The global study of geodesics in symmetric and nearly symmetric RIEMANNian manifolds.

The global study of geodesics in symmetric and nearly symmetric RIEMANNian manifolds [1])

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1. Introduction

The present article represents a further contribution to the program of research initiated in two papers of the author, RAUCH [13, 14][2]). In the first of these one finds

Theorem A. *Let* M^n, $n \geqq 2$, *be an n-dimensional complete RIEMANNian manifold of class* C^α, $\alpha \geqq 4$, *for which a constant* $K > 0$ *exists such that*

$$hK < K(P, \gamma) \leqq K\,, \quad all \quad P \in M^n, \quad all \ \gamma\,,$$

where $K(P, \gamma)$ *is the RIEMANNian curvature of the two-section* γ *with vertex at* P *and* $h \sim .75$ *is the solution of* $\sin \pi \sqrt{h} = \sqrt{h}/2$. *Then the simply connected covering* $\tilde{M}^n$ *is homeomorphic to* S^n, *the n-sphere.*

A critical review and expanded exposition of RAUCH [13], in particular the preceding result will be found in the monograph RAUCH [15]. In the interim several new contributions due to other authors have brought this particular phase of the general program to a definitive conclusion: in KLINGENBERG [12][3]) h is replaced by $h' \sim .54$ satisfying $\sin \pi\sqrt{h'} = \sqrt{h'}$, for *even* n; in TSUKAMOTO [17] this conclusion was extended to all n; and in BERGER [1] and [2] for even n, h is replaced by $h'' = 1/4$, which he proves to be best possible in the very strong sense that if the limits are actually attained and M^n is not homeomorphic to a sphere then it is a complex or quaternion projective space or the CAYLEY projective plane, each bearing its associated standard RIEMANNian metric.

BERGER's ultimate version of Theorem A shows that the method is a natural one; however, in the introduction to RAUCH [15], appearing before this result, it was pointed out that 1/4 is a natural limitation to the effectiveness of the method, quite independent of the dimension, so that 1/4 is *best possible for*

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[1]) Delivered at the International Colloquium on Topology and Differential Geometry in Zurich, June, 1960. This work was partially supported by NSF, Grant No. G 6695.

[2]) Reference to bibliography at end of paper.

[3]) The author wishes to disavow the hesitant sentiment expressed in the footnote on p. 8 of RAUCH [15]. His misunderstanding was due to a hasty reading.

the method in all dimensions and *best possible* for even n but *not* necessarily for *odd* n. In confirmation thereof is the announcement[4]) of KLINGENBERG that he can replace h by $1/4$ for *all* n but without proving that $1/4$ is best possible for *odd* n. What is missing, of course, is the natural counter-example for odd n corresponding to the projective spaces in even dimensions. Lacking such a counter-example, one is entitled to suspect that Theorem A is valid in odd dimensions with a value $h''' < 1/4$, and one is certain that the present method cannot shed any light on this question. Going further, BERGER in exhaustive unpublished calculations has confirmed that the only simply connected *homogeneous* (symmetric or non-symmetric) manifolds of *strictly positive* curvature are the spheres and the aforementioned projective spaces.[5])

Is it possible that every simply connected manifold with strictly positive curvature is homeomorphic to one of the above named models? For the reason indicated the present methods will not suffice to answer this question. But should it happen by some chance that it were answered affirmatively by another method, what would then become of "comparison theorems" such as theorem A and its improved version—would they be superseded? No, they would instead enter into a larger context by distinguishing curvature-wise among the several possible models.

Now in RAUCH [14] an extension of theorem A is envisaged in which M^n is compared with irreducible symmetric RIEMANNian manifolds other than the sphere. A holonomy condition enters; and results in BERGER [3] appearing in the interim imply (significantly, in view of what has preceded) that the only admissible models for comparison purposes are precisely the spheres and the various projective spaces. There is thus a need for a reexamination of RAUCH [14], and the present communication takes the first steps by presenting the revised hypotheses and the analytical mechanism as well as some relevant material on the global differential geometry of the general compact symmetric manifold.

2. Symmetric manifolds — curvature, geodesics, holonomy

Lack of space forbids any synoptic account of definitions and theorems; the reader is referred to CARTAN [4–7] and [8], Chapter XI, and, for a more modern treatment of certain differential geometric aspects, to the bibliography of BERGER [3]. A symmetric RIEMANNian manifold possesses

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[4]) These Commentarii, *35* (1961) 47.

[5]) Added in proof. In a letter to the author BERGER states that a reexamination of his calculations reveals the existence of two new (non-symmetric) examples, both of odd dimension (7 and 13). If confirmed, this would answer the question of the next paragraph in the negative and lend new significance to KLINGENBERGS improvement of theorem A.

Property A. *The curvature of a 2-section propagated parallelly along a geodesic is constant.*

Property B. *The restricted holonomy group coincides with the connected component containing the identity of the isotropy group* (more precisely, with the representation thereof in the tangent space at a given point).

