Some Semicontinuity Theorems for Convex Polytopes and Cell-complexes.

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Introduction

In a finite-dimensional EUCLIDean space E, a *convex polytope* P is a set which is the convex hull of a finite set; here such a set P will simply be called a *polytope*, or, if it is d-dimensional, a *d-polytope*.[2]) An *s-face* of P is an s-dimensional set (necessarily an s-polytope) which is either P itself or is the intersection of P with a supporting hyperplane. The number of s-faces of P will be denoted by $f_s(P)$ and the s-measure of their union by $\zeta_s(P)$. Our section headings are as follows:

1. The functions f_s for cell-complexes
2. A pulling process
3. The functions ζ_s for cell-complexes
4. The functions ζ_s for convex polytopes
5. Inequalities for the functions ζ_s
6. Intersection properties of simplices
7. Sections of simplices

Each polytope is a compact set, and thus the class of all polytopes in E can be metrized by the HAUSDORFF distance. Our principal aim in §1 is to show that when the polytopes are topologized in this way, each of the functions f_s is lower semicontinuous; thus for each polytope P in E it is true that all polytopes sufficiently close to P have at least as many s-faces as P. In connection with various extremal problems, this result can be used to restrict attention to polytopes whose vertices are in general position, but for that purpose the pulling process described in §2 is more advantageous.

Lower semicontinuity of the functions ζ_s is established in §§3–4.[3]) The reasoning of §3, like that of §1, applies not only to convex polytopes but also

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[1]) This paper was written at the University of Washington (Seattle, Washington, U. S. A.) while Professors EGGLESTON and GRÜNBAUM were visiting there, on leave respectively from Bedford College, London and The Hebrew University, Jerusalem. EGGLESTON's work was supported by a fellowship from the National Science Foundation (U. S. A.), GRÜNBAUM's and KLEE's by an N. S. F. grant (NSF-GP-378).

[2]) As basic references, we mention WEYL [20] for convex polytopes, ALEXANDROFF and HOPF [2] for cell-complexes, HUREWICZ and WALLMAN [12] for HAUSDORFF s-measure, and HADWIGER [11] for isoperimetric problems.

[3]) It is well-known (e. g. [11]) that for d-polytopes P, the functions $\zeta_d(P)$ and $\zeta_{d-1}(P)$ are continuous; they represent the volume and the surface area of P.

to a rather general class of cell-complexes[2]); however, it is based on a supplementary hypothesis that is removed in § 4 for the case of convex polytopes. As is shown in § 5, the lower semicontinuity of the functions f_s and ζ_s can be used to establish the existence of solutions to various isoperimetric problems. Also in § 5 is a proof that if P is a d-polytope and a and b are positive integers whose sum is at most d, then $\zeta_{a+b}(P) \leq \zeta_a(P)\zeta_b(P)$. From this and some related results we conclude that if $1 \leq r \leq s \leq d$ and if r divides s or $s = d - 1$ or $s = d$, then there exists a finite constant $\gamma(d, r, s)$ such that $\zeta_s(P)^{1/s} / \zeta_r(P)^{1/r} \leq \gamma(d, r, s)$ for all d-polytopes P. We do not determine the best values for these constants, nor even their existence except when r and s are as indicated.

From the lower semicontinuity of f_s it follows that if a decreasing sequence of d-simplices has a d-dimensional intersection S, then S is also a simplex. However, this was conjectured by Kolmogorov and proved by Borovikov [3][4]) without the assumption that S is d-dimensional; that is, the intersection of a decreasing sequence of simplices is always a simplex. § 6 contains a new proof that applies also to infinite-dimensional simplices; it is based on a characterization of simplices due to Choquet [5] and to Rogers and Shephard [19]. In § 7 a third approach is indicated, related to the fact (Brands and Laman [4], Eggleston [8], Croft [7]) that some 2-face of a tetrahedron has an area not exceeded by that of any plane section[5]); in particular, an n-dimensional generalization of this fact is established. Borovikov's proof of his theorem is shorter than either of ours, but our supplementary facts seem interesting in themselves.

1. The functions f_s for cell-complexes

Recall that the set K of all faces of a polytope forms a *cell-complex* [2]; that is, K is a finite family of polytopes in E such that each face of a member of K is itself a member of K, and such that the intersection of any two members of K is a face of both. The s-dimensional members of a cell-complex K will be called its *s-faces*, and the number of s-faces will be denoted by $f_s(K)$. Henceforth, the terms *complex* and *cell-complex* will be used interchangeably. For a polytope P or a complex K, $\sigma_s P$ or $\sigma_s K$ will denote the union of all s-faces.

When C is a convex subset of E and aff C is the smallest flat containing C, the interior of C relative to aff C is called the *relative interior* of C; it will be denoted here by ϱC, and the set C is *relatively open* if and only if $C = \varrho C$.

[4]) We are indebted for this reference to Professors T. Ganea and A. Kolmogorov.

[5]) This was posed as a problem by H. Grossman and E. Ehrhart, Amer. Math. Monthly 65 (1958) 43 and 69 (1962) 63.

For each complex K, let $|K|$ denote the union of all members of K; K will be said to have the *property* $A(s)$ provided whenever F is a face of K, C is a relatively open convex subset of $|K|$,

$$C \cap \varrho F \neq \emptyset \quad \text{and} \quad \dim C \geq \dim F = s\,,$$

then

$$C \subset F\,.$$

This condition will appear in Theorem 1.7 below. Its relevance to polytopes rests on the following fact.

1.1. *If K is a subcomplex of the complex of all faces of a polytope P, then every relatively open convex subset C of $|K|$ is contained in the relative interior of a single face of K; hence K has the property $A(s)$ for all values of s.*

Proof. We proceed by induction on the dimension d of the polytope, the assertion being obvious when $d = 1$. Suppose it is known for all $d < k$, and consider the case of a k-polytope P. If $C \subset \varrho P$, the desired conclusion holds. If C is not contained in ϱP, then C includes a relative boundary point z of P and P is properly supported at z by a hyperplane H. But then $C \subset H$, and the desired conclusion follows from the inductive hypothesis as applied to the cell-complex $\{Q \cap H : Q \in K\}$.

Sometimes the property $A(s)$ of a complex K can be conveniently verified in terms of the stronger property $A^*(s)$, where K is said to have the *property* $A^*(s)$ provided every relatively open s-dimensional convex subset of $|K|$ is contained in the relative interior of a single face of K. Although the properties $A(s)$ are independent for the various values of s, this is not true of $A^*(s)$.

1.2. *Every cell-complex K has the property $A^*(0)$. If K has the property $A^*(s)$ for some $s \geq 1$, then K has the property $A^*(t)$ for all $t \geq s$ and hence also the property $A(t)$ for all $t \geq s$. If K has the property $A^*(1)$ then K has the property $A(s)$ for all values of s.*

Proof. The first assertion is obvious. For the second, let us assume that K lacks the property $A^*(t)$. Then there is a relatively open t-dimensional convex subset J^t of $|K|$ and there are faces F and G of K such that J^t intersects both ϱF and ϱG. Choose $x \in J^t \cap \varrho F$, $y \in J^t \cap \varrho G$, and let J^s denote the intersection of J^t with an s-flat in aff J^t that contains the segment $[x, y]$. Then J^s is a relatively open s-dimensional convex subset of K and J^s intersects both ϱF and ϱG, so K lacks the property $A^*(s)$. This takes care of the second assertion. For the third assertion, it remains only to verify that $A^*(1)$ implies $A(0)$, and that is easily done.

Four lemmas are required in preparation for the main result.

1.3. *If* τ *is an affine transformation of a polytope* P *onto a polytope* Q, *then for every face* G *of* Q *there is a face* F *of* P *such that* $\tau F = G$.

Proof. It suffices to consider the case in which G is a proper face of Q, whence there is an affine functional φ on aff Q such that $\varphi G = 1$ and $\varphi < 1$ on $Q \sim G$. Then $\varphi\tau$ is an affine functional on aff P (where τ carries aff P onto aff Q), and we obtain the desired face F of P by defining

$$F \cdot = \{x \in P : \varphi(\tau x) = 1\} .$$

1.4. *If* P *is a polytope and* M *is a flat in* aff P *that intersects* P, *then* M *intersects a* (dim P — dim M)*-face of* P.

