On the Local Triviality of the Restriction Map for Embeddings

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Let V, M be C^∞ manifolds, V compact. A map $f: M \to M$ is said to have *compact support* if it agrees with the identity outside of a compact set. For $1 \leq r \leq \infty$, we consider the following spaces endowed with the C^r-topology: $\mathcal{E}^r(V, M) =$ all C^∞ embeddings of V in M; $\mathcal{C}^r(M) =$ all C^∞ maps, with compact support, of M into M; $\mathcal{D}^r(M) =$ all C^∞ diffeomorphisms, with compact support, of M onto M. We remark that $\mathcal{D}^r(M)$ is an open subset of $\mathcal{C}^r(M)$.

R. PALAIS proved [1] that if V is a submanifold of W then the restriction map $j: \mathcal{E}^r(W, M) \to \mathcal{E}^r(V, M)$ is a locally trivial fibration. Previously, R. THOM had observed [2] that j has the covering homotopy property for polyhedra. The local triviality of j follows easily from the theorem below, (see [1], or Remark 2), at the end of this note), which was also proved by J. CERF [3]. We present here a very simple proof of this theorem. For implications and applications, see the bibliography.

Theorem: *Given $f \in \mathcal{E}^r(V, M)$, there is a neighborhood U of f and a continuous map $\xi: U \to \mathcal{D}^r(M)$ such that $g = \xi(g) \circ f$ for every $g \in U$.*

Proof: We may assume that V is a submanifold of M, $f =$ inclusion, and M is embedded in some euclidean space R^k. Let $\pi': T' \to M$ be a tubular neighborhood of M in R^k and $\pi: T \to V$ a tubular neighborhood, of radius $\varepsilon > 0$, of V in R^k, with $T \subset T'$. Denote by $\frac{1}{2}T$ the tubular neighborhood of V with radius $\varepsilon/2$. Since the shortest line from a point in R^k to V is a normal segment, any line segment of length $< \varepsilon/2$ which intersects $\frac{1}{2}T$ lies entirely within T. Choose a neighborhood U' of f in $\mathcal{E}^r(V, M)$ so small that $|g(y) - y| < \varepsilon/2$ for all $g \in U'$ and all $y \in V$. Let $\lambda: R \to [0, 1]$ be a C^∞ function with $\lambda(t) = 1$ for $|t| \leq \varepsilon/4$ and $\lambda(t) = 0$ for $|t| \geq \varepsilon/2$. Define a map $\xi': U' \to \mathcal{C}^r(M)$ as follows. Given $g \in U'$, put $\xi'(g)(x) = x$, if $x \in M - T$, and $\xi'(g)(x) = \pi'\{x + \lambda(|x - \pi x|) \cdot [g(\pi x) - \pi x]\}$ if $x \in T$. One sees that ξ' is continuous and $\xi'(f)$ is the identity map of M, so $\xi'(f) \in \mathcal{D}^r(M)$. Since $\mathcal{D}^r(M)$ is open in $\mathcal{C}^r(M)$, a smaller neighborhood U of f can be chosen so that $\xi'(U) \subset \mathcal{D}^r(M)$. Put $\xi = \xi' | U$.

Remarks: 1) Let $\mathcal{D}^r_0(M) \subset \mathcal{D}^r(M)$ be the subset of C^∞ diffeomorphisms, with compact support, that are diffeotopic to the identity. It is known that

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$\mathcal{D}_0^r(M)$ is open in $\mathcal{D}^r(M)$. (This can be seen by a construction similar to, and simpler than, the above one.) So, if needed, U may be taken such that $\xi(U) \subset \mathcal{D}_0^r(M)$.

2) Given $f \in \mathcal{D}^r(V, M)$, take ξ and U as in the theorem, let $F = j^{-1}(f)$ and define a homeomorphism $\psi : F \times U \to j^{-1}(U)$ by $\psi(\bar{f}, g) = \xi(g) \circ \bar{f}$, for $\bar{f} \in F$, $g \in U$. This shows that j is a locally trivial fibration.

3) When $r = \infty$, $\mathcal{E}^\infty(V, M)$ and $\mathcal{D}^\infty(J)$ are C^∞ (infinite dimensional) manifolds, locally homeomorphic with FRÉCHET spaces. The reason why ξ is continuous is that, in the last analysis, it is obtained as a series of compositions of the variable map g with fixed C^∞ maps. Now, composition is a differentiable map in the C^∞ topology. (See [4], pages 182, 183.) So, by the same token, ξ is a C^∞ map when $r = \infty$. It follows from this and Remark 2) above that $j : \mathcal{E}^\infty(W, M) \to \mathcal{E}^\infty(V, M)$ would be a C^∞ fibration, in the sense that the local trivializing maps $\psi : F \times U \to j^{-1}(U)$ are C^∞, provided one could show that F is a differentiable manifold.

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