Every symmetric manifold is globally the product of irreducible symmetric manifolds. For the sake of simplicity, then, let E^n_+ be an irreducible symmetric manifold (dimension n) of positive curvature and E^n_- the corresponding space of negative curvature whose structure is obtained from that of E^n_+ by multiplying the infinitesimal transvections by $\sqrt{-1}$. To avoid the inconvenience of passing to the universal covering for some later topological conclusions let E^n_+ and E^n_- be assumed simply connected.

At any fixed $P \in E^n_+(E^n_-)$ consider an orthogonal frame (in the tangent space) and unit orthogonal vectors u, v having the appearance $u = (u_1, \ldots, u_{n-1}, 0)$, $v = (0, \ldots, 0, 1)$ with respect to that frame. After a suitable orthogonal transformation and renumbering of the $u's$ the curvature form at P for the section γ spanned by u and v has the appearance

$$K(P, \gamma) = \pm K_1(u_1^2 + \cdots + u_{n_1}^2) \pm \cdots \pm K_m(u_{n_{m-1}}^2 + \cdots + u_{n-1}^2), \tag{1}$$

where $K_1 > \cdots > K_m \geqq 0$ and, in general, $K_m = 0$, the plus sign occurring for E^n_+ and the minus for E^n_- [6]).

Before proceeding to use Properties **A** and **B** it will be instructive to compute 1) explicitly in terms of the structures of the Lie algebras, $\mathfrak{g}$ and $\mathfrak{k}$, of the Lie groups, G and K, of isometries and isotropy of $E^n_\pm$. Consider first the case where $E^n_+ = H$, H a simple, compact, simply connected Lie group endowed with the standard Riemannian metric for which $H \times H$ acting on H by left and right translation is the group of proper isometries to which is adjoined the symmetry $l \to ml^{-1}m$ about $m (m, l \in H)$, the holonomy ($=$ isotropy) group being AdH (i.e., AdH is the representation in $\mathfrak{h}$, the Lie algebra of H, of the inner automorphisms $l \to mlm^{-1}$, l, $m \in H$, constituting the isotropy group about the identity). Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{h}$ such that, if $X \equiv x_i X_i$ (summation convention), then the Killing form of $\mathfrak{h}$ is $-(X, X)$ where[7]) $(X, X) \equiv x_i x_i$. If X and $Y \equiv y_i X_i$ are such that $(X, X) = (Y, Y) = 1$, $(X, Y) = 0$ then, if γ is the two section spanned by X, Y,

$$K(\gamma) = -R_{ijrs} y_i x_j x_r y_s = -\tfrac{1}{4}([[X, Y], Y], X) \tag{2}$$

[6]) Because of the homogeneity of the space it is permissible to drop the P in $K(P, \gamma)$. The metric and curvature form of E^n_+ are uniquely determined up to the same multiplicative constant.

[7]) This is not the normalization used by Cartan in some of the papers referred to.

(CARTAN [6], p. 65 or p. 737). Now in $\mathfrak{h}$ introduce a new orthonormal basis $e_1, \ldots, e_n$ for which

$$[e_{2k-1}, Y] = -2\pi\omega_k(Y)e_{2k}, \tag{3.a}$$

$$[e_{2k}, Y] = 2\pi\omega_k(Y)e_{2k-1}, \quad k = 1, \ldots, m, \tag{3.b}$$

$$[e_{2m+\alpha}, Y] = 0 \qquad \alpha = 1, \ldots, l, \tag{3.c}$$

where l is the rank of $\mathfrak{h}$, $n = l + 2m$, and $\omega_k(Y)$, $k = 1, \ldots, m$, are the roots[8] (angular parameters in CARTAN's terminology) of Y. The last l vectors $e_{2m+1}, \ldots, e_n$ span a CARTAN subalgebra[9] $\mathfrak{t}$ containing Y. Clearly, if X and Y are to play the roles of u and v in (1) then because of (3c) it is permissible to put $X = \Sigma_1^{2m} u_i e_i$, $Y = \Sigma_{2m+1}^n v_i e_i$ in (2), where $u_1^2 + \cdots + u_{2m}^2 = v_{2m+1}^2 + \cdots + v_n^2 = 1$. One obtains

$$K(\gamma) = \pi^2 \sum_{k=1}^{m} (u_{2k-1}^2 + u_{2k}^2)\,\omega_k^2(Y), \tag{4}$$

where here and in (3) a root of multiplicity p is written p times[7]. Let $c_k(Y)$ be the distance (the metric reduces to the ordinary euclidean one on $\mathfrak{t}$) from the origin to the point where the vector Y crosses the hyperplane $\omega_k(\Sigma_{2m+1}^n w_i e_i) = 1$. For this point the coordinates $w_{2m+1}, \ldots, w_n$ in $\mathfrak{t}$ take the values $c_k(Y)v_{2m+1}, \ldots, c_k(Y)v_n$. One can write, therefore,

$$\omega_k(Y) = \frac{1}{c_k(Y)}\,\omega_k(c_k(Y)\,Y) = \frac{1}{c_k(Y)}.$$

Substituting in (4), one obtains

$$K(\gamma) = \sum_{k=1}^{m} (u_{2k-1}^2 + u_{2k}^2)\frac{\pi^2}{c_k^2(Y)}. \tag{5}$$

To obtain E_-^n corresponding to $E_+^n = H$ one complexifies H and $\mathfrak{h}$ and considers the subspace of H generated by the purely imaginary subspace $\sqrt{-1}\,\mathfrak{h}$ of the complex LIE algebra. One then has to change the sign of the inner product whereupon it is clear from '2) that the curvature form of E_-^n is simply the negative of that of E_+^n. The group of isometries of E_-^n is the simple complex group, the complexified H, while the holonomy group is the same as before, $Ad\,H$.