Proof. We proceed by induction on the dimension m of M, the assertion being obvious for $m = 0$ and also for $m = 1$, since the relative boundary of P is intersected by every line in aff P that intersects P itself. Suppose the theorem is known for all $m < k$, and consider the case in which $m = k > 1$. We may assume that P is a d-polytope in E^d with $0 \in P$ and that M is a k-flat in E^d with $0 \in M$. Let H be a $(d-1)$-flat such that $0 \in H \subset E^d$ and $\dim(M \cap H) = k - 1$, and let τ be a linear projection of E^d onto H that carries M onto $M \cap H$. Applying the inductive hypothesis to the polytope τP and the flat $M \cap H$, we see that $M \cap H$ must include a point z of a $(d-k)$-face G of τP. By 1.3, there is a face F of P such that $\tau F = G$. The nature of τ is such that $d - k \leq \dim F \leq d - k + 1$ and the set $\tau^{-1}(z)$ is a line in M that intersects F. If $\dim F = d - k$, the desired conclusion holds. If $\dim F = d - k + 1$, then $\tau^{-1}(z) \subset$ aff F and it follows from the inductive hypothesis that $\tau^{-1}(z)$ intersects a $(d-k)$-face of F. But then of course M intersects a $(d-k)$-face of P, and this completes the proof.

1.5. *If* τ *is an affine transformation of a polytope* P *onto an* s*-polytope* Q, *then* $\tau(\sigma_s P) = Q$.

Proof. Let $d = \dim P$ and consider an arbitrary point $q \in Q$. Then $\tau^{-1}(q)$ is a $(d-s)$-flat in aff P and hence, by 1.4, $\tau^{-1}(q)$ must intersect an s-face of P. The desired conclusion follows.

1.6. *If an* s*-polytope* S *is the limit of a sequence of polytopes* P_α, *then the sequence* $\sigma_s P_\alpha$ *is also convergent to* S.

Proof. Let τ be an affine projection of the containing space onto aff S. We note:

(*) for each $\epsilon > 0$ there exists m_ϵ such that $\|x - \tau x\| < \epsilon$ whenever $x \in P_i$ and $i > m_\epsilon$.

From (*) and the fact that $P_\alpha \to S$, it follows that $\tau P_\alpha \to S$. By 1.5, $\tau(\sigma_s P_i) = \tau P_i$ and thus $\tau(\sigma_s P_\alpha) \to S$; a second application of (*) shows that $\sigma_s P_\alpha \to S$.

In view of 1.1 we see that the first semicontinuity theorem stated in the Introduction is an immediate consequence of the following more general result.

Theorem 1.7. *Suppose K_α is a sequence of cell-complexes in E, K is a cell-complex in E having the property $A(s)$, and the sequence $|K_\alpha|$ is convergent to $|K|$. Suppose further that at least one of the following conditions is satisfied:*

(i) *the sequence $f_0(K_\alpha)$ is bounded;*

(ii) *for each i, $|K_i|$ is covered by K_i's faces of dimension $\geq s$;*

(iii) *for each point z in an s-face of K, there is a sequence z_α converging to z such that always z_i is in a face of K_i of dimension $\geq s$.*

Then $\liminf f_s(K_\alpha) \geq f_s(K)$.

Proof. Since condition (iii) is implied by (ii), it suffices to consider (i) and (iii). Suppose the desired conclusion fails. Then there exist a subsequence D_α of K_α and an integer n_s such that for all i,

$$f_s(D_i) = n_s < f_s(K)\,.$$

For $t \geq s$, each t-face of D_i is uniquely determined by the set of its own s-faces, and a t-face has at least $\binom{t+1}{s+1}$ s-faces. Consequently the sequence $f_t(D_\alpha)$ is bounded for all $t \geq s$, and of course it is identically zero for $t > d \cdot = = \dim E$. Thus we can choose a subsequence C_α of D_α, non-negative integers $n_s, \ldots, n_d$, and for each i an indexing of the faces of C_i such that the following three conditions are all satisfied:

for $s \leq t \leq d$ and for each i, the cell-complex C_i has exactly n_t t-faces $P_i^{t,1}, \ldots, P_i^{t,n_t}$;

for $s \leq t \leq d$ and for $1 \leq h \leq n_t$, the sequence $P_\alpha^{t,h}$ is convergent to a compact convex set $P^{t,h}$ of dimension $d(t, h) \leq t$;

for each i, $f_s(C_i) = n_s < f_s(K)$.

If condition (i) holds, we may assume that the sets $P_i^{t,h}$ and $P^{t,h}$ have been chosen for $0 \leq t \leq d$.

Let $Q_1, \ldots, Q_m$ be the s-faces of K. Since each of these sets is closed and none is covered by the others, there exist $\epsilon > 0$ and points x_k such that for $1 \leq k \leq m (= f_s(K))$,

$$x_k \in \varrho Q_k \sim N_{3\epsilon}(\textstyle\bigcup_{1 \leq j \leq m,\, j \neq k} Q_j)\,,$$

where N_δ of a point or set will denote its open δ-neighborhood. The s-dimensional set $N_\epsilon(x_k) \cap \varrho Q_k$ is not the union of finitely many convex sets of dimension $< s$, and consequently there exists a point

$$y_k \in (N_\epsilon(x_k) \cap \varrho Q_k) \sim \bigcup_{r\geq s,\, 1\leq j\leq n_r} \sigma_{s-1}P^{r,j}.$$

If (i) holds, then '$r \geq s$' may be replaced by '$r \geq 0$'. Hence if either (i) or (iii) is satisfied we have

$$y_k \in \varrho P^{t(k),e(k)}$$

with

$$t(k) \geq s,\ 1 \leq e(k) \leq n_{t(k)},\ d(t(k), e(k)) \geq s\,.$$

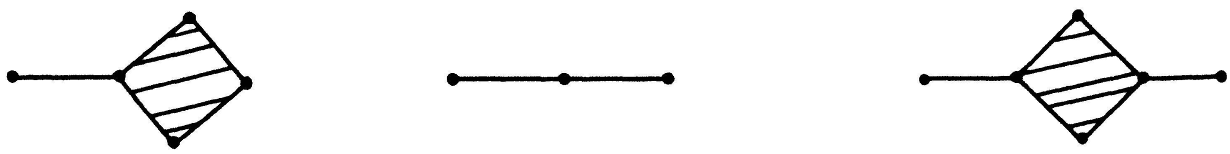
Since $y_k \in \varrho Q_k$, it follows from the property $A(s)$ of K that

$$\varrho P^{t(k),e(k)} \subset \varrho Q_k \quad (1 \leq k \leq m)\,,$$

whence $d(t(k), e(k)) = s$.

Recalling that $P^{t(k),e(k)}$ is the limit of the sequence $P_\alpha^{t(k),e(k)}$, we see from 1.6 that the sequence $\sigma_s P_\alpha^{t(k),e(k)}$ is also convergent to $P^{t(k),e(k)}$. Hence there exists n such that for each $i > n$ there is an s-face $F_{k,i}$ of $P_i^{t(k),e(k)}$ for which $F_{k,i} \subset N_\epsilon(Q_k)$ and $F_{k,i}$ intersects $N_\epsilon(y_k)$. But then $F_{k,i}$ intersects $N_{2\epsilon}(x_k)$, and because of our special choice of ϵ this implies that $F_{k,i}$ is not contained in the ϵ-neighborhood of any set Q_j with $j \neq k$. Thus (for $i > n$) the association of the $F_{k,i}$'s with the Q_k's is biunique and we have $f_s(C_i) \geq f_s(K)$, a contradiction completing the proof.

It would be interesting to find a simple *characterization* of those cell-complexes K for which 1.7 is valid. Certainly the property $A(s)$ is not necessary, for with arbitrary E and s the plane complex A below has the stated property even though A lacks the property $A(1)$. However, the property fails for B below (E arbitrary, $s = 1$) and also for C ($\dim E \geq 3$, $s = 1$).



A B C

A suitable 2-complex in E^3 shows that condition (ii) cannot be replaced by the requirement that each vertex of K_i lies in a face of dimension $\geq s$.

