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On a Class of Null-Bounded RIEMANN Surfaces[1])

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Let W be an open RIEMANN surface. Let F denote the space consisting of those harmonic functions u on W for which $*du$ is semi-exact, that is to say, has no periods around dividing cycles. As usual, we let FB and FD denote those subclasses of functions which are bounded or which have a finite DIRICHLET integral, respectively. We say that W belongs to the class O_{FD} if every function on W of class FD is constant. In the preceeding paper it was shown that an analogue of the RIEMANN-ROCH theoıem holds for surfaces of class O_{FD} and that in some respects this is the natural class of surfaces for this generalization of the RIEMANN-ROCH theorem.

Since every analytic function belongs to F, we have trivially $O_{HD} \subset O_{FD} \subset O_{AD}$. It is the purpose of the present paper to investigate some of the properties of the class O_{FD}, and to show the relationship of O_{FD} to O_{HD} and O_{AD} and to AD nullsets on compact surfaces. On occasion we mention properties of O_{FB} when these are similar to those of O_{FD}.

Functions of class F have been considered by SARIO [17] (who denotes them by K). SARIO uses an "extremal method" and proves that $O_{FD} = O_{FBD}$ and that these surfaces are characterized by the vanishing of the Q-span of W. Our present investigation of the class O_{FD} is based upon "DIRICHLET principle" methods, and neither needs nor is able to derive the results of SARIO. Thus the present investigation may be regarded as complementary to that of SARIO.

Let M denote the space of function f on W such that df vanishes identically outside a compact set. Then we show that $W \in O_{FD}$ if and only if W has the property that for each function f on W with a finite DIRICHLET integral there is a function g in M such that $D(f - g) < \varepsilon$. Using this characterization we easily prove that the class O_{FD} is preserved under quasiconformal mappings.

For surfaces of finite genus we show that the classes O_{FD} and O_{AD} coincide and that W is of class O_{AD} if and only if it is the complement in a compact surface of an AD nullset. Similarly, O_{FB} and O_{AB} coincide for surfaces of finite genus with those regions which are the complements on a compact surface of an AB nullset. In view of the invariance of O_{FD} with respect to quasiconformal mapping we obtain the corollary that for surfaces of finite genus the class O_{AD} is preserved under quasiconformal mapping. This is

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somewhat suprising, since MORI [7] has given an example of two surfaces (of infinite genus, of course) which are quasiconformally equivalent and such that one belongs to O_{AD} and the other does not. Moreover, the results of AHLFORS and BEURLING enable one to construct two quasiconformally equivalent plane domains such that one is of class O_{AB} while the other is not. This shows that the class O_{FB} is not preserved under quasiconformal equivalence.

In section 2 I have included some results on the classes O_{HD} and O_G which I believe give better insight into the methods used for the class O_{FD} in the first section.

It will be convenient to assume that the functions with which we deal are complex valued. Consequently, we must define the mixed DIRICHLET integral $D(f, g)$ as $\iint df \wedge * d\bar{g}$. We set $D(f) = D(f, f)$. Two functions f and g are said to be orthogonal if $D(f, g) = 0$, and a sequence f_n will be said to converge to f in the sense of the DIRICHLET integral if $D(f_n - f) \to 0$.

1. The class O_{FD} and property P. As before we say that a differentiable function f on W is of class M if df vanishes identically outside some compact set. Equivalently, f belongs to M if there is a compact region Ω such that f is constant on each component of the complement of Ω. At one point in section 3 we shall find it convenient to assume that M contains in addition to the differentiable functions also functions which have generalized square integrable derivatives in the sense of FRIEDRICHS. This extra generality poses no essential difficulties, and in general we shall ignore it where it is not relevant.

Let $\bar{M}$ denote those functions on W which have a finite DIRICHLET integral and are such that for each $\varepsilon > 0$ there is a function g in M such that $D(f - g) < \varepsilon$. Note that the harmonic functions in $\bar{M}$ form the space HM of harmonic measures in the sense of AHLFORS [3]. We begin with the following lemma, which is contained implicitly in [3]:

Lemma 1. On an arbitrary RIEMANN surface the spaces $\bar{M}$ and FD are the orthogonal complements of each other in the space D of all functions on W with a finite DIRICHLET integral.