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[8]) As a result of the fact noted in [7]) the roots here will not have the traditional appearance (for all matters pertaining to roots the reader is referred here to de SIEBENTHAL [16] and references there). In addition, the roots here are evaluated for the parameters having square sum 1.

[9]) The presumption here is that Y is general so that $\mathfrak{t}$ is unique. If Y is singular, then take any one $\mathfrak{t}$.

To obtain the other symmetric manifolds in question consider a compact, connected LIE group G with LIE algebra $\mathfrak{g}$, an involutive automorphism τ of G (the symmetry), and the maximal compact connected subgroup K pointwise invariant under τ with LIE algebra $\mathfrak{k} \subset \mathfrak{g}$. Let $X_1, \ldots, X_q$, $q = \dim g$, be a basis for $\mathfrak{g}$ normalized as before and, in addition, so that $X_1, \ldots, X_n$ form a basis for $\mathfrak{k}^\perp$, the orthogonal complement of $\mathfrak{k}$. The one-parameter subgroups generated by $\mathfrak{k}^\perp$ form $E^n_+ = G/K$. Given $Y \in \mathfrak{k}^\perp$ and a CARTAN subalgebra $\tilde{\mathfrak{t}} \subset \mathfrak{k}^\perp$ of $\dim \lambda < l$, where $l + 2m = q$ one can as before choose a new orthogonal basis $e_1, \ldots, e_q$ in $\mathfrak{g}$ such that $e_{2k-1} \in \mathfrak{k}^\perp$, $k = 1, \ldots, m$, and $e_{2m+\alpha}$, $\alpha = 1, \ldots, \lambda$ span $\tilde{\mathfrak{t}}$ the two sets together spanning $\mathfrak{k}^\perp$ (so that $m + \lambda = n$) and, in addition, formulae 3) hold with $\omega_k(Y)$, the root with respect to $\mathfrak{t}$, replaced by $\tilde{\omega}_k(Y)$, the root with respect to $\tilde{\mathfrak{t}}$, i.e. in particular, a linear form in $v_{2m+1}, \ldots, v_{2m+\lambda}$ only. One then has, corresponding to (4) and (5)

$$K(\gamma) = \pi^2 \sum_{k=1}^{m} u^2_{2k-1} \tilde{\omega}^2_k(Y) \tag{6}$$

and

$$K(\gamma) = \sum_{k=1}^{m} u^2_{2k-1} \frac{\pi^2}{\tilde{c}^2_k(Y)}, \tag{7}$$

where $\tilde{c}_k(Y)$ is defined analogously to $c_k(Y)$. The consideration of $\sqrt{-1}\,\mathfrak{k}^\perp$ and reversal of the sign of the inner product (in this paragraph just the first n terms of the KILLING form are used) show that $K(\gamma)$ for E^n_- is just the negative of that for E^n_+. The holonomy (isotropy) group remains K but the group of isometries becomes the open real simple group with LIE algebra $\mathfrak{k} + \sqrt{-1}\,\mathfrak{k}^\perp$.

Let $P \in E^n_+ (E^n_-)$ be arbitrary but fixed, and let σ be a fixed geodesic ray issuing from P. With the conventions used in (1) let v be the unit tangent vector to σ at P so that u lies in the $(n-1)$-plane orthogonal to v. Introduce FERMI coordinates along σ and consider the JACOBI equations (cf. RAUCH [15], Chapter 3). Property A implies that the latter take the shape

$$\begin{aligned} &\eta''_\alpha \pm K_1 \eta_\alpha = 0\,, \qquad \alpha = 1, \ldots, n_1\,, \\ &\quad \cdot \quad \cdot \quad \cdot \\ &\eta''_\alpha \pm K_m \eta_\alpha = 0\,, \qquad \alpha = n_{m-1} + 1, \ldots, n-1\,, \end{aligned} \tag{8}$$

where the prime denotes differentiation with respect to s, the arc-length measured on σ from P.[10])

[10]) In order to reconcile formulae (6), (7) with (1), (8) one must replace $u_1, u_3, \ldots, u_{2m-1}$ in the former by $u_1, u_2, \ldots, u_{n-\lambda}$ and $u_{2m+1}, \ldots, u_{2m+\lambda}$ by $u_{n-\lambda+1}, \ldots, u_{n-1}$.