2. A pulling process

We know by 1.7 that for all s, every polytope Q sufficiently close to a given polytope P has at least as many s-faces as P; however, except for this restriction the facial structure of Q may be radically different from that of P. In various circumstances, it is convenient to be able to choose Q so that its facial structure is well-behaved in some specific way (for example, so that all of Q's proper faces are simplices) and at the same time is closely related to the facial structure of P. This may be accomplished with the aid of the 'pushing process' of [14] or the 'pulling process' described below. For some purposes [10, 14, 15] these are interchangeable; for others [16, 17], the pulling process is preferable because its effect on facial structure is more easily described. Let us denote the convex hull of a set X by con X, and the set of all vertices or extreme points of a polytope P by ex P. When X is the set of all vertices of a d-polytope con X in E^d, and q is one of these vertices, we say that *X' is obtained from X by pulling* (resp. *pushing*) *q to q'* provided $X' = (X \sim \{q\}) \cup \{q'\}$, where q' is a point of E^d such that the half-open segment $]q, q']$ does not intersect any hyperplane determined by points of X and such that $q \in \operatorname{int} \operatorname{con} X'$ (resp. $q' \in \operatorname{int} \operatorname{con} X$). It is clear that such a pulling and pushing are always possible, and that $\operatorname{ex}(\operatorname{con} X') = (X \sim \{q\}) \cup \{q'\}$. The following result may be compared with 2.4 of [14].

Theorem 2.1. *Suppose X is the set of all vertices of a d-polytope* con X *in E^d, and X' is obtained from X by pulling q to q'. For $0 \leq s \leq d-1$, the s-faces of* con X' *are exactly the sets F^s and the sets* $\operatorname{con}(G^{s-1} \cup \{q'\})$, *where F^s is an s-face of* con X *such that $q \notin F^s$ and G^{s-1} is an $(s-1)$-face of a $(d-1)$-face K^{d-1} of* con X *such that $q \in K^{d-1} \sim G^{s-1}$.*

Proof. We recall that if S is a proper subset of the set T of all vertices of a polytope $P = \operatorname{con} T$, then the following three assertions are equivalent: S is the set of all vertices of some face of P; $\operatorname{aff} S \cap \operatorname{con}(T \sim S) = \emptyset$; T admits a supporting hyperplane H for which $T \cap H = S$. These characterizations will be used without further explicit reference.

Note that in the statement of 2.1, F^s may be described alternatively as an s-face of a $(d-1)$-face J^{d-1} of con X such that $q \notin J^{d-1}$. Clearly q lies in an open halfspace Q determined by the hyperplane $\operatorname{aff} J^{d-1}$, and from the definition of pulling it follows that also $q' \in Q$. Now if F^s were not a face of con X', then $\operatorname{aff} F^s$ would intersect the set $\operatorname{con}(X' \sim \operatorname{ex} F^s)$ and hence (since $\operatorname{aff} F^s \subset \operatorname{aff} J^{d-1}$ and $q' \in Q$) $\operatorname{aff} F^s$ would intersect $\operatorname{con}(X \sim \operatorname{ex} F^s)$, contradicting the fact that F^s is a face of con X. It follows that F^s is a face of con X'; and from the fact that $q \in \operatorname{int} \operatorname{con} X'$ we deduce easily that every face of con X' that is not incident to q' must also be a face of con X.

Next, consider G^{s-1} and K^{d-1} as described, and let G^{d-2} be a $(d-2)$-face of K^{d-1} such that $q \in K^{d-1} \sim G^{d-2}$. By the preceding paragraph, G^{d-2} is a $(d-2)$-face of both con X and con X', and hence is incident to exactly two $(d-1)$-faces of each of these d-polytopes. Since one of the former two faces (K^{d-1}) includes q, one of the latter must include q'. That is, the set

$G^{d-2} \cup \{q'\}$ lies in a supporting hyperplane H of X'. By the definition of pulling, $H \cap (X \sim G^{d-2}) = \emptyset$, and consequently the set $\mathrm{con}\,(G^{d-2} \cup \{q'\})$ is a $(d-1)$-face of $\mathrm{con}\, X'$. But then $\mathrm{con}\,(G^{s-1} \cup \{q'\})$ is an s-face of $\mathrm{con}\, X'$.

To complete the proof we must show that if F^s_* is an s-face of $\mathrm{con}\, X'$ that is incident to q', then $F^s_* = \mathrm{con}\,(F^{s-1} \cup \{q'\})$ where F^{s-1} is an $(s-1)$-face of a $(d-1)$-face K^{d-1} of $\mathrm{con}\, X$ such that $q \in K^{d-1} \sim F^{s-1}$. Let F^{d-1}_* be a $(d-1)$-face of $\mathrm{con}\, X'$ that contains F^s_*; let $W \cdot = \mathrm{ex}\, F^s_* \sim \{q'\}$ and $Z \cdot = \mathrm{ex}\, F^{d-1}_* \sim \{q'\}$. If $\mathrm{aff}\, W$ is s-dimensional, then $q' \in \mathrm{aff}\, W$ and this contradicts the definition of pulling; thus $\mathrm{aff}\, W$ is $(s-1)$-dimensional and the set $F^{s-1} \cdot = \mathrm{con}\, W$ is an $(s-1)$-face of $\mathrm{con}\, X'$, whence also of $\mathrm{con}\, X$. Similarly, $\mathrm{con}\, Z$ is a $(d-2)$-face of $\mathrm{con}\, X'$ and of $\mathrm{con}\, X$. Let $H \cdot = = \mathrm{aff}\,(Z \cup \{q\})$, a hyperplane in E^d. Then H supports X, for otherwise some point of X would lie on the q' side of H, whence $]q, q'] \cap \mathrm{aff}\,(Z \cup \{x\}) \neq \emptyset$ and the definition of pulling is contradicted. With H supporting X, we see that the set

$$(\mathrm{aff}\,(W \cup \{q\})) \cap \mathrm{con}\, X$$

is a face of $\mathrm{con}\, X$; denoting by K^{d-1} any $(d-1)$-face of $\mathrm{con}\, X$ containing it, we see that F^{s-1} is a face of K^{d-1}. This completes the proof.

3. The functions ζ_s for cell-complexes

In the present section, μ_s will denote an arbitrary positive-valued real function subject to the following four conditions:

(a) the domain of μ_s is the family U_s of all nonempty subsets of E that can be expressed as the union of finitely many s-polytopes;

(b) if $U \in \mathsf{U}_s$ and $P_1, \ldots, P_n$ are s-polytopes in E such that $U \subset \bigcup_{i=1}^n P_i$, then $\mu_s(U) \leq \Sigma_{i=1}^n \mu_s(P_i)$;

(c) if $P_1, \ldots, P_n$ are distinct s-faces of a complex in E, then $\mu_s(\bigcup_{i=1}^n P_i) = = \Sigma_{i=1}^n \mu_s(P_i)$;

(d) if a sequence P_α of s-polytopes in E is convergent to an s-polytope P, then $\mu_s(P) \leq \liminf \mu_s(P_\alpha)$.

For each complex K, $\zeta_s(\mathsf{K}) \cdot = \mu_s(\sigma_s \mathsf{K})$, the s-measure of the s-skeleton $\sigma_s \mathsf{K}$.

Obviously conditions (b) and (c) are satisfied by the s-dimensional HAUSDORFF measure[2]) based on a EUCLIDean metric for E. To see that condition (d) is also satisfied, we denote by π the orthogonal projection of E onto the flat $\mathrm{aff}\, P$, whence πP_α is convergent to P and hence $\mu_s(P) = \lim \mu_s(\pi P_\alpha)$ by a well-known result [11] on the continuity of volume. On the other hand, $\mu_s(\pi P_i) \leq \mu_s(P_i)$ because the transformation π is a metric contraction.

A complex K will be said to have the *property* $A_*(s)$ provided there is no s-dimensional face F of K whose relative interior ϱF intersects a relatively open convex subset C of $|K|$ of dimension $> s$; this amounts to replacing the condition '$C \subset F$' in the definition of $A(s)$ by the condition that '$\dim C \leq \leq \dim F$'. Thus $A^*(s) \Longrightarrow A(s) \Longrightarrow A_*(s)$. The three complexes pictured in § 1 all have the property $A_*(1)$ but not the property $A(1)$.

Theorem 3.1. *Suppose* K_α *is a sequence of cell-complexes in* E, K *is a cell-complex in* E, *and the sequence* $|K_\alpha|$ *is convergent to* $|K|$. *Suppose further that the sequence* $f_0(K_\alpha)$ *is bounded, and that* K *has the property* $A_*(s)$ *or* $\liminf \sigma_s K_\alpha \supset \sigma_s K$.

Then $\liminf \zeta_s(P_\alpha) \geq \zeta_s(P)$.

