On sphere-bundles over spheres.

On sphere-bundles over spheres[1]

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§ 1. Introduction

In 1953, J. H. C. Whitehead and myself made an investigation into the homotopy theory of sphere-bundles over spheres (see [6], [7]). The purpose of this note is to add something to that theory, especially in the case when torsion occurs in the homology of the total space. Our main application is

Theorem 1.1. *Let B be an $(n-1)$-sphere bundle over S^n, where $n \geq 2$. If B is an H-space then B is homeomorphic to S^3, to real projective 3-space, or to S^7.*

More generally, let B be a q-sphere bundle over S^n, where $n, q \geq 1$. If B admits a cross-section and is an H-space then B has the homotopy type of $S^n \times S^q$, by[2] Theorem (1.23) of [6]; and so $n, q \in \{1, 3, 7\}$, by Adams' theorem [1]. I do not know whether B is homeomorphic to $S^n \times S^q$ under these conditions. There always exists a cross-section when $n \leq q$. In case $n > q + 1$ I understand[3] from Dr. Adams that the theory of secondary operations enables it to be shown that B is not an H-space unless either (i) $q = 3$ and $n = 5$ or 7, or (ii) $q = 7$ and $n = 11$ or 15. Examples of (i) are the special unitary group $SU(3)$ and the symplectic group $\mathrm{Sp}(2)$. I do not know any examples of (ii) but it might be worthwhile to investigate the quaternionic Stiefel manifold $X_{3,2}$ and the octonionic Stiefel manifold $Y_{2,2}$ (see [3]).

§ 2. Torsion q-spheres

Let X be a space and let q be a positive integer such that $H_r(X)$ is a torsion group for $0 < r < q$ and $H_q(X)$ is cyclic infinite. Then we describe X as a *torsion q-sphere* and we define its *spherical index* $I(X)$ to be the index of $\theta\pi_q(X)$ in $H_q(X)$, where θ denotes the Hurewicz homomorphism. If X is 1-connected then it follows from the Serre-Hurewicz theorem (Theorem 1 on page 271 of [9]) that $I(X) > 0$.

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[1]) This paper was presented at the International Colloquium on Differential Geometry and Topology at Zurich, June 1960.

[2]) A direct proof will be found in § 8 below.

[3]) I am most grateful to Dr. Adams for this interesting information.

If X is a torsion q-sphere then SX, the suspension of X, is a torsion $(q+1)$-sphere and $I(X)$ is a multiple of $I(SX)$. We prove

Theorem 2.1. *Let K be a countable connected CW-complex. Suppose that K is a torsion q-sphere and also an H-space. Then $I(K) = I(SK)$.*

Let $K * K$ denote the join of K with itself and let $h: K * K \to SK$ denote the map obtained by the Hopf construction from the multiplication on K. Then the following diagram is commutative, where h_* and $h_\#$ denote induced homomorphisms and E denotes suspension.

$$\begin{array}{ccccc}
\pi_q(K) & \xrightarrow{E} & \pi_{q+1}(SK) & \xleftarrow{h_*} & \pi_{q+1}(K * K) \\
\big\downarrow{\scriptstyle\theta} & & \big\downarrow{\scriptstyle\theta} & & \big\downarrow{\scriptstyle\theta} \\
H_q(K) & \xrightarrow[E]{} & H_{q+1}(SK) & \xleftarrow[h_\#]{} & H_{q+1}(K*K)
\end{array}$$

By (2.3) of [5] we have

$$\pi_{q+1}(SK) = E\pi_q(K) + h_*\pi_{q+1}(K * K) .$$

Also $H_{q+1}(K * K)$ is a torsion group, by the Künneth formula, and hence $h_\#$ is trivial. Therefore $\theta h_* = h_\# \theta = 0$ and consequently

$$\theta\pi_{q+1}(SK) = \theta E\pi_q(K) = E\theta\pi_q(K) .$$

Hence (2.1) follows at once.