The first conclusion from (8) for E_+^n is

Lemma 1. *The first point Q on σ conjugate to P (briefly, first conjugate point) is at the distance $s = \pi/\sqrt{K_1}$ from P.* See RAUCH [15], Chapter 3.

The second conclusion for E_+^n is stronger:

Theorem 1. *The geodesic issuing from P whose initial tangents lie in the hyperplane $\delta: u_{n_1+1} = \cdots = u_{n-1} = 0$ all have the same length $\pi/\sqrt{K_1}$, all meet at the first conjugate point Q on σ, and form a metric sphere with constant curvature K_1 and dimension n_1+1 in the natural induced metric.*

There are two proofs both based on Property B.

First proof (group-theoretic): given $u \in \delta$ there exists a one-parameter subgroup of the holonomy group taking v into u (in particular) because the curvature K_1 of the section γ spanned by v and u is non-zero. By Property B this subgroup is realized by a one-parameter subgroup $T(\theta)$ of K, $0 \leqq \theta \leqq 2\pi$. $T(\theta)$ rotates σ into the geodesic $\sigma(\theta)$ issuing from P whose initial tangent $u(\theta)$ lies in γ. Denote the point on $\sigma(\theta)$ at distance s from P by $T(\theta, s)$. Then, by JACOBI's theorem, $y = T_\theta(0, s)$ is a vector solution of (6) for which $\eta(0) = 0$, $\eta'_\alpha(0) = 0$, $\alpha = n_1+1, \ldots, n-1$, so that $T_\theta(0, \pi/\sqrt{K_1}) = 0$. From the group property of $T(\theta)$ one deduces immediately that $T(\theta)Q = Q$ identically in θ. Applying this reasoning for all $u \in \delta$ one draws the first two conclusions of Theorem 1. Furthermore, since (1) on δ is precisely the curvature form for a sphere of dimension n_1+1 and curvature K_1 and the holonomy group takes v into any $u \in \delta$ and leaves the form (1) invariant (Property B) one sees that at P all sections γ have the same curvature K_1. The transvections then carry the conclusion to every point of the geodesics in question.

Second proof (differential-geometric): one deduces from (6) (RAUCH [15], p. 21 and p. 41) that if Σ is the unit n_1-sphere in the subspace of the tangent space at P spanned by δ and v and $d\Sigma^2$ is the line element on Σ then the line element $d\sigma^2$ on E_+^n satisfies

$$d\sigma^2 = ds^2 + \frac{\sin^2 \sqrt{K_1} s}{K_1} d\Sigma^2 \tag{9}$$

along σ. The argument at the beginning of the first proof shows that (9) holds along any geodesic whose initial tangent at P lies in δ. But (9) is the classic expression for the metric of a sphere in geodesic polar coordinates and Theorem 1 is deduced in an obvious fashion.

Before continuing with the analysis of E_+^n one can dispose of the topological implications of the curvature structure of E_-^n by observing that (8) (with the minus sign) show immediately the absence of points conjugate to P and hence by a familiar reasoning the existence of a homeomorphism of E_-^n on euclidean n-space. In fact this is a special case of the theorem of HADAMARD and CARTAN, a proof of which can be found in RAUCH [15].

The key role played by the curvature through its intermediary, the conjugate locus in E_+^n, is shown by

Theorem 2. *Let C be the locus of first conjugate points of any fixed $P \in E_+^n$. Then $E_+^n - C$ is an n-cell.*

Two proofs of Theorem 2 will be given. The first gives a very detailed description of C but it uses the full force of the group structure and, in particular, makes use of (4) and (5). The second is purely differential-geometric using Properties A and B only and not (4), (5), (6), and (7). While it gives less information it is more elementary and is entirely in the spirit of the author's methods.

As for the first proof Theorem 2 is a corollary of

Theorem 2'. *C, the locus of first conjugate points of $P \in E_+^n$, is the orbit under the isotropy group K about P*[11]*) of a polyhedron Π obtained as follows: consider a maximal torus T^λ through P and the dominant root $\tilde{\omega}$ (of $\mathfrak{k}^\perp$) with respect to T^λ (i.e., with respect to $\tilde{\mathfrak{t}}^\lambda$, tangent space of T^λ); consider the $(\lambda - 1)$-plane $\tilde{\omega} = 1$ and the images thereof under the WEYL group of E_+^n; the intersection of the open half spaces bounded by these planes and containing P form an λ-cell D whose boundary is a polyhedron $\tilde{\Pi}$; identifying faces of $\tilde{\Pi}$ equivalent under the translations of the group of the diagram of E_+^n, one obtains Π. $E_+^n - C$ is the orbit of D under K and is, in particular, an n-cell.*