Proof. Suppose the desired conclusion fails. Then as in 1.7, there exist a subsequence C_α of K_α, a number $\delta \in]0,1[$, integers $n_0, \ldots, n_d$, and for each i an indexing of the faces of C_i such that the following three conditions are all satisfied:

for $0 \leq t \leq d$ and for each i, the cell-complex C_i has exactly n_t t-faces $P_i^{t,1}, \ldots, P_i^{t,n_t}$;

for $0 \leq t \leq d$ and for $1 \leq h \leq n_t$, the sequence $P_\alpha^{t,h}$ is convergent to a polytope $P^{t,h}$ of dimension $d(t,h) \leq t$;

$$\liminf \mu_s(\sigma_s C_\alpha) < \delta \mu_s(\sigma_s K) . \tag{1}$$

Now suppose first that $\liminf \sigma_s K_\alpha \supset \sigma_s K$, whence of course $\liminf \sigma_s C_\alpha \supset \sigma_s K$. Since $f_0(C_\alpha)$ is bounded, it follows with the aid of 1.6 that

$$\sigma_s K \subset \textstyle\bigcup_{j \in J} P^{s,j} , \tag{2}$$

where J is the set of all indices j such that $1 \leq j \leq n_s$ and $d(s,j) = s$. Since the sequence $P_\alpha^{s,j}$ converges to $P^{s,j}$, it follows from (d) that for all sufficiently large i we have

$$\mu_s(P_i^{s,j}) > \delta \mu_s(P^{s,j}) \quad (\text{all } j \in J) . \tag{3}$$

Then from (b) and (c) in conjunction with (2) and (3) we see that

$$\mu_s(\sigma_s K) \leq \Sigma_{j \in J}\, \mu_s(P^{s,j}) < \frac{1}{\delta}\, \Sigma_{j \in J}\, \mu_s(P_i^{s,j}) \leq \frac{1}{\delta}\, \mu_s(\sigma_s C_i) ,$$

contradicting (1) and completing the proof.

Finally, suppose K has the property $A_*(s)$ and consider an arbitrary s-face Q of K. Let

$$Q^* \cdot = \varrho Q \sim \textstyle\bigcup_{r \geq 0,\, 1 \leq j \leq n_r} \sigma_{s-1} P^{r,j} .$$

Then Q^* is covered by the sets $\sigma_t P^{r,j}$ with $t \geq s$, whence (since K has the property $A_*(s)$) Q^* is covered by the sets $P^{r,j}$ for which $d(r,j) = s$. But then Q is also covered by these sets, we are led again to (2) above and the proof proceeds as before.

Note that if μ_1 is one of the (equivalent) standard 1-measures related to the EUCLIDean metric for E, and if each of the complexes K_i is connected, then the validity of 3.1 for $s = 1$ does not require boundedness of the sequence $f_0(K_\alpha)$. For even without the boundedness, $\sigma_1 K_\alpha$ is a sequence of continua in E and it is known (NÖBELING [18]) that $\mu_1(M) \leq \liminf \mu_1(\sigma_1 K_\alpha)$ for every continuum $M \subset \liminf \sigma_1 K_\alpha$. In the next section is proved that for arbitrary s, the boundedness of $f_0(K_\alpha)$ may be abandoned when we are concerned only with the faces of convex polytopes rather than more general cell-complexes.

4. The functions ζ_s for convex polytopes

The proof to be presented here employs more properties of the s-measure μ_s than were required in the previous section; for the sake of simplicity we shall assume that E is equipped with a EUCLIDean metric and μ_s is the s-dimensional HAUSDORFF measure based on that metric. For a polytope P, $\zeta_s(P) \cdot = \mu_s(\sigma_s P)$.

Theorem 4.1. *If P_α is a sequence of convex polytopes in E converging to a polytope P, then* $\liminf \zeta_s(P_\alpha) \geq \zeta_s(P)$.

Proof. We may assume that all of the polytopes P_i and P are of the same dimension d as E, and hence have nonempty interior in E. For let E' denote the smallest flat containing P and let π denote the orthogonal projection of E onto E'. Then the sequence πP_α is convergent to P. Further, $\zeta(\pi P_i) \leq \zeta(P_i)$, for π is a metric contraction and from 1.3 and 1.5 it follows that $\pi\sigma_s P_i \supset \sigma_s \pi P_i$. Thus the desired conclusion for the sequence $P_\alpha \to P$ is implied by that for the sequence $\pi P_\alpha \to P$, and we assume henceforth that the sets E, P, and P_i are all d-dimensional.

We may assume that $P_i \supset P$. For let z be an interior point of P and note that for each $\epsilon \in [0,1]$ it is true that

$$z + (1-\epsilon)(P - z) \subset P_i \subset z + (1+\epsilon)(P - z)$$

for almost all i. Thus for almost all i there exists a smallest positive number β_i such that $P \subset P_i' \cdot = x + \beta_i(P - x)$. It is evident that $\beta_\alpha \to 1$ and hence $\mu(P_\alpha') / \mu(P_\alpha) \to 1$, so the desired conclusion for the sequence $P_\alpha \to P$ is implied by that for the sequence $P_\alpha' \to P$. We assume henceforth that $P_i \supset P$ for all i.

Let $Q_1, \ldots, Q_m$ be the s-faces of P, where $m = f_s(P)$. The sets ϱQ_j $(1 \leq j \leq m)$ are pairwise disjoint, so each point x of their union is in a unique set $\varrho Q_{j(x)}$. We shall denote by $F(x)$ the $(d-s)$-dimensional flat in E that is orthogonal to $Q_{j(x)}$ and intersects $Q_{j(x)}$ only at x. Then $F(x)$ intersects the interior of P and hence the intersection $P \cap F(x)$ is a $(d-s)$-polytope, to be denoted here by $P(x)$. Similarly, we define the $(d-s)$-polytopes $P_i(x) \cdot = P_i \cap F(x)$. It is evident that x is a vertex of $P(x)$ and that $P_i(x) \supset P(x)$; it is easily verified that $P_\alpha(x) \to P(x)$.

For $x \in \bigcup_{j=1}^m \varrho Q_j$ and for $i = 1, 2, \ldots$, let $d_i(x)$ denote the minimum distance from x to a vertex of the polytope $P_i(x)$; that is,

$$d_i(x) \cdot = \inf \{ \| x - v \| : v \in \operatorname{ex} P_i(x) \} .$$

We claim that $d_\alpha(x) \to 0$. Indeed, let z be a relatively interior point of $P(x)$ and for each i let γ_i be the largest positive number such that

$$P(x) \supset P_i^*(x) \cdot = z + \gamma_i (P_i(x) - z) .$$

Then $\gamma_\alpha \to 1$, $P_\alpha^*(x) \to P(x)$, and for the desired conclusion it suffices to show that as $i \to \infty$, the minimum distance from x to a vertex of $P_i^*(x)$ converges to 0. Since x is a vertex of $P(x)$, every neighborhood U of x relative to $P(x)$ contains the closure of $J \cap P(x)$ for some open halfspace $J \ni x$. Since $P_\alpha^*(x) \to P(x)$, it is true for almost all i that $P_i^*(x)$ intersects J, and then since J is a halfspace it must include a vertex u of $P_i(x)$. But then $u \in U$ and the desired conclusion follows.

Now consider an arbitrary $\epsilon > 0$. We want to show that $\zeta_s(P_i) > \zeta_s(P) - 2\epsilon$ for almost all i. Note first that if

$$Q_k^\delta \cdot = Q_k \sim N_{2\delta} (\textstyle\bigcup_{1 \leq j \leq m,\, j \neq k} Q_j) , \tag{1}$$

then $\mu_s(Q_k^\delta) \to \mu_s(Q_k)$ as $\delta \to 0$. Thus we can choose $\tau > 0$ so that

$$\Sigma_{j=1}^m \mu_s(Q_j^\tau) > \Sigma_{j=1}^m \mu_s(Q_j) - \epsilon = \zeta_s(P) - \epsilon .$$

The set Q_j^τ will be denoted henceforth by G_j. For each positive integer h, $G_j(h)$ will denote the set of all points x of G_j such that for all $i \geq h$, the set $N_\tau(G_j)$ includes at least one vertex of $P_i(x)$. Obviously $G_j(1) \subset G_j(2) \subset \ldots$, and by the previous paragraph the union of these sets is G_j itself. Hence $\mu_s(G_j(\alpha)) \to \mu_s(G_j)$, and there exists an integer h_0 such that

$$\Sigma_{j=1}^m \mu_s(G_j(h)) > \Sigma_{j=1}^m \mu_s(G_j) - \epsilon \quad \text{for all} \quad h > h_0 .$$

Now consider an arbitrary integer $h > h_0$, and for $x \in G_j(h)$ (with $1 \leq j \leq m$) let $W_h(x) \cdot = N_\tau(G_j) \cap \operatorname{ex} P_h(x)$; since each point of $W_h(x)$ is a vertex of the polytope $P_h(x) = P_h \cap F(x)$, it follows that $W_h(x) \subset \sigma_s P_h$. Define

$$T_{j,h} \cdot = \textstyle\bigcup_{x \in G_j(h)} W_h(x) .$$

We claim that the sets $T_{j,h} (1 \leq j \leq m)$ are pairwise disjoint, are $G_{\delta\sigma}$ sets and hence μ_s-measurable, and that always $\mu_s(T_{j,h}) \geq \mu_s(G_j(h))$. When we have established these claims we can conclude that

$$\zeta_s(P_h) = \mu_s(\sigma_s P_h) \geq \Sigma_{j=1}^m \mu_s(T_{j,h}) \geq \Sigma_{j=1}^m \mu_s(G_j(h))$$
$$> \Sigma_{j=1}^m \mu_s(G_j) - \epsilon \geq \Sigma_{j=1}^m \mu_s(F_j) - 2\epsilon = \zeta_s(P) - 2\epsilon$$

and the proof will be complete.

