

$$f(\pm s) = A s^{-\alpha} I_{1-\alpha}^{\alpha} \tilde{\tilde{u}}(\pm s) \tag{8.5}$$

holds. We now apply the differentiation process $s\dfrac{d}{ds}$ on the right hand side of (8.5) twice, and under the integral sign. We find by using (6.14),

$$\left(s\frac{d}{ds}\right)^2 f(\pm s) = A s^{-\alpha} I_{1-\alpha}^{\alpha} u(\pm s)\,. \tag{8.6}$$

Because by the assumption $u(\pm s)$ is Hoelder continuous with exponent μ, it follows from (8.6) that $s^{\alpha}\left(s\dfrac{d}{ds}\right)^2 f(\pm s)$ is Hoelder continuous with exponent $\mu + \alpha$. By the theorem of Privaloff [16], the initial values of the analytic function $\psi(W) \equiv W^{\alpha}\left(W\dfrac{d}{dW}\right)^2 g(W)$ are Hoelder continuous with the same exponent[11]), hence $\left(W\dfrac{d}{dW}\right)^2 g(W) \in \mathfrak{G}_1'(\mu)$. But by lemma 5.2 $B\left(W\dfrac{d}{dW}\right)^2 g(W) = z$. Hence z and consequently z^* belongs to $Z_1'(\mu)$, and the theorem is proved.

By an analogous reasoning one has

Theorem 3′. Let $v^*(W)$ be a solution of (1.2)′ of class C_2 in D and continuous up to l. Let the initial values $v^*(y)$ have the properties (a) $v^*(0) = 0$ and (b) $v^*(y)$ is Hoelder continuous with exponent $\mu + 2\alpha, 0 < \mu < 1 - 2\alpha$. Then the solution $z^* = u + iv^*$ belongs to the class $Z_1'(\mu)$ where u is the conjugate solution of v^* in (2.3) with $u(0) = 0$.

8.2. Analyticity. The following theorem has been proved by A. Huber, using method of axially symmetric potentials [9a]. On the basis of our main theorem we shall show that Huber's theorem is equivalent to theorems 3 and 3′.

[11]) For the function $\varphi(W) = e^{-i\alpha\pi} W^{\alpha} \psi(W)$ one has $\operatorname{Re}\varphi(\pm s) = O s^{2\alpha+\mu}$, $\Delta_h \operatorname{Re}\varphi(\pm s) = O s^{\alpha} h^{\alpha+\mu}$. By using a proof of Privaloff's theorem one shows that $\operatorname{Im}\varphi(\pm s)$ has the same properties as mentioned for $\operatorname{Re}\varphi(\pm s)$. One can then conclude that $W^{-\alpha}\varphi(W) \in H_{\alpha+\mu}$.

Theorem 4. Let $u(W)$ be a solution of (1.2) of class C_2 in D continuous up to l. Let $u(y) = 0$. Then $v^*(y)$ is regular analytic on l, and $u(W) = X^{1-2\alpha} \sum_0^\infty S_n(y) X^{2n}$, where the $S_n(y)$ are analytic in y.

Theorem 4'. Let $v^*(W)$ be a solution of (1.2) of class C_2 in D continuous up to l. Let $v^*(y) = 0$. Then $u(y)$ is regular analytic on l and $v^*(W) = X^{1+2\alpha} \sum_{n=0}^\infty T_n(y) X^{2n}$, where the $T_n(y)$ are analytic in y.

Proof. 1.) From theorem 3 follows theorem 4. Because of $u(y) = 0$ the assumption in theorem 3 is satisfied and hence the solution $z^* = u + iv^*$ belongs to $Z_1'(\mu)$ and the corresponding $g(W) \in \mathfrak{G}_1'(\mu)$ for any value of $0 < \mu < 1 - 2\alpha$. By (7.1) $f(y) = 0$, hence $\operatorname{Re} e^{-i\alpha\pi} W^{2\alpha} g(W) = 0$ on l. Therefore $g_1(y) = \cos \alpha\pi \mid y \mid^{-2\alpha} \Phi(y)$ where Φ is analytic in y and $\Phi(0) = 0$. Solving the second equation of (7.1)' one finds $v^*(y)$ analytic in y. For the series expansion of u see BERGMAN [1].