Proof of Theorem 2': Let σ be any geodesic issuing from P. It belongs to at least one T^λ. To T^λ one can apply the analysis of (6) and (7) as modified in the second paragraph, and one sees from (7), and (8) that the first conjugate point is at the distance $\pi/\sqrt{K_1} = \min \tilde{C}_k(Y)$, Y being the initial tangent of σ and the minimum being taken over k. Now by definition $\min \tilde{C}_k(Y) = 1/\max \tilde{\omega}_k(Y)$, and one can write $\tilde{\omega}_k(Y) = a_{kl} v_{2m+1} + \cdots + a_{k\lambda} v_{2m+\lambda} = |\vec{\omega}_k|\,|Y| \cos\theta$ where $|\vec{\omega}_k| = (a_{k1}^2 + \cdots + a_{kl}^2)^{1/2}$ is the norm of the root vector $\vec{\omega}_k = (a_{kl}, \ldots, a_{k\lambda})$, norm $Y = 1$, and θ is the angle

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[11]) Again because of homogeneity it is sufficient to confine one's attention to one P, which in this case may be taken to be the projection of the identity element of G.

between $\vec{\omega}_k$ and Y. Thus $\max_k \tilde{\omega}_k(Y)$ is attained for the root vector with maximum norm. But if one returns to the usual coordinates[8]) in $\tilde{\mathfrak{t}}$ then every root vector can be written as a linear combination with non-negative integral coefficients of fundamental root vectors. One easily verifies that increasing any one of these coefficients increases the norm of the vector with respect to $e_{2m+1}, \ldots, e_{2m+\lambda}$ so that the *dominant* root vector has largest norm and $\min \tilde{c}_k(Y) =$ distance from P to the point where Y cuts $\tilde{\omega} = 1$. Thus the part of this plane cut out by the fundamental chamber is in C, and the facts that the Weyl group is a group of isometries and all maximal tori are conjugate under K complete the proof of Theorem 2′ except for the last statement. Notice that if Y lies in several tori so that it is *singular* in $\tilde{\mathfrak{t}}$ there is no ambiguity, the dominant curvature K_1 being uniquely defined and hence also the first conjugate point. Such a point lies on the intersection of at least two planes of Π and their intersection with one or more planes of the infinitesimal diagram or on the intersection of one plane with the infinitesimal diagram.

To prove the last statement of Theorem 2′ lay off on each ray issuing from P in the tangent space at P a segment of length $\pi/\sqrt{K}$ where K_1 belongs to the initial direction of the ray. Because K_1 is continuous, bounded from above, and has a positive minimum as a function of that direction the totality of these segments form a cell F. Introducing geodesic polar coordinates at P one obtains a map $g_P: F \to E^n_+$ which is, thanks to the absence of conjugate points on each geodesic issuing from P up to length $\pi/\sqrt{K_1}$, a *local* homeomorphism *onto* the orbit of D under K, that being the totality of geodesic arcs issuing from P and free of conjugate points as the preceding analysis has shown (see Rauch [15] for the local homeomorphism property of g_P). E^n_+ being complete, every point of it can be joined to P by an *absolutely minimizing* geodesic, no interior point of which, therefore, can be conjugate to P. Thus g_P maps F *onto* $E^n_+ - C$, which can now be identified with the orbit of D under K. However, g_P is $1-1$ on F as one sees by imbedding E^n_+ in AdG (Cartan [6], p. 425 or 947) and then using the uniqueness of the canonical representation for general adjoint matrices (Cartan [10]), i.e., precisely those in the present case which belong to $g_P(F)$.

The second proof of Theorem 2, promised above, uses a method which is quite characteristic of the author's researches and consists of the introduction of the *space of geodesics* $\tilde{E}^n_+$ of E^n_+ (a concept introduced in Rauch [13], p. 297). $\tilde{E}^n_+$ consists of all the geodesic segments issuing from P and terminating in c and C itself. This point set is topologized as follows: every point, including P, not on C has a neighborhood defined in the natural way by any one branch of (the a priori multi-valued) g_P^{-1}. The key part of the argument is to topologize a neighborhood of C. But Theorem 1, formula (6), and Property B provide the necessary tools. Indeed, given any one segment σ and the corresponding point $Q \in C$, one obtains an n-dimensional neighborhood of Q from short arcs of the geodesics meeting at Q and forming the $(n_1 + 1)$-dimensional sphere of Theorem 1 and from the neighboring (transverse) points of C given by varying the initial direction v of σ by allowing small increments of $u_{n_1+1}, \ldots, u_{n-1}$. E_+^n thus topologized is a manifold, simply connected and with the same local differential geometry as E_+^n and furthermore is a smooth unbounded covering of E_+^n as one sees by applying the c-process (RAUCH [15], Chapter 2) to any curve γ in E_+^n. Any potential difficulties in applying the c-process at C are eliminated by the preceding. But E_+^n is assumed simply connected. Hence $\tilde{E}_+^n = E_+^n$ and Theorem 2 follows.