Note that $T_{j,h} \subset N_\tau(G_j) = N_\tau(Q_j^\tau)$. From the definition (1) it follows that $N_\tau(Q_j^\tau) \cap N_\tau(Q_k^\tau) = \emptyset$ when $j \neq k$, and consequently the sets $T_{j,h}(1 \leq j \leq m)$ are pairwise disjoint.

If x is a point of the set $G_j(h)$, then the $(d-s)$-flat $F(x)$ is orthogonal to the s-flat aff Q_j and the two flats intersect only at x. Since $\emptyset \neq W_h(x) \subset P_h(x) \subset F(x)$, we see that the orthogonal projection of E onto aff Q_j is a metric contraction that carries $T_{j,h}$ onto $G_j(h)$. Thus surely $\mu_s(T_{j,h}) \geq \mu_s(G_j(h))$, and it remains only to show that the set $T_{j,h}$ is s-measurable.

Note that for each i, the set-valued function $P_i(x) \mid x \in \varrho Q_j$ is continuous, whence the function ex $P_i(x) \mid x \in \varrho Q_j$ is lower semicontinuous. By using this semicontinuity in conjunction with the definition of $G_{j,h}$ and the fact that $N_\tau(G_j)$ is open, one verifies that $G_j(h)$ is a G_δ set relative to the closed set G_j and hence $G_j(h)$ is a G_δ set in F_j; further, the set-valued function $W_h(x) \mid x \in G_j(h)$ is lower semicontinuous. For each n, let C_n denote the set of all points $x \in G_j(h)$ for which the set $W_h(x)$ consists of exactly n points. From the lower semicontinuity it follows that each function $W_h(x) \mid x \in C_n$ is continuous, and that each set C_n is the difference of two relatively closed subsets of $G_j(h)$. But then C_n is a G_δ set in aff F_j and the set $\bigcup_{x \in C_n} F(x)$ is a G_δ set in E. The continuity of W_h on C_n guarantees that the set $\bigcup_{x \in C_n} W_h(x)$ is closed relative to the set $\bigcup_{h \in C_n} F(x)$ and hence is a G_δ set in E. But of course

$$T_{j,h} = \bigcup_{n=1}^\infty \bigcup_{x \in C_n} W_h(x),$$

so $T_{j,h}$ is a $G_{\delta\sigma}$ set and the proof is complete.

It seems probable that the convexity assumption in 3.1 can be considerably weakened, although we have no satisfactory result in that direction. Suppose X is a subset of E and $\varphi_0, \varphi_1, \varphi_2, \ldots$ is a sequence of homeomorphisms of X into E such that φ_α converges uniformly to φ_0. Suppose that for $i = 0, 1, 2, \ldots, K_i$ is a complex such that $|K_i| = \varphi_i X$, and suppose that K_0 has the property $A_*(s)$. Is it then necessarily true that $\liminf \zeta_s(K) \geq \zeta_s(K_0)$? An affirmative answer would provide a generalization of 4.1. It is clear that for $s \geq 2$, some sort of regularity condition must be imposed on the convergence of $|K_\alpha|$ to $|K_0|$, even when the individual complexes are well-behaved. For example, if Z_α is a sequence of polygonal arcs in E^3, converging to the boundary B of a simplex, we may surround each set Z_i by a polyhedral 2-sphere S_i in such a way that $S_\alpha \to B$ but $\mu_2(S_\alpha) \to 0$.

5. Inequalities for the function ζ_s

In preparation for our next main result, we require three lemmas.

5.1. *For each a-face F^a of an $(a+b)$-polytope P, there is a b-face F^b of P such that the polytope* con $(F^a \cup F^b)$ *is $(a+b)$-dimensional.*

Proof. We may assume that the containing space E is of dimension $a+b$. Let G be a b-flat orthogonal to $\operatorname{aff} F^a$ in E and let π denote the orthogonal projection of E onto G. It suffices to choose F^b so that πF^b is b-dimensional, and the existence of such an F^b follows from 1.5.

5.2. *Suppose P is a d-polytope, $1 \leq a \leq d$, y is a point of an $(a-1)$-face of P, and C is the smallest convex cone which has vertex y and contains every a-face of P that includes y. Then C contains P.*

Proof. Let F be the lowest-dimensional face that includes y, whence $y \in \varrho F$. It suffices to consider the case in which the containing space E is of dimension d, $y = 0$, and $\dim F = a-1$, so that $\operatorname{aff} F$ is an $(a-1)$-dimensional linear subspace of E. Let π denote the orthogonal projection of E onto the orthogonal supplement M of $\operatorname{aff} F$, and let K denote the cone from y over P (that is, $K \cdot = \bigcup_{p \in P} [0, \infty[(p)$. Since $\operatorname{aff} F \subset C \cap K$, it is easy to verify that

$$C = \operatorname{aff} F + \pi C \quad \text{and} \quad K = \operatorname{aff} F + \pi K ,$$

so to show that $C \supset P$ it suffices to prove that $\pi C \supset \pi K$. Now $\pi K = K \cap M$, a polyhedral convex cone that is pointed and hence is the convex hull of its extreme rays. For each extreme ray J of $K \cap M$ there is a linear functional f on M such that

$$K \cap M \subset \{u \in M : f(u) \geq 0\} \quad \text{and} \quad J = \{u \in K \cap M : f(u) = 0\} .$$

Then $f\pi$ is a linear functional g on E such that

$$K \subset \{v \in E : g(v) \geq 0\} \quad \text{and} \quad (\operatorname{aff} F) + J = \{v \in K : g(v) = 0\} .$$

The intersection $(\operatorname{aff} F + J) \cap P$ is an a-face of P that contains F and whose image under π includes a point of $J \sim \{0\}$. Thus πC contains every extreme ray of πK, whence $\pi C \supset \pi K$ and the proof is complete.

5.3. *Suppose P is a d-polytope, $0 < a < d$, $x \in P$, y is a point of $\sigma_a P$ nearest to x, and F is an a-face of P that includes y. Then y is relatively interior to F and the segment $[x, y]$ is orthogonal to the flat* $\operatorname{aff} F$.

Proof. Let B denote the EUCLIDean ball of center x and radius $\| y - x \|$, H the hyperplane through y orthogonal to $[x, y]$, and Q the closed halfspace that is bounded by H and misses the interior of B. Clearly $\sigma_a P$ does not intersect the interior of B. Since H is tangent to B, the set $\sigma_a P$ cannot contain any line segment $]y, z[$ for which $z \in E \sim Q$, and consequently Q contains every a-face of P that includes y. But this implies that $y \in \varrho F$ (whence $F \subset H$ and $[x, y]$ is orthogonal to aff F), for otherwise y would lie in an $(a-1)$-face of P and 5.2 would lead to the impossible conclusion that $P \subset Q$.

Theorem 5.4. *If K is a cell-complex in E and a and b are nonnegative integers then*

$$\zeta_{a+b}(K) \leq \zeta_a(K)\zeta_b(K).$$

Proof. We assume that $a > 0 < b$, for the remaining cases are trivial. For each s, let the s-faces of K be $F_1^s, \ldots, F_{n(s)}^s$, where of course $n(s) \cdot = f_s(K)$. Let A denote the set of all ordered triples (i, j, k) of positive integers that satisfy the following three conditions:

(i) $1 \leq i \leq n(a)$, $1 \leq j \leq n(b)$, $1 \leq k \leq n(a+b)$;
(ii) F_i^a and F_j^b are both faces of F_k^{a+b};
(iii) the set con $(F_i^a \cup F_j^b)$ is of dimension $a+b$, and hence intersects ϱF_k^{a+b}.