§ 3. The suspension of B

Let $v: S^{n-1} \to S^q$ be a map such that v is constant if $n \leq q$. Let $u: S^q \times S^{n-1} \to S^q$ be a map such that

$$u(x, e) = x , \quad u(e, y) = vy \quad (x \in S^q,\ y \in S^{n-1}) ,$$

where e denotes the basepoint in all cases. Let V^n denote an n-element bounded by S^{n-1} and let $W^n = V^n - S^{n-1}$. Let B be the space obtained from the disjoint union of S^q and $S^q \times V^n$ by identifying points of $S^q \times S^{n-1}$ with their images under u. The images of $e \times W^n$ and $(S^q - e) \times W^n$ under the identification map are open cells which we denote by e^n and e^{n+q}, respectively, so that

$$B = S^q \cup e^n \cup e^{n+q} .$$

In this cellular decomposition e^n is attached by $\theta \in \pi_{n-1}(S^q)$, the homotopy class of v, and e^{n+q} is attached by $\beta \in \pi_{n+q-1}(S^q \cup e^n)$, say. It is shown in § 3 of [6] that, with a suitable choice of u, the total space of any q-sphere bundle over S^n can be constructed in this way. In general the construction yields a quasi-fibration of (B, S^q) over S^n, as described in § 1 of [5], and it is not difficult to extend the theory of [6] and [7] appropriately.

Now consider the suspended complex

$$SB = S^{q+1} \cup e^{n+1} \cup e^{n+q+1} ,$$

where e^{n+1} is attached by $E\theta \in \pi_n(S^{q+1})$ and e^{n+q+1} is attached by $E\beta \in \pi_{n+q}(S^{q+1} \cup e^{n+1})$. Let $f : S^q * V^n \to SB$ be obtained by the Hopf construction from the identification map $S^q \times V^n \to B$. Then f has degree ± 1, and agrees with the map $g : S^q * S^{n-1} \to S^{q+1}$ which is obtained by the Hopf construction from u. Hence $E\beta = \pm j_* \mu$, where $\mu \in \pi_{n+q}(S^{q+1})$ is the class of g and j_* denotes the injection. We choose orientations so that

$$E\beta = j_* \mu . \tag{3.1}$$

One application of this relation can be seen in § 4 of [1].

§ 4. The kernel of j_*

Let us recall from [10], [11] and [12] the main facts known about $\pi_{2n-1}(S^n)$ when n is even. First we recall Freudenthal's theorem that $E\pi_{2n-2}(S^{n-1})$ coincides with the kernel of the Hopf invariant. Let w_n denote the Whitehead product $[\iota_n, \iota_n]$, which has Hopf invariant 2.

If $\alpha \in \pi_{2n-1}(S^n)$ has Hopf invariant h then

$$\varphi_r \alpha = r\alpha + \tfrac{1}{2} h r (r-1) w_n , \tag{4.1}$$

where φ_r denotes the endomorphism induced by mapping S^n to itself with degree r. By Adams' theorem [1], $\pi_{2n-1}(S^n)$ contains no element of odd Hopf invariant unless $n = 2, 4$ or 8. If $n = 2, 4$ or 8 then $w_n = 2\gamma_n + E\beta_n$, where γ_n denotes the Hopf class and β_n generates the cyclic group $\pi_{2n-2}(S^{n-1})$. The order of β_n is 1, 12 or 120 according as $n = 2$, 4 or 8.

We continue to suppose that n is even. Let Y_m^{n+1} denote the complex formed by attaching an $(n+1)$-cell to S^n with a map of degree m. Chang [2] has shown that the kernel of the injection

$$j_* : \pi_{2n-1}(S^n) \to \pi_{2n-1}(Y_m^{n+1})$$

is generated by mw_n together with the elements of $\varphi_m \pi_{2n-1}(S^n)$.