2.) From theorem 4 follows theorem 3. Let the given initial values of u be $u(y) = \varphi(y)$, and let $A s^{-\alpha} I_{1-\alpha}^\alpha \varphi(\pm s) = F(\pm s)$, so that $s^\alpha F(\pm s)$ is HOELDER continuous with exponent $\mu + \alpha$. One can construct an analytic function $G(W) = G_0 + iG_1$ in D (or in a subdomain D adjacent to l), such that $G_0 \pm \tan \alpha\pi G_1(\pm s) = F(\pm s)$. It follows that $G \in \mathfrak{G}_1'(\mu)$, and hence to G there corresponds a solution $Z^* = U + iV^* \in Z_1'(\mu)$. Since $u - U = 0$ on l, we have by theorem 4, $v^*(y) - V^*(y)$ analytic in y, hence $v^*(y)$ is HOELDER continuous with exponent $\mu + 2\alpha$, the same as does $V^*(y)$. From the analytical behavior of $(u - U)$ and of $(v^* - V^*)$ in D we conclude $z^* = u + iv^* \in Z_1'(\mu)$.

By using theorems 4, 4' and the main theorem in the same way as in the preceeding proof, we obtain the following extensions of theorems 3 and 3'.

Theorem 5. The assumptions on $u(W)$ are the same as in theorem 3, except that the HOELDER exponent of $u(y)$ is now either μ' with $1 - 2\alpha < \mu' < 1 - \alpha$ or μ'' with $1 - \alpha < \mu'' < 1$. Then $v^*(y)$ is differentiable, let $v_y^*(0) = c$. The solution $z^* = (u - (1 - 2\alpha)^{-1} c X^{1-2\alpha}) + i(v^* - cy)$ with $z^*(0) = 0$ belongs to $Z_2'(\mu')$ or to $Z_3'(\mu'')$.

Theorem 5'. The assumptions on $v^*(W)$ are the same as in theorem 3', except those on the initial value v^*. We assume now that $v^*(y)$ is a differentiable function with $v_y^*(0) = 0$, and $v_y^*(y)$ is HOELDER continuous either with exponent ν', $0 < \nu' < \alpha$, or with exponent ν'', $\alpha < \nu'' < 2\alpha$. Then the solution $z^* = u + iv^*$ with $z^*(0) = 0$ belongs to $Z_2'(\mu')$ or to $Z_3'(\mu'')$ with $\mu' = \nu' - 2\alpha + 1$, $\mu'' = \nu'' - 2\alpha + 1$.

8.3. HOELDER continuity of z^* in D. By the definition of H_μ § 6, functions z^* which belong to the classes Z_k' are HOELDER continuous along rays issued from the origin. The question naturally arises as to the HOELDER continuity of z^* in D. From theorems 3 and 3′ one immediately derives the following

Theorem 6. Let the initial value $u(y)$ (or $v^*(y)$) be HOELDER continuous with exponent μ, (or $\mu + 2\alpha$) with $0 < \mu < 1 - 2\alpha$. Then both $u(W)$ and $v^*(W)$ are HOELDER continuous respectively with exponents μ and $\mu + 2\alpha$ in any closed subdomain D' in D.

Proof. One may take any point P on l as the origin and apply theorems 3, 3′ to conclude that $z^*(W) - z^*(P)$ has the HOELDER continuity property along rays issued from P. It is easy to see that the HOELDER constants can be made independent of P.

§ 9. Mixed type equations

9.1. The results of § 7, 8 and 9 are now summarized and reformulated in terms of the solutions $U(x, y)$ of the equations

$$x^p U_{yy} + U_{xx} = 0\,, \qquad p > 0\,, \qquad \alpha = \frac{p}{2(p+2)}\,. \tag{9.1}$$

For brevity we omit similar statements on solutions V of the equation $x^q V_{yy} + V_{xx} = 0$ with $0 > q > -1$.

Let D be a domain in the half-plane $x > 0$ with the boundary segment l on the y-axis. We shall consider solutions $U(x, y)$ of class C_2 in D with continuous initial data $U(0, y) = U_0(y)$ and $U_x(0, y) = U_1(y)$.

In restating the previous theorems for U both the assumptions and conclusions on U are obtained from those on $u(X, y)$ where $X^{(1-2\alpha)} = (1 - 2\alpha)^{-1} x$, $u(X, y) = U(x, y)$, see (2.1). Since the transformation $(X, y) \to (x, y)$ changes the scale of X and retains the scale of y, the domain D will be mapped into a domain D in (X, y) plane with the same boundary segment l on the y-axis. We shall not reformulate HOELDER continuity properties in D or along rays issued from the origin in D[12]). Our main interest will be those continuity properties of the initial data. A function $\varphi(y)$ on l is said to belong to the class h_μ if it is HOELDER continuous in y with exponent μ, and $\varphi(0) = 0$.