Proof. Let $u \in D$, and suppose $D(u, f) = 0$ for all $f \in M$. Since M contains all smooth functions which vanish outside a compact set, u must be harmonic by the WEYL lemma. We must show that $*du$ is semi-exact. Let C be a dividing cycle on W. Then there is a domain R consisting of a finite number of annuli such that the sum of the inner boundaries is homologous to C. Let f be a smooth function which is identically constant in the complement of R and increases from zero to one in each annulus as we go from the inner boundary to the outer. Then $f \in M$, and $0 = D(f, u) = \iint df \wedge * d\bar{u} = \int_C * d\bar{u}$. Thus $* d\bar{u}$ is semi-exact. But this implies that $* du$ is semi-exact, proving that the orthogonal complement of M is contained in FD.

On the other hand, suppose that $f \in M$ and $u \in F$. Let Ω be a compact region bounded by smooth Jordan curves such that f is constant in each component of the complement of Ω. Let Γ be the boundary of Ω, and Γ_i the boundaries of the components of the complement of Ω. Then $\Gamma = \cup \Gamma_i$, and f is constant on each Γ_i. Thus

$$D(f, u) = \iint df \wedge * d\bar{u} = \int_\Gamma f * d\bar{u} = \Sigma c_i \int_{\Gamma_i} * d\bar{u} = 0 ,$$

since each Γ_i is a dividing cycle and $* du$ is semi-exact. Thus M is orthogonal to FD, and the orthogonal complement of M contains FD.

Thus FD is the orthogonal complement of M. Since $\bar{M}$ is the closure of M with respect to the Dirichlet integral, we see that $\bar{M}$ and FD are orthogonal complements of one another, proving the lemma.

We say that a Riemann surface has property P if $\bar{M} = D$, i. e. if every function with a finite Dirichlet integral can be approximated arbitrarily well in the sense of the Dirichlet integral by functions in M. The following proposition is an immediate consequence of our lemma:

Proposition 1. A Riemann surface W is of class O_{FD} if and only if it has property P.

By a semi-exact differential we mean a closed differential whose periods around each dividing cycle is zero. Proposition 2 expresses a useful integration formula for semi-exact differentials on a surface of class O_{FD}. For the proof we shall need the following lemma, whose elementary proof we omit:

Lemma 2. On the Riemann surface W let Ω be a compact region (i. e. connected open set with compact closure) whose boundary Γ consists of a finite number of smooth Jordan curves, and let α be a closed differential defined in a neighborhood of Γ and satisfying $\int_\Gamma \alpha = 0$. Then we can extend α to be a closed differential in all of Ω.

Proposition 2. Let O be an open set with a compact closure on a Riemann surface W of class O_{FD}. Let the boundary Γ of O consist of a finite number of smooth Jordan curves. Let f be a smooth function with a finite Dirichlet integral in the complement $\tilde{O}$ of O and α a semi-exact differential which is continuous and square integrable in $\tilde{O}$. Then

$$\iint_{\tilde{O}} df \wedge \alpha = - \int_\Gamma f \alpha .$$

Proof. Since α is semi-exact, its integral around the boundary of each component of $\tilde{O}$ vanishes, and so we may extend α by lemma 2 to be a closed square integrable differential on all of W. Let us also extend f to be a smooth function on all of W. Then $\int\limits_{\Gamma} f\alpha = \iint\limits_{O} df \wedge \alpha$ by STOKES' theorem, and our proposition is equivalent to proving that $\iint\limits_{W} df \wedge \alpha = 0$.

Let g be a function in M, and let Ω be a compact region containing $\bar{O}$ and bounded by a finite number of smooth JORDAN curves such that g is constant on each component of $\tilde{\Omega}$. Then g is a constant c_i on the boundary C_i of a component of $\tilde{\Omega}$, and

$$\iint\limits_{W} dg \wedge \alpha = \iint\limits_{\Omega} dg \wedge \alpha = \Sigma c_i \int\limits_{C_i} \alpha = 0 ,$$

since α is semi-exact in $\tilde{O}$.

Thus $\iint\limits_{W} dg \wedge \alpha = 0$ for each $g \in M$. Since $W \in O_{FD}$, there is a $g \in M$ such that $D(f - g) < \varepsilon^2$, and we have

$$| \iint\limits_{W} df \wedge \alpha | = | \iint d(f - g) \wedge \alpha | < \varepsilon \, || \alpha ||$$

where $|| \alpha ||^2 = \iint\limits_{W} \alpha * \bar{\alpha}$. Since ε is arbitrary, we have $\iint df \wedge \alpha = 0$, proving the proposition.