We reduce this to

Theorem 4.2. *If* m *is odd or if* $n \neq 2, 4, 8$ *then*

$$j_*^{-1}(0) = m\pi_{2n-1}(S^n) \, .$$

If m *is even and* $n = 2$, 4 *or* 8 *then*

$$j_*^{-1}(0) = 2m\pi_{2n-1}(S^n) + \frac{m}{2} E\pi_{2n-2}(S^{n-1}) \, .$$

Suppose first that $\pi_{2n-1}(S^n)$ contains no elements of odd Hopf invariant and therefore is generated by w_n and suspension elements. Then since $\varphi_m E = mE$ and $\varphi_m w_n = m^2 w_n$ we conclude from Chang's result that $j_*^{-1}(0)$ is generated by mw_n together with the elements of $mE\pi_{2n-2}(S^{n-1})$. This proves (4.2) unless $n = 2$, 4 or 8. Now suppose that $n = 2, 4$ or 8. If m is odd it follows from (4.1), with $\alpha = \gamma_n$, and from Chang's result that $j_*^{-1}(0)$ is generated by mw_n and $m\gamma_n$. If m is even it follows similarly that $j_*^{-1}(0)$ is generated by mw_n and $\frac{1}{2}m(2\gamma_n - w_n)$, and hence is generated by mw_n and $\frac{1}{2}mE\beta_n$. Since $\pi_{2n-1}(S^n)$ is generated by γ_n and $E\beta_n$, this completes the proof of (4.2).

We regard Y_m^{n+1} as the suspension of Y_m^n and consider the commutative diagram shown below, where i_* is the injection.

$$\begin{tikzcd}
\pi_{2n-2}(S^{n-1}) \arrow[r, "i_*"] \arrow[d, "E"'] & \pi_{2n-2}(Y_m^n) \arrow[d, "E"] \\
\pi_{2n-1}(S^n) \arrow[r, "j_*"] & \pi_{2n-1}(Y_m^{n+1})
\end{tikzcd}$$

Notice that $i_* \varphi_m = 0$, and that

$$m\pi_{2n-1}(S^n) \frown E\pi_{2n-2}(S^{n-1}) = E\varphi_m\pi_{2n-2}(S^{n-1}) \, .$$

Hence and from (4.2) we deduce

Corollary 4.3. *Let* $\xi \in \pi_{2n-2}(S^{n-1})$ *be an element such that* $Ei_*\xi = 0$. *Suppose that* m *is odd or* $m \equiv 0 \bmod 2n$ *or* $n \equiv 2 \bmod 4$ *or* $n \geq 10$. *Then* $i_*\xi = i_*\eta$, *where* $E\eta = 0$.

§ 5. The spherical index

Let n be even and let $m \neq 0$. We take $q = n - 1$ in the construction of § 3, and suppose that v is a map of degree m. Consider the composition

$$\pi_{2n-1}(S^n) \xrightarrow{\Delta} \pi_{2n-2}(S^{n-1}) \xrightarrow{E} \pi_{2n-1}(S^n),$$

where Δ denotes the transgression operator associated with B. We prove

Theorem 5.1. *The kernel of $E\Delta$ contains an element of* Hopf *invariant* 1 *if either* (i) m *is odd, or* (ii) $n = 2$, *or* (iii) $n = 4$ *and* $m \equiv 0 \bmod 8$, *or* (iv) $n = 8$ *and* $m \equiv 0 \bmod 16$. *In all other cases the kernel contains an element of* Hopf *invariant* 2 *but does not contain an element of* Hopf *invariant* 1.