Some examples should prove instructive. Consider first the group spaces. Comparing (4) and Theorem 1 one sees that every geodesic (read: 1-parameter-subgroup) issuing from (say) the identity is one meridian of a 3-sphere formed of geodesics issuing from the identity and meeting again at the first conjugate point—if the initial element of the LIE algebra is *not singular* since the dominant root has multiplicity 1. This 3-sphere is, of course, a simple orthogonal group isomorphic to the covering group of the group of all rotations in three variables. In case the initial element is singular then there is, as noted before, a degeneration—some of the roots become equal and some vanish. One observes from the shape of (4) that in such a case the sphere of Theorem 1 is always odd dimensional. Maximal degeneration occurs at the vertices of the fundamental cell (i.e., the positive fundamental domain of the WEYL group) of the diagram. These are obtained by setting $\omega = 1$ and any $l - 1$ of the fundamental roots equal to zero. The orbits of these vertices under K form the *antipodal loci* observed by CARTAN in CARTAN [5, 6], which loci are then simply special subsets of C.

Similar remarks apply to the more general symmetric spaces of rank higher than one, the phenomena being appropriately more complex.

Of particular interest, however, are the symmetric spaces of rank one, namely, the spheres, S^n, of both even and odd dimensions with metrics of constant curvature, and the projective spaces: $P^n(C) = SU(n+1)/U(n)$, the complex projective space with the hermitian elliptic metric, $P^n(Q) = Sp(n+1)/Sp(n) \times Sp(1)$, the quaternion projective space (dimension $4n$) with an analogous metric, and P^2 (CAYLEY), the CAYLEY projective plane (dimension 16) with an analogous metric. Because of their explicit and elementary geometric and analytical definition it is not necessary to use formulae (6) and (7) to compute (1) and (8), which can instead be deduced directly from said definitions (for the projective spaces see BERGER [2], p. 59). One obtains the results in summary form in which is included the shape of the metrics in appropriate geodesic polar coordinates along a geodesic issuing from P (as in (9)) (cf. RAUCH [15], Chapter 3):

$$S^n : K(u_1^2 + \cdots + u_{n-1}^2)\,, \tag{10}$$

$$\eta_\alpha'' + K\eta_\alpha = 0 \qquad \alpha = 1, \ldots, n-1\,,$$

$$ds^2 + \frac{\sin^2 \sqrt{K}s}{K} dy_\alpha dy_\alpha \quad \text{(summation convention on } \alpha\text{)};$$

$$P^n(C) : Ku_1^2 + \frac{K}{4}(u_2^2 + \cdots + u_{2n-1}^2)\,,$$

$$\eta_1'' + K\eta_1 = 0$$

$$\eta_\alpha'' + \frac{K}{4}\eta_\alpha = 0 \qquad \alpha = 2, \ldots, 2n-1\,,$$

$$ds^2 + \frac{\sin^2 \sqrt{K}s}{K} dy_1^2 + \frac{4\sin^2(\sqrt{K}s/2)}{K} dy_\alpha dy_\alpha\,;$$

$$P^n(Q) : K(u_1^2 + u_2^2 + u_3^2) + \frac{K}{4}(u_4^2 + \cdots + u_{4n-1}^2)\,,$$

$$\eta_\alpha'' + K\eta_\alpha = 0 \qquad \alpha = 1, 2, 3$$

$$\eta_\beta'' + \frac{K}{4}\eta_\beta = 0 \qquad \beta = 4, \ldots, 4n-1\,,$$

$$ds^2 + \frac{\sin^2 \sqrt{K}s}{K} dy_\alpha dy_\alpha + \frac{4\sin^2(\sqrt{K}s/2)}{K} dy_\beta dy_\beta\,;$$

$$P^2\,(\text{CAYLEY}): \quad K(u_1^2 + \cdots + u_7^2) + \frac{K}{4}(u_8^2 + \cdots + u_{15}^2)\,,$$

etc.

The normalization of the curvature has been deliberately avoided in (10). The spheres of Theorem 1 are respectively S^n itself, $S^2 = P^1(C)$, $S^4 = P^1(Q)$, and $S^8 = P^1$ (CAYLEY) while C is respectively the antipode of P, and the respective planes ($P^{n-1}(C)$, $P^{n-1}(Q)$, P^1 (CAYLEY) $= S^8$) at infinity with respect to P.

3. Comparison Theorems

Theorem A and its subsequent versions are compounded of two elements, the geodesic structure of S^n as given by (10) and Theorem 1 and a metric comparison theorem (cf. RAUCH [15], Chapter 4) for the "unknown" M^n. In seeking to find analogues of Theorem A one finds in the preceding section the necessary generalization of the first element for the symmetric spaces of positive curvature so that those spaces become candidates for the role of standard model. In extending the metric comparison theorem in RAUCH [14], the author found that the symmetric models indeed fitted the needs of his analytic machinery with an additional restriction—that the holonomy group of the non-symmetric manifold be a subgroup of that of the symmetric model. According to the results of BERGER [3], however, the only irreducible holonomy groups of non-symmetric manifolds are precisely those of the spaces of rank one enumerated in Section 2. In view of this it will be well to reexamine RAUCH [14] and to begin the reexamination with the metric comparison theorem there (Theorem 2, p. 305).