2. The classes O_{HD} and O_G. We establish in this section some analogues for O_{HD} and O_G surfaces of the two propositions in the last section. The results derived here are fairly well known, but we include them for comparison. Let K be the class of functions which vanish outside a compact set, and denote[2]) by $\bar{K}$ those functions f such that for each $\varepsilon > 0$, there is a $g \in K$ such that $D(f - g) < \varepsilon$. An immediate consequence of the WEYL lemma is the following lemma:

Lemma 3. On an arbitrary RIEMANN surface the space $\bar{K}$ and the space HD of harmonic functions with a finite DIRICHLET integral are the orthogonal complements of one another in the space D.

We say that a RIEMANN surface has property P_0 if $\bar{K} = D$, i. e. if every function with a finite DIRICHLET integral can be approximated in the sense of the DIRICHLET integral by a function in K. We say that $W \in O_{HD}$ if the space HD contains only constants. Lemma 3 implies the following proposition which is analogous to Proposition 1:

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[2]) This space $\bar{K}$ differs slightly from the one introduced in [14], since we make no provision here for the exclusion of the constants. The space $\bar{K}$ as defined here always includes the constants.

Proposition 3. A Riemann surface W is of class O_{HD} if and only if it has property P_o.

We denote by O_G the class of parabolic Riemann surfaces. There are various equivalent definitions of parabolic surfaces (cf. [10] or [13]). Parabolic surfaces are characterized by the following property which we may use here as a definition: The Riemann surface W is parabolic if and only if there is a compact region $\Omega \subset W$ such that every function which is harmonic and has a finite Dirichlet integral in the complement of $\tilde{\Omega}$ and which vanishes on the boundary of Ω vanishes identically. Let $K(\Omega)$ denote the class of functions which vanish outside some compact set and which also vanish on Ω. Let $\bar{K}(\Omega)$ denote the closure of $K(\Omega)$ in the sense of the Dirichlet integral. Then we have easily the following lemma:

Lemma 4. The space $\bar{K}(\Omega)$ and the space of functions in $\tilde{\Omega}$ which are harmonic and have a finite Dirichlet integral and which vanish on the boundary of Ω are the orthogonal complements of one another in the space of all functions in $\tilde{\Omega}$ which have a finite Dirichlet integral and vanish on the boundary of Ω.

We say that W has the property $P_{oo}(\Omega)$ if all functions in the complement of Ω which have a finite Dirichlet integral and which vanish on the boundary of Ω belong to $\bar{K}(\Omega)$. If W has the property $P_{oo}(\Omega)$ for some compact region Ω, then clearly it has the property $P_{oo}(\Omega_1)$ for each compact region Ω_1 which either contains or is contained in Ω. Hence the property P_{oo} does not depend on the region Ω chosen, and we indicate it simply by P_{oo}. Lemma 4 implies the following proposition:

Proposition 4. A Riemann surface is parabolic if and only if it has the property P_{oo}.

Proposition 5. Let W be a parabolic Riemann surface, and O an open set whose closure is compact and whose boundary Γ consists of a finite number of smooth Jordan curves. Let f be a smooth function defined on the complement $\tilde{O}$ of O and having a finite Dirichlet integral. Let α be a closed differential which is continuous and square integrable over $\tilde{O}$. Then $\int_\Gamma f\alpha = -\iint_{\tilde{O}} df \wedge \alpha$.

Proof. Let Ω be a compact region containing $\bar{O}$, and let the boundary of Ω be C. Then

$$\int_\Gamma f\alpha + \iint_{\tilde{O}} df \wedge \alpha = \int_C f\alpha + \iint_{\tilde{\Omega}} df \wedge \alpha ,$$

by STOKES' theorem, and so it suffices to prove that

$$\int_C f\alpha = -\iint_{\tilde{\Omega}} df \wedge \alpha .$$

Since this equality holds for functions which vanish outside some compact set, and every f differs from one which vanishes on C by a function which vanishes outside a compact set, we see that it suffices to consider the case in which f is identically zero on C. Since W is parabolic, it has property $P_{00}(\Omega)$, and each f which vanishes on C and has a finite DIRICHLET integral can be approximated by functions g which also vanish outside compact sets. For these g we have $\iint dg \wedge \alpha = 0$ by STOKES' theorem, whence $\iint df \wedge \alpha = 0$, proving the proposition.