As in § 3 let μ denote the element of Hopf invariant m in $\pi_{2n-1}(S^n)$ which is obtained from u by the Hopf construction. If $\alpha \in \pi_{2n-1}(S^n)$ has Hopf invariant h then

$$E\Delta(\alpha) = \varphi_m(\alpha) - hm\mu, \tag{5.2}$$

by (2.5) of [5]. Hence and from (4.1) it follows that $E\Delta$ annihilates $2\mu + (1 - m)w_n$, which has Hopf invariant 2. Suppose that $h = 1$. Then $n = 2$, 4 or 8 and we write

$$\alpha = \gamma_n + sE\beta_n, \qquad \mu = m\gamma_n + tE\beta_n,$$

where s and t are integers. By (4.1) and (5.2) we have

$$E\Delta(\alpha) = m(s - t)E\beta_n + \tfrac{1}{2}m(m - 1)E\beta_n.$$

Thus we can choose s so as to make $E\Delta(\alpha) = 0$ if, and only if, $\frac{1}{2}m(m - 1)$ is divisible by the greatest common divisor of m and the order of β_n. Hence (5.1) follows at once.

The spherical index of B is defined, since $m \neq 0$. By (10.1) of [4] we have

$$I(B) = k\,|\,m\,|, \tag{5.3}$$

where k denotes the index, in the group of integers, of the image of $\Delta^{-1}(0)$ under the Hopf invariant. We recall from (3.5) of [4] that

$$E : \pi_{2n-2}(S^{n-1}) \to \pi_{2n-1}(S^n)$$

is a monomorphism if $n = 2$ or $n \equiv 0 \bmod 4$, and that in any case $2E^{-1}(0) = 0$. Therefore some information about k can be obtained from (5.1), and by substituting this in (5.3) we obtain

Corollary 5.4. *If* $n = 2$ *or if* m *is odd then* $I(B) = |m|$. *If* $n = 4$ *or* 8 *and if* m *is divisible by* $2n$ *then* $I(B) = |m|$. *If* $n = 4$ *or* 8 *and if* m *is even but not divisible by* $2n$ *then* $I(B) = 2|m|$. *If* $n \equiv 0 \bmod 4$ *and* $n \geq 12$ *then* $I(B) = 2|m|$. *If* $n \equiv 2 \bmod 4$ *and* $n \geq 6$ *then* $I(B) = 2|m|$ *or* $4|m|$.

Every case is covered in (5.4). By the methods used in § 7 of [4] it is possible to decide between the alternatives when $n \equiv 2 \bmod 4$ and $n \geq 6$, but this need not detain us here.

We further deduce

Corollary 5.5. *Suppose that* $I(B) = I(SB)$. *Then either* (i) $|m| \leq 2$ *and* $n = 2$, *or* (ii) m *is odd and* $n = 4$ *or* 8.

We can use (4.2) to determine $I(SB)$, since it follows from (3.1) that $I(SB)$ is equal to the least positive multiple of μ which is contained in the kernel of j_*. When $n = 6$ or $n \geq 10$ we find that $I(SB)$ is a divisor of m. When $n = 2$ we find that $I(SB) = 1$ or 2 according as m is odd or even. When $n = 4$ or 8 we find that $I(SB)$ is not divisible by the greater of the numbers 4 and $(m, 2n)$. Hence it follows from (5.4) that $I(SB) \neq I(B)$ unless m and n are as in (5.5).

§ 6. Proof of the main theorem

Let B be constructed as in § 3, with $q = n - 1$ and v a map of degree m. We prove

Lemma 6.1. *If* B *is an H-space then either* (i) $n = 2$ *and* $m = \pm 1$, ± 2 *or* (ii) $n = 4$ *and* $m = \pm 1$.