Let M^n be a non-symmetric simply connected RIEMANNian manifold with holonomy group N. Let $T(P)$ be the tangent space at $P \in M^n$, and let $N(P)$ be N acting on $T(P)$. If $P, Q \in M^n$ and δ is a C^1 curve joining P to Q, let $k_\delta(P, Q): T(P) \to T(Q)$ be the linear map generated by parallel transport along δ. Clearly $N(P) = k_\delta^{-1}(P, Q) N(Q) k_\delta(P, Q)$. Let E_+^n be a compact symmetric manifold of *rank one* with curvature unnormalized and holonomy group K, and let T be the tangent space at any one fixed point of E_+^n (because of homogeneity it matters not which point).

Definition 1. *M^n is c-close to E_+^n if there exist a value K_0 for the highest curvature of E_+^n, and for each $P \in M^n$ a linear map $h(P): T \to T(P)$ such that*

$$h^{-1}(Q) k_\delta(P, Q) h(P) \in K, \qquad \text{all } P, Q \in M^n, \text{ all } \delta, \tag{11}$$

$$cK(\gamma) < K(P, h(P)\gamma) \leqq K(\gamma), \qquad \text{all } P \in M^n, \text{ all } \gamma, \tag{12}$$

where γ is a 2-section in T, $h(P)\gamma$ is its image in $T(P)$, $K(\gamma)$ and $K(P, \gamma)$ are, as before, respectively the sectional curvatures of E_+^n and M^n, and $0 < c < 1$ is a constant.

Lemma 2. *In Definition 1 $h(P)$ may be replaced by $h(P)k$, $k \in K$.*

Proof: (11) is verified by a trivial group-theoretic calculation and (12) is verified because $K(\gamma)$ is invariant under K.

Some elaboration will be useful in grasping the full import of Definition 1.

Setting $P = Q$ in (11) and letting δ range over all curves one sees that if M^n is c-close to E_+^n then its holonomy group, in particular, is (isomorphic to) a subgroup of K. It will be convenient, not to say more vivid, to call a manifold whose holonomy group[12] is $U(n)$ *almost-KAEHLERian*, is $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ *almost-quaternionic*, and is Spin (9) *almost-CAYLEYan*, all in the current spirit. Thus in Theorem A and its subsequent versions M^n might very well *a priori* be almost-KAEHLERian, etc. That it cannot be so *a posteriori* but must, in fact, have holonomy group $SO(n)$ (should one say, *almost-spherical?*) is a consequence of deep topological restrictions due to the more restricted holonomy group (for almost-KAEHLERian, see CHERN [11]). *Thus the curvature restrictions imply restrictions on the holonomy group.* It would be interesting to see this confirmed directly by an analysis of the curvature tensor.

Theorem 2 of RAUCH [14] can, in the special cases at hand, be sharpened to the following theorem in which $P^n(C)$ may be replaced by $P^n(Q)$ or P^2 (CAYLEY) and the term almost-KAEHLERian by almost-quaternionic or almost-CAYLEYan with the appropriate and obvious changes drawn from formula (10).

Theorem 3. *Let M^{2n} be c-close (Definition 1) to $P^n(C)$ so that M^{2n} is almost-KAEHLERian (in particular, possibly almost-quaternionic). Let $P \in M^{2n}$ be arbitrary but fixed. Let γ be the 2-section in T tangent to a projective line in $P^n(C)$, and let σ be a geodesic issuing from P whose initial tangent lies in $h(P)\gamma$ in $T(P)$, where $h(P)$ is the linear map of Definition 1. Let the geodesic polar coordinates at P (vide supra) be so chosen that $y_2 = y_{2n-1} = 0$ are the equations of $h(P)\gamma$. Then*

$$ds^2 + a_{11}(s)\,dy_1^2 < \tag{13}$$

$$< ds^2 + \left(\frac{\sin^2\sqrt{cK_0}\,s}{cK_0}\right)^{1 - \sin^2\theta_0} \left(\frac{4\sin^2(\sqrt{cK_0}/2)s}{cK_0}\right)^{\sin^2\theta_0} dy_1^2\,,$$

where the left-hand side is the metric of M^{2n} written in polar form along σ with $y_2 = \cdots = y_{2n-1} = 0$; and where $0 \leqq s \leqq \lambda\pi/\sqrt{K_0}$, $0 < \lambda < 1$ and $\theta_0 = \theta_0(\lambda, c)$ for fixed λ tends to 0 as $c \to 1$.

The proof of Theorem 3 is contained except for minor notational changes in pages 312–319 of RAUCH [14]. However, the first three pages of that proof are somewhat awkward, and the author would prefer to lay a secure groundwork here leaving the reader with the one or two remarks necessary to complete the proof by reading the remainder there.

Along σ introduce FERMI coordinates so that P is at $s = 0$ and $z_2 = \cdots = z_{2n-1} = 0$ at $s = 0$ is the hyperplane $y_2 = \cdots = y_{2n-1} = 0$.

[12]) With appropriate torsion vanishing.