1) Let U be a solution of class C_2 in D, and U and U_x are continuous up to l. Let (a) $U_0 \in h_\mu$, $U_1 \in h_{\mu+2\alpha-1}$ hold for $1 - 2\alpha < \mu < 1 - \alpha$. Then there exists an analytic function $g(W) = g_0 + i g_1$ in D, $W = y + iX$, such that (b)

[12]) The omission of these properties as required in the assumptions of theorems 2, 2′, 3 and 3′, is justified by theorems 5 and 5′.

$|y|^{\alpha} g(y) \in h_{\mu+\alpha}$, and with $f(\pm s) = g_0(\pm s) \pm \tan \alpha\pi g_1(\pm s)$ the following relations hold: (c) $s^{\alpha} f(\pm s) = A I^{\alpha}_{1-\alpha} U_0(\pm s)$, $s^{\alpha} g_1(\pm s) = \mp A_2 I^{1-\alpha}_{\alpha} U_1(\pm s)$, where $s = |y|$, $A = \Gamma(2\alpha)/\Gamma(\alpha)$, $A_2 = A^{-1} 4^{\alpha-1}(1-2\alpha)^{-2\alpha}$. Conversely, to each $g(W)$ in D with (b) there is a $U(x, y)$ in D with (a), such that (c) is true. For the notation I^{δ}_{β} see § 6.4.

2) A similar relation between the initial values of $g(W)$ and the data U_0, U_1 can be derived from (7.3)′, if the range of μ in 1) is changed to $1-\alpha<\mu<1$, and $U_0 \in h_{\mu}$, $U_1 \in h_{\mu+2\alpha-1}$, while the corresponding $g(W)$ has continuous derivatives on l, with $s^{\alpha} \dfrac{dg(\pm s)}{ds} \in h_{\mu+\alpha-1}$. Further correspondences between g and U can be established for U with continuous U_x, U_y on l, etc.

It is of interest to note that U can have a normal derivative on the initial line while the tangential derivative may not exist there. The situation is reversed if instead of U, solutions V are considered.

3) Reflection law. If U is of class C_2 in D and continuous up to l, with $U_0(y) \equiv 0$, then $U_1(y)$ is analytic in y. Moreover, for integers n let $x_n = x_0 \exp(2n\pi i/p + 2)$, then $U(x_n, y) = U(x_0, y) . \exp. (n\pi i/p + 2)$. If U_x is continuous up to l and $U_1(y) \equiv 0$, then $U_0(y)$ is analytic and moreover, $U(x_n, y) = U(x_0, y)$. This generalizes the classical reflection principle of harmonic functions ($p = 0$), by extending the argument x to complex values.

4) If U is of class C_2 in D and continuous up to l, with $U_0 \in h_{\mu}$ with $1-2\alpha < \mu < 1-\alpha$ or $1-\alpha < \mu < 1$, then U_x is continuous up to l and $U_1 \in h_{\mu+2\alpha-1}$. The converse is also true.

9.2. Equations (9.1) with positive odd integer p. In this case the equation is hyperbolic for $x < 0$[13]. Consider a solution $U(x, y)$ of the hyperbolic equation with U and U_x continuous up to l, $U(o, y) = U_0(y)$, $U_x(o, y) = U_1(y)$. The question as to whether or not this solution can be continuously extended to the elliptic part of the equation with continuous normal derivative, can be answered by our results characterizing the initial data in 9.1, However there is another way of describing the continuation across the initial line which makes direct use of the analytic extension of $U(x, y)$ in the elliptic part of the solution.