If $m = 0$ then the cohomology of B is an exterior algebra on generators of dimension n and $n - 1$. Since one of these numbers is even we have an immediate contradiction with the Hopf-Leray theorem. Therefore $m \neq 0$, and consequently n is even. If $n = 2$ then (6.1) follows immediately from (2.1) and (5.5). Let $n \geq 4$. Suppose, to obtain a contradiction, that m is divisible by some odd prime p. Then $H^n(B, Z_p)$ contains a non-zero element x, say, and since x is primitive, for dimensional reasons, it follows from elementary Hopf algebra theory that $x^2 \neq 0$. But $H^{2n}(B, Z_p) = 0$, and so we have a contradiction. Since m is odd, by (2.1) and (5.5), we conclude that $m = \pm 1$. Therefore B is a homotopy $(2n - 1)$-sphere and moreover it follows from Adams' theorem [1] that $n = 4$. This completes the proof of (6.1).

Suppose further that B is an $(n-1)$-sphere bundle over S^n. If $n = 2$ then the classification of fibre bundle theory shows that B is homeomorphic to S^3 if $m = \pm 1$, to real projective 3-space if $m = \pm 2$. If $n = 4$ and $m = \pm 1$ then MILNOR [8] has shown that B is homeomorphic to S^7. This completes the proof of (1.1).

§ 7. Homotopy classification

Let B and B' be $(n-1)$-sphere bundles over S^n, where n is even, such that

$$H_{n-1}(B) \approx Z_m \approx H_{n-1}(B') \qquad (m \geq 1)\,.$$

Then B and B' can be constructed as described in § 3 by attaching $(2n-1)$-cells to Y^n_m. Hence B and B' have the same homotopy type if, and only if, there exists a cellular homotopy equivalence $g : Y^n_m \to Y^n_m$ such that[4])

$$g_* \beta = \pm \beta' \,, \tag{7.1}$$

where $\beta, \beta' \in \pi_{2n-2}(Y^n_m)$ denote the attaching classes and g_* denotes the automorphism induced by g. Suppose that (7.1) is satisfied where g is a map of degree r. Then $f_* E\beta = \pm E\beta'$, where f_* is the automorphism of $\pi_{2n-1}(Y^{n+1}_m)$ induced by the suspension of g, and hence

$$\varphi_r \mu \equiv \pm \mu' \bmod j_*^{-1}(0) \tag{7.2}$$

by (3.1), where $\mu' \in \pi_{2n-1}(S^n)$ corresponds to B' as μ does to B. By (5.1) of [7],

$$k_* \beta = w'_n = k_* \beta' \,, \tag{7.3}$$

where $w'_n \in \pi_{2n-2}(Y^n_m, S^{n-1})$ denotes the relative WHITEHEAD product of ι_{n-1} with a generator of $\pi_n(Y^n_m, S^{n-1})$ and k_* denotes the injection. Also it follows from (10.1) of [4] that the order of w'_n is either m or $2m$ according as $\pi_{2n-1}(S^n)$ does or does not contain an element of HOPF invariant 1. The automorphism of $\pi_{2n-2}(Y^n_m, S^{n-1})$ induced by g transforms w'_n into $r^2 w'_n$. Hence and from (7.1) it follows that

$$\left.\begin{aligned} r^2 &\equiv \pm 1 \bmod m \quad \textit{if} \quad n = 2, 4 \textit{ or } 8\,; \\ &\equiv \pm 1 \bmod 2m \quad \textit{otherwise.} \end{aligned}\right\} \tag{7.4}$$

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[4]) The signs $\pm$ in this section are linked together, i.e. they stand for plus in every case or for minus in every case.

Thus (7.2) and (7.4) are satisfied for some value of r if B and B' have the same homotopy type.