It should also be noted that the proposition also holds under the hypothesis that $W \in O_{HD}$ if we require α to satisfy $\int_\Gamma \alpha = 0$.

3. Properties of the ideal boundary and behavior under quasiconformal mapping. We say that two RIEMANN surfaces W and W' are quasiconformally equivalent if there is a one-to-one quasiconformal mapping φ of W onto W'. We use here only the following property of quasiconformal mappings:

Lemma 5. Let φ be a quasiconformal mapping of W onto W'. Then the adjoint mapping takes the space of functions with a finite DIRICHLET integral on W' onto the space of functions with a finite DIRICHLET integral on W, and we have

$$\frac{1}{K} D(u) \leqslant D(u \circ \varphi) \leqslant K D(u) .$$

For differentiable mappings this lemma follows directly from the definition of quasiconformality. For general quasiconformal mappings it is easily established using generalized derivatives (cf. [6] and [8]).

Using this lemma we see that the adjoint mapping φ^* defined by $\varphi^*(u) = u \circ \varphi$ maps the space $M(W')$ onto $M(W)$ and that if every function on W' with a finite DIRICHLET integral can be approximated in the sense of the DIRICHLET integral on W' by a function in $M(W')$ then every function W with a finite DIRICHLET integral can be approximated in the sense of the DIRICHLET integral on W by a function in $M(W)$. Thus if W' has property P, so does W. This implies by Proposition 1 that if $W' \in O_{FD}$ then $W \in O_{FD}$. Using a similar argument involving the properties P_0 and P_{00}, we have the following theorem:

Theorem 1. *The classes O_{FD}, O_{HD}, and O_G are preserved under quasiconformal equivalence.*

This property for O_G is due to PFLUGER [11]. For O_{HD} see [15]. MORI [7] has given an example which shows that the classes O_{AB} and O_{AD} are not preserved under quasiconformal equivalence. The behavior of O_{HB} under quasiconformal mapping remains open.

Following AHLFORS [1], we say that two RIEMANN surfaces W and W' have the same ideal boundary if there are open sets $O \subset W$ and $O' \subset W'$ with compact closures such that $W \sim \bar{O}$ and $W' \sim \bar{O}'$ are conformally equivalent. A property of RIEMANN surfaces is said to be a property of the ideal boundary if whenever two surfaces have the same ideal boundary either both possess the property or else neither possesses it. We shall show that the property of belonging to the class O_{FD} is a property of the ideal boundary. Let O be an open set on W with a compact closure and bounded by a finite number of smooth JORDAN curves. We shall say that property I holds for O if the conclusion of Proposition 2 holds for every f and α satisfying the hypothesis of the proposition. Thus Proposition 2 says that property I holds for every O on W if W is of class O_{FD}. Suppose, on the other hand, that property I holds for some O on W. Then if $u \in FD$, we have $*d\bar{u}$ semi-exact, and so

$$\iint_{\tilde{O}} du * d\bar{u} = -\int u * d\bar{u}$$
$$= -\iint_{O} du * d\bar{u} .$$

Thus $D(u) = 0$, and u is a constant, whence $W \in O_{FD}$. Suppose now that W and W' are such that $W \sim \bar{O}$ and $W' \sim \bar{O}'$ are conformally equivalent. Then if $W \in O_{FD}$, property I must hold on W for O. Since this is in reality a property of $W \sim \bar{O}$, it must hold on W' for O'. Thus $W' \in O_{FD}$, and we have shown that the property of belonging to the class O_{FD} is a property of the ideal boundary. A similar argument shows that the properties of belonging to O_G and O_{HD} are properties of the ideal boundary (cf. [13]). Thus we have the following proposition:

Proposition 6. The properties of belonging to the classes O_{FD}, O_{HD} and O_G are all properties of the ideal boundary.

4. Planar surfaces and function-theoretic nullsets. We shall refer to surfaces of genus zero (i. e. surfaces on which every cycle is a dividing cycle) as planar surfaces. According to the KOEBE uniformization theorem, each planar surface is conformally equivalent to a domain in the complex plane, and we shall often find it convenient to speak of properties of the complementary set of the plane domain. Throughout this section we shall use E to denote a compact set in the plane whose complement is connected. We shall call E an AD (or AB) nullset if some open set O containing E has the property that each function f which is analytic in $O \sim E$ and has a finite DIRICHLET integral in $O \sim E$ (or which is bounded in $O \sim E$) may be extended to E in such a manner that it is analytic in O. Note that if this property holds for some open set containing E then it holds for each larger open set containing E, and as a consequence of Proposition 7 it holds for every open set containing E if it holds for the complex sphere. Thus the property of being an AD or AB nullset does not depend on the open set O used in its definition.