Note that if $(i, j, k) \in A$ and $(i, j, k') \in A$, then $k = k'$, for different faces of K have no relative interior points in common.

With $(i, j, k) \in A$, let $\pi_{(i,j,k)}$ denote the transformation which projects aff F_k^{a+b} orthogonally onto a b-flat $G(i, j, k)$ that is orthogonal to aff F_i^a in aff F_k^{a+b}. According to 1.5, the set $\pi_{(i,j,k)} F_k^{a+b}$ is not merely the projection of F_k^{a+b} but is the projection of $\sigma_b F_k^{a+b}$, and except possibly for a set of zero b-measure each point of $\pi_{(i,j,k)} F_k^{a+b}$ lies in the projection of a b-face F_m^b of F_k^{a+b} such that $(i, m, k) \in A$. It follows that

$$\mu_{a+b}(F_i^a \oplus \pi_{(i,j,k)} F_k^{a+b}) \leq \mu_a(F_i^a) \Sigma_{(i,m,k) \in A} \mu_b(F_m^b), \tag{1}$$

where it is convenient to let '$\oplus$' denote vector addition with respect to an origin at the intersection of aff F_i^a with $G(i, j, k)$.

For each k, let $I(k)$ denote the set of all i for which F_i^a is a face of F_k^{a+b}, and for each $i \in I(k)$ let $j(i)$ be such that $(i, j(i), k) \in A$. The existence of $j(i)$ follows from 5.1. Then with the above interpretation of $\oplus$, it is true that

$$F_k^{a+b} \subset \bigcup_{i \in I(k)} (F_i^a \oplus \pi_{(i,j(i),k)} F_k^{a+b}). \tag{2}$$

To see this, we consider an arbitrary point $x \in \varrho F_k^{a+b}$ and let y be a point of $\sigma_a F_k^{a+b}$ nearest to x. By 5.3, there is a unique i such that $y \in F_i^a$ and $[x, y]$ is orthogonal to the flat aff F_i^a, and then we have

$$x = y \oplus \pi_{(i,j(i),k)}(x) \in F_i^a \oplus \pi_{(i,j(i),k)} F_k^{a+b}.$$

From (1) and (2) it follows that

$$\begin{aligned}\mu_{a+b}(F_k^{a+b}) &\leq \Sigma_{i\in I(k)}\mu_{a+b}(F_i^a \oplus \pi_{(i,j(i),k)}F_k^{a+b}) \leq \\ &\leq \Sigma_{i\in I(k)}[\mu_a(F_i^a)\,\Sigma_{(i,m,k)\in A}\mu_b(F_m^b)] \leq \\ &\leq \Sigma_{(i,j,k)\in A}\mu_a(F_i^a)\mu_b(F_j^b)\,.\end{aligned}$$

But then

$$\zeta_{a+b}(K) = \Sigma_{k=1}^{n(a+b)}\mu_{a+b}(F_k^{a+b}) \leq \Sigma_{k=1}^{n(a+b)}\,\Sigma_{(i,j,k)\in A}\mu_a(F_i^a)\mu_b(F_j^b)\,,$$

and since no pair (i,j) can be associated in A with more than one value of k it follows that

$$\zeta_{a+b}(K) \leq (\Sigma_{i=1}^{n(a)}\mu_a(F_i^a))(\Sigma_{j=1}^{n(b)}\mu_b(F_i^b)) = \zeta_a(K)\zeta_b(K)\,.$$

Corollary 5.5. *Suppose that r and s are positive integers and r is a divisor of s. Then*

$$\frac{\zeta_s(K)^{1/s}}{\zeta_r(K)^{1/r}} \leq 1$$

for every cell-complex K in a EUCLIDean *space.*

Proof. Suppose $s = mr$. From 5.4 it follows that

$$\zeta_s(K) \leq \zeta_r(K)\zeta_{s-r}(K) \leq \zeta_r(K)^2\zeta_{s-2r}(K) \leq \ldots \leq \zeta_r(K)^m\,,$$

whence $(\zeta_s(K)/\zeta_r(K)^m)^{1/s} \leq 1$ and this is the desired inequality.

When $r < s$ but r is not a divisor of s, we are unable to determine whether the quotient $\zeta_s(K)^{1/s}/\zeta_r(K)^{1/r}$ is uniformly bounded as K ranges over all cell-complexes in EUCLIDean spaces. However, we are primarily interested in convex polytopes, and there a little more information is available.

Theorem 5.6. *Suppose $1 \leq r \leq s \leq d$, and $s = d$ or $s = d-1$ or r is a divisor of s. Then there is a smallest finite constant $\gamma(d,r,s)$ such that $\zeta_s(P)^{1/s}/\zeta_r(P)^{1/r} \leq \gamma(d,r,s)$ for all* EUCLIDEAN *d-polytopes P.*

Proof. When r divides s, we merely apply 5.5. Note that for all d, r and s, the existence of $\gamma(d,r,s)$ is equivalent to the condition:

$\Gamma(d,r,s)$: For each family of d-polytopes P such that $\zeta_r(P)$ is uniformly bounded, $\zeta_s(P)$ is also uniformly bounded.

Obviously $\Gamma(d,r,s)$ is implied by the existence of $\gamma(d,r,s)$. Conversely, $\Gamma(d,r,s)$ implies the existence of a finite constant that bounds the quotients $\xi(P) \cdot= \zeta_s(P)^{1/s}/\zeta_r(P)^{1/r}$ for d-polytopes P such that $\zeta_r(P) = 1$, and since $\xi(\lambda P) = \xi(P)$ for all $\lambda > 0$, this is in fact a general bound.

Now we recall the classical isoperimetric inequality ([11], p. 195) and CAUCHY's formula for surface area ([11], p. 208), both applying to n-polytopes and asserting respectively that

$$(I_n) \quad \zeta_n(P)^{1/n} \leq \frac{1}{n\omega_n^{1/n}} \zeta_{n-1}(P)^{1/(n-1)}$$

and

$$(C_n) \quad \zeta_{n-1}(P) = \frac{1}{\omega_{n-1}} \int \zeta_{n-1}(\pi_u P)\, du\,,$$

where ω_n is the n-measure of the unit ball in E^n, π_u denotes the orthogonal projection of E^n onto a hyperplane orthogonal to the line Ru, and u ranges over the unit sphere in E^n. Let A_d denote the assertion that $\Gamma(d, r, s)$ is valid whenever $1 \leq r \leq s \in \{d-1, d\}$. Then A_1 is trivial and A_2 is an immediate consequence of I_2. Suppose A_{d-1} is known and consider a family of d-polytopes P in E^d for which $\zeta_r(P)$ is uniformly bounded. Since $\zeta_r(\pi_u P) \leq \zeta_r(P)$ (use 1.5 and a basic property of HAUSDORFF measure), $\zeta_r(\pi_u P)$ is also uniformly bounded (for all u and for all P in the family) whence $\zeta_{d-1}(\pi_u P)$ is uniformly bounded by the inductive hypothesis, $\zeta_{d-1}(P)$ is uniformly bounded by (C_d), and $\zeta_d(P)$ is uniformly bounded by (I_d). This completes the proof.

We have not determined the existence of $\gamma(d, r, s)$ except in the cases covered by 5.6; even in those cases we know the values of the constants only when $r = d-1$ and $s = d$. A theorem of ABERTH [1] implies that $\gamma(3, 1, 2) \leqq (6\pi)^{-1/2}$.

Let C^d denote the space of all convex bodies in E^d, metrized by the HAUSDORFF distance, and let P^d denote the dense subspace consisting of all d-polytopes in E^d. Since the functions ζ_s are all lower semicontinuous on P^d, they can be extended in the usual way to all of C^d; specifically,

$$\zeta_s(C) \cdot = \liminf_{P \in P^d,\, P \to C} \zeta_s(P) \tag{*}$$

for each $C \in C^d$. The extended functions ζ_s are also lower semicontinuous, and they provide a natural way of assigning such quantities as total edge-length and total area of 2-faces to an arbitrary convex body C in E^d. It would be interesting to find a simple geometric characterization of the members of C^d_s, where this is the set of all $C \in C^d$ for which $\zeta_s(C) < \infty$. Note that $C^d_o = P^d$, while $C^d_{d-1} = C^d_d = C^d$; for $1 \leq s \leq d-2$, C^d_s properly contains P^d but is of the first category in C^d. Note that if r, s, and d are as in 5.6 and if $C \in C^d_r$, then $\zeta_s(C)^{1/s}/\zeta_r(C)^{1/r} \leq \gamma(d, r, s)$.