Conversely, let r be an integer such that (7.2) and (7.4) are satisfied. Let $g : Y^n_m \to Y^n_m$ be a cellular map of degree r. Then

$$Eg_*\beta = \pm E\beta', \quad k_*g_*\beta = \pm k_*\beta',$$

by (3.1) and (7.3). Hence $\pi_{2n-2}(S^{n-1})$ contains an element ξ such that $Ei_*\xi = 0$ and $g_*\beta = \pm\beta' + i_*\xi$. Suppose that either m is odd or $m \equiv 0 \bmod 2n$ or $n \equiv 2 \bmod 4$ or $n \geq 10$. Then it follows from (4.3) and from (5.3) of [13] that $\xi \equiv \eta \bmod i_*^{-1}(0)$, where η denotes the Whitehead product of ι_{n-1} with some element of $\pi_n(S^{n-1})$. By (3.5) of [7] there exists a cellular map $g' : Y^n_m \to Y^n_m$, also of degree r, such that

$$g_*\beta - g'_*\beta = i_*\eta = i_*\xi .$$

Then g', like g, is a homotopy equivalence, because r is prime to m. Since $g'_*\beta = \pm\beta'$ this proves that B and B' have the same homotopy type.

As an application we deduce

Theorem 7.5. *Let $n = 6$ or let $n \geq 10$. Then B and B' have the same homotopy type if, and only if, there exists an integer r, where $r^2 \equiv 1 \bmod 2m$, such that*

$$rE\mu \equiv E\mu' \bmod m\pi_{2n}(S^{n+1}) .$$

Suppose that B and B' have the same homotopy type. Then (7.2) and (7.4) are satisfied, as we have seen. Since m is even, by Adams' theorem, and since -1 is not a quadratic residue of 4 it follows from (7.4) that $r^2 \equiv 1 \bmod 2m$. By (7.2), therefore, $\varphi_r\mu \equiv \mu' \bmod j_*^{-1}(0)$, and since $j_*^{-1}(0) = m\pi_{2n-1}(S^n)$, by (4.2), it follows by suspension that

$$rE\mu \equiv E\mu' \bmod m\pi_{2n}(S^{n+1}).$$

Conversely, suppose that $r^2 \equiv 1 \bmod 2m$ and $rE\mu \equiv E\mu' \bmod m\pi_{2n}(S^{n+1})$. Certainly (7.4) is satisfied. The suspension

$$E : \pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1})$$

is an epimorphism, and its kernel is generated by w_n. Hence it follows that

$$r\mu - \mu' \equiv \tfrac{1}{2}m(r-1)w_n \bmod m\pi_{2n-1}(S^n) ,$$

and therefore

$$\varphi_r\mu - \mu' \equiv \tfrac{1}{2}m(r^2-1)w_n \equiv 0 \bmod m\pi_{2n-1}(S^n) ,$$

since $r^2 - 1$ is even. Hence and from (4.2), (7.2) follows at once. Therefore B and B' have the same homotopy type, which completes the proof of (7.5).

It is clear that for some values of m a classification theorem is obtainable by these methods in case $n = 4$ or 8, but more elaborate techniques are necessary before all values of m can be treated. The appropriate theory has been developed by Dr. M. G. BARRATT (unpublished). However, except for these cases (7.5) fills the gap in the classification of sphere-bundles over spheres by homotopy type of the total space as given in [6] and [7].

§ 8. Appendix

Let E, F, B be spaces, with $F \subset E$, and let $p : E \to B$ be a map which is constant on F. Suppose that p induces an isomorphism

$$p_* : \pi_r(E, F) \to \pi_r(B) \qquad (r = 0, 1, \ldots)$$

and that there exists a map $f : B \to E$ such that $pf \simeq 1$. Then we obtain the direct sum decomposition

$$i_* \oplus f_* : \pi_r(F) \oplus \pi_r(B) \approx \pi_r(E) ,$$

where f_* is induced by f and i_* is the injection. Hence if E is an H-space it follows that

$$h_* : \pi_r(F \times B) \approx \pi_r(E) ,$$

where $h : F \times B \to E$ is the map defined by

$$h(x, y) = x \cdot f(y) \qquad (x \in F, y \in B) .$$

Suppose in addition that E and $F \times B$ have the homotopy type of CW-complexes. Then h is a homotopy equivalence, by Theorem 1 of [14]. Hence $F \times B$ is an H-space, and hence F and B are H-spaces.

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