These nullsets have been extensively studied by AHLFORS and BEURLING [4] and by SARIO [16] and [17]. We give here a few elementary properties which will be useful in the sequel. The following lemma plays a fundamental role in our treatment:

Lemma 6. Let E be a compact set in the complex sphere S, and let O be an open set containing E. Then each analytic function f in $O \sim E$ can be expressed as $f = f_1 - f_2$, where f_1 is analytic in O and f_2 is analytic in $S \sim E$.

Proof. Let O_1 be an open set containing E, whose closure is contained in O, and whose boundary Γ consists of a finite number of smooth JORDAN curves. Define

$$f_1(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta & z \in O_1 \\[2ex] \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta + f(z) & z \in O \sim \bar{O}_1\,, \end{cases}$$

and

$$f_2(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) & z \in O_1 \sim E \\[2ex] \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta & z \in S \sim \bar{O}_1\,. \end{cases}$$

We see that the functions f_1 and f_2 are independent of the choice of Γ in virtue of the CAUCHY integral theorem. Thus f_1 and f_2 can be defined on Γ so that they are analytic across Γ, and we see that f_1 is analytic in O and f_2 is analytic in $S \sim E$. Since $f = f_1 - f_2$, the lemma is established.

If f has a finite DIRICHLET integral in $O \sim E$, we see that f_1 has a finite DIRICHLET integral in O and f_2 has a finite DIRICHLET integral in $S \sim E$. Similarly, if f is bounded, then so are f_1 and f_2.

If E is such that the complement of E in S is of class O_{AD}, and if f has a finite DIRICHLET integral in $O \sim E$, then f_2 has a finite DIRICHLET integral over $S \sim E$, and so must be constant. Hence f differs from f_1 by a constant and therefore admits of an analytic extension to all of O. Consequently E is an AD nullset. But if E is an AD nullset and f an analytic function in $S \sim E$

with a finite DIRICHLET integral, then f can be extended to be analytic in all of S and so must be constant. Thus we have shown that E is an AD nullset if and only if its complement is of class O_{AD}. Since a similar consideration applies to bounded functions, we have the following proposition [4]:

Proposition 7. A compact set E in the complex sphere is an AD (or AB) nullset if and only if its complement is of class O_{AD} (or O_{AB}).

Let E be a compact set on the sphere. A point p of E is said to belong to the AD kernel of E if there is an analytic function defined in the complement of E in some neighborhood of p which has a finite DIRICHLET integral and which is singular at p, i. e. admits of no extension which is analytic at p. One sees that the AD kernel of E is a perfect set contained in E and is empty if and only if E is an AD nullset. Moreover, the intersection of each open disk with E is either empty or else is not an AD nullset. Thus if E is not an AD nullset, we can express E as the union of N disjoint closed sets none of which is an AD nullset. Since similar considerations apply for the class of AB functions, we have the following lemma:

Lemma 7. Let E be a compact set in the plane which is not an AD (or AB) nullset. Then for each integer N, we may express E as the union of N disjoint closed sets none of which is an AD (or AB) nullset.

5. RIEMANN surfaces of finite genus. We say that a RIEMANN surface W has finite genus if there are a finite number of cycles $C_1, \ldots, C_g$ such that each cycle on W which does not intersect any of the C_k is a dividing cycle. Let Ω be a compact region containing the C_k. Then $\tilde{\Omega}$ is a planar surface and so is conformally equivalent to a plane domain G. The boundary contours of Ω correspond to a finite number of boundary continua $\Gamma_1, \ldots, \Gamma_n$ of G, and we may choose G (by performing an auxiliary conformal mapping, if necessary) so that these continua are simple analytic curves. Call the union of these continua Γ. Then W can be constructed by means of a suitable identification between the points of Γ and the boundary of Ω. Let G_0 be that component of the complement of Γ which contains G. Then G_0 is a domain bounded by a finite number of JORDAN curves, and if we make our previous identification of these curves with the boundary of Ω, we obtain a compact RIEMANN surface W_0 which contains W as a subdomain. If we attach to G the components of the complement of Γ which do not contain G, we obtain a planar surface which has the same ideal boundary as W in the sense of AHLFORS. We have thus established the following lemma:

Lemma 8. Every RIEMANN surface of finite genus is conformally equivalent to a subregion of a compact RIEMANN surface. For every RIEMANN surface of finite genus there is a planar surface with the same ideal boundary.