Many problems of isoperimetric type are of interest not only for the class P^d of all d-polytopes in E^d, but also for certain subclasses Q, such as the class $P^{d,v}$ ⟨resp. $P^{d,f}$⟩ of all members of P^d that have exactly d edges incident to each vertex ⟨resp. d $(d-2)$-faces incident to each $(d-1)$-face⟩. (The letters

v and f are to suggest regularity of behavior at vertices and maximal faces respectively.) In connection with isoperimetric problems for $\mathbf{Q}$, it may be useful to extend ζ_s in a different way:

$$\zeta_s(\mathbf{Q}, C) \cdot = \liminf_{Q \in \mathbf{Q},\, Q \to C} \zeta_s(Q)$$

for each body C in the closure of $\mathbf{Q}$. The functions $\zeta_s(\mathbf{Q}, \cdot)$ are also lower semicontinuous, are defined on all of $\mathbf{C}^d$ when $\mathbf{Q}$ is dense in $\mathbf{P}^d$, and of course they agree with ζ_s on $\mathbf{Q}$ itself. However, the functions $\zeta_s(\mathbf{Q}, \cdot)$ may fail to agree with ζ_s on the set $\mathbf{P}^d \sim \mathbf{Q}$; this is true in particular when $\mathbf{Q} = \mathbf{P}^{d,v}$ and when $\mathbf{Q} = \mathbf{P}^{d,f}$. Various relationships among the functions ζ_s as restricted to $\mathbf{Q}$ yield the corresponding relationships among the extended functions $\zeta_s(\mathbf{Q}, \cdot)$. In particular, it can be proved that when $1 \le r \le s \le d$ the quotient $\zeta_s(P)^{1/s}/\zeta_r(P)^{1/r}$ is uniformly bounded as P ranges over $\mathbf{P}^{d,v}$ and consequently $\zeta_s(\mathbf{P}^{d,v}, C)^{1/s}/\zeta_r(\mathbf{P}^{d,v}, C)^{1/r}$ is uniformly bounded as C ranges over $\mathbf{C}^d$. We do not know whether the same statement is valid with $\mathbf{P}^{d,v}$ replaced by $\mathbf{P}^{d,f}$, though of course its validity is guaranteed by 5.6 when $s = d$ or $s = d - 1$ or r is a divisor of s.

In the rest of this section, we shall regard the functions f_s and ζ_s as defined on all of $\mathbf{C}^d$ by means of the natural lower semicontinuous extension (as in (*) above) from $\mathbf{P}^d$. This will simplify certain statements, and seems to be of genuine interest for the functions ζ_s when $0 < s$. On the other hand, $f_s(C) = \infty$ whenever $s < d$ and $C \in \mathbf{C}^d \sim \mathbf{P}^d$, and of course the same is true of $\zeta_0 (= f_0)$.

Theorems 1.7, 3.1 and 5.6 lead to the existence of solutions for a wide variety of isoperimetric problems, involving polytopes, analogous to those discussed in [9] and [11]. A few of these are described below. We shall not actually determine the solutions, but merely prove their existence.

Proposition 5.7. *Suppose B and C are convex bodies in E^d, with $B \subset C$, and s is an integer with $0 \le s \le d$. Among all the convex bodies P for which $B \subset P \subset C$, there is one* (*or more*) *for which $f_s(P)$ is a minimum and there is one* (*or more*) *for which $\zeta_s(P)$ is a minimum.*

Proof. Blaschke's selection theorem implies the compactness of the class of all convex bodies P for which $B \subset P \subset C$. Then apply the lower semicontinuity of f_s and ζ_s.

In preparation for the next proposition, a lemma is required.

5.8. *Suppose $0 < t \le d$, $\tau < \infty$, and B is a convex body in E^d. Then there exists a convex body C in E^d such that $P \subset C$ whenever P is a convex body in E^d with $B \subset P$ and $\zeta_t(P) \le \tau$.*

Proof. The assertion is obvious when $t = d$ and also when $d \leq 2$. To handle the general case, we proceed by induction on d. If the lemma is false for some $d > 2$ and some τ and B, there is a sequence C_α of convex bodies in E^d such that

$$B \subset C_i,\ \zeta_t(C_i) \leq \tau \quad (i = 1, 2, \ldots)$$

and

$$\lim_{i\to\infty} \sup \{\| x \| : x \in C_i\} = \infty .$$

Assuming for notational convenience that $0 \in B$, we employ the definition of ζ_t and a straightforward compactness argument to produce a sequence P_α of polytopes, a ray $[0, \infty[x_0$ emanating from 0 (where $x_0 \in E \sim \{0\})$, and a sequence of real numbers $\beta_\alpha \to \infty$ such that

$$[0, \beta_i]x_0 \subset P_i \quad (i = 1, 2, \ldots) .$$

Now let E^{d-1} be a hyperplane in E^d such that $[0, \infty[x_0 \subset E^{d-1}$, and let π be the transformation that projects E^d orthogonally onto E^{d-1}. Then of course

$$\pi B \cup [0, \beta_i]x_0 \subset \pi P_i \quad (i = 1, 2, \ldots) ,$$

and from 1.5 in conjunction with the basic property of HAUSDORFF measure that we have used several times earlier, it follows that

$$\zeta_t(\pi P_i) \leq \zeta_t(P_i) \leq \tau \quad (i = 1, 2, \ldots) .$$

This shows that the lemma fails in E^{d-1} if it fails in E^d.

Theorem 5.9. *Suppose B is a convex body in E^d, r, s, t and k are integers, and τ is a real number. Suppose $0 \leq r \leq d$, $0 \leq s < d$, $0 < t \leq d$, and there exists a polytope P in E^d such that $B \subset P$, $f_s(P) \leq k$, and $\zeta_t(P) \leq \tau$. Then among all such polytopes P, there is one for which $\zeta_s(P)$ is a minimum, and if $s = d$ or $s = d - 1$ there is also one for which $\zeta_r(P)$ is a maximum.*

Proof. Use 5.8, BLASCHKE's selection theorem, the lower semicontinuity of f_s, ζ_t, and ζ_s, and the continuity of ζ_s when $s = d$ or $s = d - 1$.

Theorem 5.10. *Suppose r, s, d and k are integers with $1 \leq r < d, 0 \leq s < d$, and $k \geq \binom{d+1}{s+1}$. Then among the convex bodies C of unit volume in E^d, there are those for which $\zeta_r(C)$ is equal to its minimum value $1/\gamma(d, r, d)^r$. And among the d-polytopes P of unit volume for which $f_s(P) \leq k$, there are those for which $\zeta_r(P)$ is a minimum.*

Proof. We discuss only the first assertion, for the second is similar. Since $\gamma(d, r, d)$ is the largest constant subject to the requirement that $\zeta_r(C)^{1/r} \geq$ $\geq \zeta_d(C)^{1/d}/\gamma(d, r, d)$ for all convex bodies C in E^d, it follows from the definition of ζ_r, the semicontinuity of ζ_r and the continuity of ζ_d that there is a sequence P_α of polytopes in E^d such that $\mu_d(P_\alpha) \to 1$ and $\zeta_r(P_\alpha) \to 1/\gamma(d,r,d)^r$.

In view of the translation-invariance of μ_d and ζ_r, we may assume that each polytope P_i includes the origin. Now suppose for the moment that the sequence {width of P_α} is bounded from 0. Then, since $\mu_d(P_\alpha) \to 1$, the sequence diam (P_α) is bounded from ∞ and with $0 \in P_i$ the sequence P_α has a subsequence that converges to a convex *body* C_0 in E^d; C_0 is of unit volume and $\zeta_r(C_0) = 1/\gamma(d, r, d)^r$.

It remains only to show that the sequence {width of P_α} is bounded away from 0. Suppose the contrary, whence there exists a subsequence Q_α of P_α and a sequence H_α of hyperplanes in E^d such that the width of Q_i in the direction orthogonal to H_i converges to zero as $i \to \infty$. Let πQ_i denote the orthogonal projection of Q_i onto H_i. Since $\mu_d(Q_\alpha) \to 1$ it is evident that $\mu_{d-1}(\pi Q_\alpha) \to \infty$ and it then follows from 5.6 that $\zeta_r(\pi Q_\alpha) \to \infty$. But then with the aid of 1.5 we see that $\zeta_r(Q_\alpha) \to \infty$, and this is a contradiction completing the proof.