The JACOBI equations of M^{2n} along σ will be

$$\eta''_\alpha + K_{\beta\alpha}(s)\eta_\beta\,, \quad \alpha = 1, \ldots, 2n-1 \tag{14}$$
(β summed over the same range),

where $K_{\alpha\beta}(s) z_\alpha z_\beta = R_{2n\alpha\,2n\beta}(s) z_\alpha z_\beta$, $z_\alpha z_\alpha = 1$, is the curvature for the section defined by the unit tangent to σ at s and the vector $(z_1, \ldots, z_{2n-1}, 0)$ normal to σ at s. Then it is sufficient (see RAUCH [15], p. 21 and p. 36) to prove that

$$\bar{\eta}(s)\cdot\bar{\eta}(s) = \bar{\eta}_\alpha(s)\bar{\eta}_\alpha(s) < \left(\frac{\sin^2 \sqrt{cK_0}s}{cK_0}\right)^{1-\sin^2\theta_0} \left(\frac{4\sin^2(\sqrt{cK_0}/2)s}{cK_0}\right)^{\sin^2\theta_0}, \tag{15}$$
$$0 \leqq s \leqq \lambda\pi/\sqrt{K_0}\,,$$

where $\bar{\eta}$ is the vector solution of (14) such that $\bar{\eta}(0) = 0$, $\bar{\eta}_\alpha(0) = \delta^1_\alpha$.

First, one observes that

$$c\left\{K_0 z_1^2 + \frac{K_0}{4}(z_2^2 + \cdots + z_{2n-1}^2)\right\} < K_{\alpha\beta}(s) z_\alpha z_\beta \leqq \tag{16}$$
$$\leqq K_0 z_1^2 + \frac{K_0}{4}(z_2^2 + \cdots + z_{2n-1}^2)\,, \qquad 0 \leqq s < \infty\,.$$

In fact (16) follows from the c-closeness of M^{2n} to $P^n(C)$. First of all, (16) is true at $s = 0$ as one sees by choosing a frame in T for which γ is given by $z_2 = \cdots = z_{2n-1} = 0$ (i.e., z_1 and z_{2n} free). (The choice of z as symbol for coordinates in T will be justified immediately.) If one then chooses as basis vectors in $T(P)$ the images of those in T under $h(P)$ and momentarily assumes $z_\alpha z_\alpha = 1$ then (16) follows immediately for $s = 0$ from (12) and (10). But the choice of FERMI coordinates implies that the z at Q, distance s along σ from P, are coordinates relative to the frame that has been transported to Q by parallelism along 0 from the frame at P. Thus $k_\sigma(P, Q)$ is represented by the identity matrix in the FERMI coordinates. However, if $h(Q)$ is a mapping for which (12) holds then $k_\sigma(P,Q)h(P) = h(Q)h^{-1}(Q)k_\sigma(P,Q)h(P) = h(Q)h$, $h \in U(n)$, so that (16) is valid for Q, too.

Consequently one has by repeated application of the fundamental lemma (RAUCH [15], p. 32 and pp. 37–38)

$$J(z, r) \leqq I(z, r) < J_c(z, r)\,, \qquad 0 \leqq r < \pi/\sqrt{K_0}\,, \tag{17}$$

where

$$J(z, r) = \int_0^r \left(\mu'_\alpha \mu'_\alpha - K_0 \mu_1^2 - \frac{K_0}{4}(\mu_2^2 + \cdots + \mu_{2n-1}^2)\right) ds\,,$$

α summed from 1 to $2n-1$ and μ being the vector solution of the first equations of (10) with $\mu(0)=0$, $\mu(r)=z$; $J_c(z, r)$ is the same with K_0 replaced by cK_0; and

$$I(z, r) = \int_0^r (\eta'_\alpha \eta'_\alpha - K_{\alpha\beta}\eta_\alpha\eta_\beta)ds = f_{\alpha\beta}(r) z_\alpha z_\beta ,$$

$$\alpha, \beta \text{ summed from 1 to } 2n-1 ,$$

where η is a vector solution of (14) with $\eta(0)=0$, $\eta(r)=z$ (cf. RAUCH [15], p. 28 and p. 43). Evaluating the right and left sides of (17) explicitly one has

$$(\sqrt{K_0}\cot\sqrt{K_0}r)z_1^2 + \left(\frac{\sqrt{K_0}}{2}\cot\sqrt{\frac{K_0}{2}}\,r\right)(z_2^2 + \cdots + z_{2n-1}^2) \tag{18}$$
$$\leqq f_{\alpha\beta}(r)z_\alpha z_\beta <$$
$$(\sqrt{cK_0}\cot\sqrt{cK_0}r)z_1^2 + \left(\frac{\sqrt{cK_0}}{2}\cot\frac{\sqrt{cK_0}}{2}\,r\right)(z_2^2 + \cdots + z_{2n-1}^2) .$$

From here on the reader may follow the indicated pages of RAUCH [14] with the one precaution that the estimate in the eighth line from the top of p. 317 is no longer needed and a resulting sharpening takes place by suitably modifying the following estimates.

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