Since a plane domain is of class O_G or O_{HD} if and only if its complement has capacity zero [9], and the properties of belonging to the class O_G or O_{HD} are properties of the ideal boundary, we have the following corollary, where we make use of the inclusion $O_G \subset O_{HB} \subset O_{HD}$:

Corollary. For a RIEMANN surface of finite genus the classes O_G, O_{HB}, and O_{HD} coincide and are characterized as those surfaces which are obtained from compact surfaces by the removal of a set of capacity zero.

The principal goal of this section is to show that a similar situation holds for RIEMANN surfaces of finite genus of class O_{AD} and O_{AB}, namely that these surfaces are obtained from compact surfaces by the removal of an AD or AB nullset. This result, and lemma 7 as well, seems to be a part of the folklore of open RIEMANN surfaces, but since I have not seen a proof, I include one here for completeness.

Unfortunately, lemma 6 becomes false if we replace S by a compact surface of positive genus. Hence we begin by constructing an analogue of lemma 6. Let W be a compact RIEMANN surface and Ω a region on W. Let f be a function on Ω, and define the differential $\bar{\partial} f$ to be $\frac{\partial f}{\partial \bar{z}}\, d\bar{z}$ in terms of a local uniformizer z. We say that the function f is *semi-analytic* if $\bar{\partial} f$ is equal in Ω to the conjugate of an everywhere analytic differential on W. Since the analytic functions on Ω are characterized by $\bar{\partial} f = 0$, we see that the analytic functions are semi-analytic. Since the LAPLACE operator can be expressed as $4 \frac{\partial^2}{\partial z\, \partial \bar{z}}$, we see that the semi-analytic functions are harmonic in Ω.

On the compact surface W, let $G(p, p_0; q, q_0)$ be the fundamental potential, i. e. that function of p which is harmonic except at q and q_0 where it has the singularities $-\log |z(p) - z(q)|$ and $\log |z(p) - z(q_0)|$, respectively, and which is normalized by $G(p_0, p_0; q, q_0) = 0$. Then $G(p, p_0; q, q_0) = G(q, q_0; p, p_0)$, and, in its dependence on q, $\partial_q G$ is an analytic differential except at p and p_0, where it has simple poles with residues $+\frac{1}{2}$ and $-\frac{1}{2}$. Moreover, $\bar{\partial}_p \partial_q G$ is, in its dependence on p, the conjugate of an everywhere analytic differential on W. With these preliminaries we are able to prove the following lemma:

Lemma 9. Let W be a compact RIEMANN surface and E a closed subset of W. Let O be an open set containing E and f an analytic function in $O \sim E$.

Then we may express f as $f = f_1 - f_2$, where f_1 is semi-analytic in O and f_2 semi-analytic in $W \sim E$.

Proof. Let O_1 be on open set containing E whose closure is contained in O and whose boundary Γ consists of a finite number of smooth JORDAN curves. Let p_0 be a point in $W \sim \bar{O}$, and write $G(p, q)$ for $G(p, p_0; q, q_0)$. Define

$$f_1(p) = \begin{cases} \dfrac{1}{\pi i} \displaystyle\int_\Gamma f(q)\, \partial_q G(p, q) & p \in O_1 \\ \dfrac{1}{\pi i} \displaystyle\int_\Gamma f(q)\, \partial_q G(p, q) + f(p) & p \in O \sim \bar{O}_1 \end{cases}$$

$$f_2(p) = \begin{cases} \dfrac{1}{\pi i} \displaystyle\int_\Gamma f(q)\, \partial_q G(p, q) - f(p) & p \in O_1 \sim E \\ \dfrac{1}{\pi i} \displaystyle\int_\Gamma f(q)\, \partial_q G(p, q) & p \in W \sim \bar{O}_1 . \end{cases}$$

It follows from the CAUCHY theorem that the definitions of f_1 and f_2 are independent of the choice of Γ, (i. e. of O_1) and so they can be defined on Γ so that they are smooth in O and $W \sim E$, respectively. Since $\bar{\partial} f = 0$, we have $\bar{\partial} f_1 = \dfrac{1}{\pi i} \int_\Gamma f(q)\, \bar{\partial}_p \partial_q G$. But $\bar{\partial}_p \partial_q G$ is, in its dependence on p, the conjugate of an everywhere analytic differential on W. Thus $\bar{\partial} f_1$ is also the conjugate of an everywhere analytic differential on W, and so f_1 is semi-analytic in O. Similarly, f_2 is semi-analytic in $W \sim E$.