Let us denote by $U_e(x, y)$ and $U_h(x, y)$ respectively the elliptic and hyperbolic parts of the solution. We say that U_e and U_h are continuation of each other, if they and their x-derivatives are continuous across l, with the common initial data $U_0(y)$ and $U_1(y)$. For the consideration which follows, we use X as in (2.1) for $x > 0$, and similarly use X_1 with $X_1^{1-2\alpha} = (1-2\alpha)^{-1}(-x)$ for $x < 0$, so that $U_e(x, y) = u_e(X, y)$ and $U_h(x, y) = u_h(X_1, y)$. We have

[13]) For hyperbolic equations with a singular line cf. HELWIG [8], for the general theory of hyperbolic equations with analytic coefficients cf. LERAY [13].

$$\frac{\partial^2 u_e}{\partial X^2} + \frac{\partial^2 u_e}{dy^2} + \frac{2\alpha}{X}\frac{\partial u_e}{\partial X} = 0\,, \tag{9.1}$$

and

$$-\frac{\partial^2 u_h}{\partial X_1^2} + \frac{\partial^2 u_h}{\partial y^2} - \frac{2\alpha}{X_1}\frac{\partial u_h}{\partial X_1} = 0\,. \tag{9.2}$$

Recall the analytic extension of $u_e(X, y)$ as given by $\Omega(Z, W)$ in § 3. It is analytic in a domain $D^* = D \times \overline{D}$. If both $-\overline{Z}$ and W vary independently along the boundary segment l, then they describe a boundary surface $l^* = l \times l$ of D^*. Let l be given by $y_0 > y > y_1$, a point on l^* may be denoted by (σ, τ) with $-\sigma$ and τ both varying between y_0 and y_1. On l^*, Ω will have the boundary values $\Omega(\sigma, \tau)$. From the relation $\Omega(Z,W) = u_e(\tilde{X}, \tilde{Y})$ in § 3, we can also express the boundary values on l^* by

$$\Omega(\sigma, \tau) = u_e(iX', y) \text{ with } X' = -\tfrac{1}{2}(\sigma + \tau),\ y = \tfrac{1}{2}(\tau - \sigma). \tag{9.3}$$

We know the explicit expression of the boundary values in terms of $U_0(y)$ and $U_1(y)$ for at least one case, namely, when $Z = 0$ and $W = \tau$ varies along l. In this case, indeed, $\Omega(o, \tau) = g(\tau)$ where $g(W)$ is the analytic function which corresponds to u_e by the operator L. Since the origin may be chosen anywhere on l, we can derive from the expression of $g(y)$ in (7.2)$'$ or in 1) § 9.1, the general formula for $\Omega(\sigma, \tau)$ as follows:

$$\begin{aligned} u_e(iX', y) = {} & (iX')^k C_1 \int_0^1 t^{-\alpha}(1-t)^{-\alpha} U_1(y + (2t-1)X')\, dt \\ & + C_0 \int_0^1 t^{\alpha-1}(1-t)^{\alpha-1} U_0(y + (2t-1)X')\, dt\,, \end{aligned} \tag{9.4}$$

with $k = (1 - 2\alpha)$, $C_1 = \left(\frac{1}{k}\right)^k \frac{\Gamma(2-2\alpha)}{\Gamma^2(1-\alpha)}$ and $C_0 = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)}$.

One can also verify (9.4) by the following reasoning. $u_e(iX', y)$ satisfies (9.2) with X_1 replaced by X'. Moreover $u_e(o, y) = U_0(y)$ and $(iX'/k)^2 \frac{\partial u_e}{\partial X'} \to U_1(y)$ as $X' \to 0$. The initial value problem of (9.2) is then solved by (9.4).

Since $u_h(X_1, y)$ satisfies (9.2) with the conditions that $u_h(o, y) = U_0(y)$ and $(X_1/k)^2 \frac{\partial u_h}{\partial X_1} \to -U_1(y)$ as $X_1 \to 0$, so after identifying X' with X_1 we find $u_e(iX_1, y)$ and $u_h(X_1, y)$ both satisfy (9.2) with initial conditions differing by a constant factor. Hence it follows

$$u_h(X_1, y) = \operatorname{Re} u_e(iX_1, y) - (\sec \alpha\pi + \tan \alpha\pi) \operatorname{Im} u_e(iX_1, y)\,. \tag{9.5}$$

As an application of the above formula we prove the following statement: If $u_h = 0$ along the characteristic $X_1 + y = 0$, $0 > y > \frac{1}{2}y_1$ and if u_h can be continued to the elliptic part of the domain continuously with continuous normal derivative, then $U_0(y)$ and $U_1(y)$ are analytic in y for $0 > y > y$.

For the proof one notes that $u_h = 0$ along the characteristic implies, since $u_e(-iy, y) = g(2y)$, that a linear combination of $g_0(y)$ and $g_1(y)$ is zero on l, hence $g_0(y)$ and $g_1(y)$ are analytic in y for $0 > y > y_1$. Therefore the same holds for $U_0(y)$ and $U_1(y)$.

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