Corollary. The functions f_1 and f_2 in lemma **9** can be taken to be analytic if and only if for some (and hence for every) curve Γ which separates E from the complement of O, we have $\int_\Gamma f \alpha = 0$ for every differential α which is everywhere analytic on W.

Proof. If $\int_\Gamma f \alpha = 0$ for every analytic α, then the functions f_1 and f_2 constructed in the proof of lemma **9** are actually analytic, since $\bar{\partial}_p \partial_q G$ is an analytic differential in q. If on the other hand we can choose f_1 and f_2 to be analytic, then $\int_\Gamma f_1 \alpha = 0$ by applying the CAUCHY theorem to the open set O, while $\int_\Gamma f_2 \alpha = 0$ by applying the CAUCHY theorem to $W \sim E$. Thus $\int_\Gamma f \alpha = 0$, proving the corollary.

Since the definition we gave in the last section for E to be an AD or AB

nullset depends only on the relation of E to a neighborhood of E, it carries over unchanged to sets E on a compact RIEMANN surface. We formulate the following theorem:

Theorem 2. *For RIEMANN surfaces of finite genus the classes O_{AD} and O_{FD} coincide and consist of those surfaces which are obtained by deleting an AD nullset from a compact surface. Similarly, the classes O_{AB} and O_{FD} coincide for surfaces of finite genus and consist of precisely those surfaces which are obtained by deleting an AB nullset from a compact surface.*

Proof. It is trivial that surfaces obtained by deleting an AD (or AB) nullset from a compact surface are of class O_{AD} (or O_{AB}). Suppose that E is an AD nullset, and that h is a function of class FD in $W \sim E$. Let u be the real part of h and let Ω be a region which contains E and has the property that each cycle in Ω divides on $W \sim E$. The existence of such a domain follows from the fact that we can find a compact set in $W \sim E$ which carries a basis for the non-dividing cycles of $W \sim E$. The fact that $^*du$ is semi-exact implies that u is the real part of an analytic function f in $\Omega \sim E$. Since the DIRICHLET integral of f is twice that of u, f has a finite DIRICHLET integral and hence is regular on E. Thus u is regular on E and so must be constant, since it is harmonic on all of W. Similarly, the imaginary part of h is constant, and $W \sim E$ is of class O_{FD}.

Let E be an AB nullset on W and Ω as before. Let h be of class FB in $W \sim E$, and let u be the real part of h. Then as before u is the real part of a function f which is analytic in $\Omega \sim E$. If m is a bound for u, then the function $(f - 2m)^{-1}$ is a bounded analytic function in $\Omega \sim E$ and so must be regular on E. Thus u is regular on E and therefore constant, since it is everywhere harmonic. Similarly, the imaginary part of h is constant, and $W \sim E$ is of class O_{FB}.

This shows that surfaces obtained by removing an AD (or AB) nullset from a compact surface are of class O_{FD} (or O_{FB}). Since $O_{FB} \subset O_{AB}$ and $O_{FD} \subset O_{AD}$, our theorem will be proved if we show that every surface of finite genus of class O_{AD} (or O_{AB}) is obtained by removing an AD (or AB) null set from a compact surface.

Suppose that we have a surface of finite genus of class O_{AD}. Then by lemma 8 it can be obtained from a compact surface W by the deletion of a closed set E. Let W have genus g, and suppose that E is not an AD nullset. Then by lemma 7 we can find $g + 1$ disjoint closed subsets $E_1, \ldots, E_{g+1}$ of E, none of which is an AD nullset. Let f_k be a function which is analytic and has a finite DIRICHLET integral in some neighborhood Ω_k of E_k but which is not regular on E_k. By lemma 9 we can find a function h_k which is semi-analytic in $W \sim E_k$ and which differes from f_k by a function which is semi-analytic in a neighborhood of E_k. Thus h_k has a finite DIRICHLET integral in $W \sim E_k$ and can not be extended to a harmonic function in all of Ω_k, since such an extension would give us a harmonic extension of f_k to all of Ω_k, and this harmonic function would be analytic in Ω_k, since it is analytic in $\Omega_k \sim E$.

The number of everywhere analytic differentials on W is g. Hence there are constants $c_1, \ldots, c_{g+1}$, not all zero, such that, if we set $h = \Sigma c_k h_k$, then $\bar{\partial} h = 0$, i. e. h is analytic. If $c_k \neq 0$, then h does not admit a harmonic extension to E_k, since h_k does not and the functions h_j, $j \neq k$, are harmonic on E_k. Thus h is a non-constant analytic function with a finite DIRICHLET integral in $W \sim E$, and so $W \sim E$ is not of class O_{AD}. This shows that if $W \in O_{AD}$, then E is an AD nullset. A similar proof applies to the AB case, proving the theorem.

Since the class O_{FD} is preserved under quasiconformal mappings, we obtain as an immediate corollary the following theorem:

Theorem 3. *For* RIEMANN *surfaces of finite genus the class O_{AD} is preserved under quasiconformal equivalence.*

MORI [7] has given an example which shows that the above theorem is false if one omits the hypothesis that the surfaces are of finite genus. Since O_{FD} is always preserved under quasiconformal equivalence, and one of the two surfaces of MORI is not of class O_{AD}, it follows that the other surface belongs to O_{AD} but not to O_{FD}. In the next section we show that even for planar surfaces the class $O_{FB} = O_{AB}$ is not preserved under quasiconformal equivalence.

6. A counterexample. AHLFORS and BEURLING [5] have constructed an example of a mapping φ of the unit disk onto itself which is quasiconformal and which is not absolutely continuous on the circumference. Consequently, φ must take a closed set E_1 of zero linear measure on the circumference into a closed set E_2 of positive linear measure. Extend φ by reflection to be a mapping of the full RIEMANN sphere S onto itself. As a result of AHLFORS [2], this extended φ is again quasiconformal. Thus the planar RIEMANN surfaces $S \sim E_1$ and $S \sim E_2$ are quasiconformally equivalent. But the first is of class O_{AB} while the second is not, since a necessary and sufficient condition that a closed subset of the unit circumference be an AB nullset is that it have linear measure zero [4].

7. A criterion of the SARIO-PFLUGER type for the class O_{FD}. Let W be an open RIEMANN surface and $d\sigma = \mu(z) \mid dz \mid$ a conformal metric on W whose curvature is non-positive as in [10] section 10.106, and let $\varrho(p)$ denote the distance to a point p from a fixed point p_0. We assume further that $\varrho_1 \leqslant \infty$ has the property that for $\varrho_0 < \varrho_1$, the set of p such that $\varrho(p) \leqslant \varrho_0$ is compact.

Then the "circles" $\varrho(p) = \varrho$ consist in general of a finite number of closed curves. Let these curves be grouped into sets Γ_i such that each set Γ_i is the relative boundary of a component of $\{p : \varrho(p) > \varrho\}$. Set $\Lambda(\varrho) = \max_i \int_{\Gamma_i} d\sigma$. Since each Γ_i is a dividing cycle on W, and consequently $\int_{\Gamma_i} * du = 0$ for each $u \in F$, an immediate modification of PFLUGER's proof [12] of the SARIO-PFLUGER criterion for O_{AD} gives us the following proposition (cf. also [16] and [10]):

Proposition 8. A sufficient condition that $W \in O_{FD}$ is the divergence of the integral

$$\int^{\varrho_1} \frac{d\varrho}{\Lambda(\varrho)} .$$

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Added in proof (October 26, 1959): After the submission of this paper I have been informed that K. OIKAWA has previously demonstrated the invariance under quasiconformal mappings of planar surfaces of class O_{AD}. This proof occurs in an article in Japanese entitled "Some properties of quasiconformal mappings" Sugaku, 9 (1957), 13–14. The proof of OIKAWA depends on several deep results including: STREBEL's result on the extension of quasiconformal mappings to AD nullsets; the AHLFORS-BEURLING [4] characterization of a AD nullset as a closed set such that every conformal image its complement has complementary area zero; the theorem of BERS [6] that a quasiconformal mapping carries sets of zero area into sets of zero area.

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