In the first assertion of 5.10, we do not know whether the minimum value of ζ_r is actually attained by a polytope. We are also unable to decide whether 5.10 remains valid when the condition of unit volume is replaced by that of unit surface area, though this can be established without difficulty when $r = 1$.

6. Intersection properties of simplices

By definition, a *d-simplex* (in a real linear space L) is a set that is the convex hull of $d + 1$ affinely independent points, its vertices. It was apparently Choquet [5] who first noticed that among the finite-dimensional compact convex sets, the simplices are exactly those sets S that have the following property:

(Σ) *whenever the intersection of two homothets of S is at least one-dimensional, then the intersection is itself a homothet of S.*

(Here a *homothet* of S is a set of the form $x + \alpha S$ for $x \in L$ and $\alpha > 0$.)

Choquet then used the condition (Σ) to *define* the notion of a simplex in the infinite-dimensional case, and established the equivalence of (Σ) to various other conditions (see Choquet and Meyer [6] and Kendall [13]). Independently of Choquet but a bit later, Rogers and Shephard [19] showed that the simplices are characterized by the following property, obviously implied by (Σ):

(Σ') *whenever the intersection of two translates of S is at least one-dimensional, then the intersection is a homothet of S.*

(Actually, the conditions (Σ) and (Σ') are known to be equivalent, but that fact will not be used here.)

With the aid of these characterizations, it is a simple matter to prove the theorem of Kolmogorov and Borovikov [3] stated in the Introduction, and to free it from the assumption of finite-dimensionality.

Theorem 6.1. *Suppose that $\mathcal{S}$ is a family of compact sets in a* HAUSDORFF *linear space L, that each set $S \in \mathcal{S}$ has the property (Σ) $\langle$resp. $(\Sigma')\rangle$, and that the intersection of any two members of $\mathcal{S}$ contains a member of $\mathcal{S}$. Then the set $\bigcap \mathcal{S}$ also has the property (Σ) $\langle$resp. $(\Sigma')\rangle$.*

Proof. An equivalent formulation of condition (Σ) $\langle$resp. $(\Sigma')\rangle$ is that whenever the intersection of two homothets $\langle$resp. translates$\rangle$ of S is nonempty, then it has the form $x + \alpha S$ for some $x \in E$ and $\alpha \in [0,1]$. Now let $K \cdot = \bigcap \mathcal{S}$ and consider points $y_i \in L$ and numbers $\beta_i > 0$ such that the intersection $(y_1 + \beta_1 K) \cap (y_2 + \beta_2 K)$ is nonempty; when considering $(\Sigma)'$, we assume further that $\beta_1 = \beta_2 = 1$. For each $S \in \mathcal{S}$,

$$(y_1 + \beta_1 S) \cap (y_2 + \beta_2 S) \supset (y_1 + \beta_1 K) \cap (y_2 + \beta_2 K) \neq \emptyset ,$$

so by hypothesis there exist $x_S \in L$ and $\alpha_S \in [0,1]$ such that

$$(y_1 + \beta_1 S) \cap (y_2 + \beta_2 S) = x_S + \alpha_S S . \tag{1}$$

Choose $S' \in \mathcal{S}$, define $J \cdot = y_1 + \beta_1 S' - [0,1]S'$, and note that $x_S \in J$ whenever $S \subset S'$. Since the set $J \times [0,1]$ is compact, some point (x_0, α_0) of $J \times [0,1]$ is a cluster point of the net $(x_S, \alpha_S) \mid S \in \mathcal{S}$. From (1) we see that

$$(y_1 + \beta_1 K) \cap (y_2 + \beta_2 K) \subset x_S + \alpha_S S \quad \text{for all } S \in \mathcal{S} , \tag{2}$$

$$(y_1 + \beta_1 S) \cap (y_2 + \beta_2 S) \supset x_S + \alpha_S K \quad \text{for all } S \in \mathcal{S} , \tag{3}$$

whence with the aid of compactness it follows in a straightforward manner from (2) that

$$(y_1 + \beta_1 K) \cap (y_2 + \beta_2 K) \subset x_0 + \alpha_0 K$$

and from (3) that

$$(y_1 + \beta_1 K) \cap (y_2 + \beta_2 K) \supset x_0 + \alpha_0 K .$$

Corollary 6.2. *The intersection of a decreasing sequence of simplices is itself a simplex.*

In CHOQUET's approach, some measure theory and the notion of a vector lattice are involved in showing that the simplices are characterized by (Σ). In the ROGERS and SHEPHARD proof [19] that the simplices are characterized by (Σ'), the most difficult and lengthy part is the demonstration that if a d-polytope S in a d-dimensional EUCLIDean space E has the property (Σ'), then S is a simplex. We shall now indicate an alternative proof of this fact, proceeding by induction on the dimension d. The statement is obvious for $d \leq 1$. Assuming $d > 1$ we note that if a $(d-1)$-face F of S is contained in the supporting hyperplane H, then

(4) F has property (Σ') and hence is a $(d-1)$-simplex by the inductive hypothesis;

(5) S is intersected in a single point v by its supporting hyperplane H' that is parallel to H but different from H.

For (4) we may assume that $0 \in F$ and denote by Q the halfspace that contains S and is bounded by H. With $x, y \in H$, we are interested in the intersection $(x + F) \cap (y + F)$, assumed to be nonempty. From (Σ') and the nonemptiness of $(x + S) \cap (y + S)$ we have

$$(x + S) \cap (y + S) = z + \alpha S \quad \text{with} \quad z \in E\,, \alpha \in [0,1]\,.$$

But then $z \in x + S \subset Q$ and since

$$H \supset (x + F) \cap (y + F) \subset z + \alpha S \subset z + Q\,,$$

it follows that $z \in H$. Hence

$$\begin{aligned}(x + F) \cap (y + F) &= (x + S \cap H) \cap (y + S \cap H) = (x + S) \cap (y + S) \cap H = \\ &= (z + \alpha S) \cap H = z + \alpha(S \cap H) = z + \alpha F\,,\end{aligned}$$

and (4) is established.

For (5) we note that if $S \cap H'$ should contain a segment parallel to the segment $[0, x]$, then for a sufficiently small $\epsilon > 0$ the set $S \cap (\epsilon x + S)$ would have both H and H' as supporting hyperplanes, whence this set, being a homothet of S by (Σ'), would actually be a translate of S; this is impossible, for the width of $S \cap (\epsilon x + S)$ in the direction x is less than that of S.

Now let F and v be as in (4) and (5), let w be the centroid of F, and let $S_\beta := \beta(v - w) + S$. For $\beta < 1$ sufficiently near to 1, the only face of S_β intersecting S is $\beta(v - w) + F$ and the only vertex of S contained in S_β is the vertex v. Thus the set $S_\beta \cap S$ is a pyramid with apex v and base contained in $\beta(v - w) + F$. By (Σ'), the set $S_\beta \cap S$ is homothetic to S and thus its base is homothetic to F. Since F is a $(d - 1)$-simplex, $S_\beta \cap S$ must be a d-simplex and hence S is a d-simplex as claimed.

7. Sections of simplices

It was recently proved (Brands and Laman [4], Eggleston [8], Croft [7]) that among all the plane sections of a tetrahedron in Euclidean 3-space E^3, some 2-face has the greatest area. This suggests the following questions.

(1) *If S is a d-simplex in E^d and P is an r-polytope in S, must S have an r-face whose r-measure is at least that of P?*

(2) *If S is a d-simplex in E^d and P is an r-polytope in S having at most $r + 2$ vertices, must S have an r-face whose r-measure is at least that of P?*

An affirmative answer to (1) implies one to (2). The answer to (1) is clearly affirmative if $r = 0$, $r = 1$, or $r = d$, and by the result quoted above it is affirmative if $d \leq 3$. In 7.1 below, we establish an affirmative answer to (2) when $r = d - 1$. The relevance to 6.2 lies in the fact that an affirmative answer to (2) (for a given d and r) can be used to show that if an r-dimensional set S is the intersection of a decreasing sequence $S_1 \supset S_2 \supset \ldots$ of d-simplices, then S is an r-simplex. For suppose S is not a simplex. Then S contains an r-polytope P that has $r + 2$ vertices and has greater r-measure than the largest r-simplex in S. For $i = 1, 2, \ldots$, S_i has an r-face F_i whose r-measure is at least that of P, and the sequence F_α admits a subsequence $F_{n(\alpha)}$ that converges to an r-simplex T in S. But then

$$\mu_r(T) = \lim \mu_r(F_{n(\alpha)}) \geq \mu_r(P) > \mu_r(T),$$

a contradiction completing the proof.

Theorem 7.1. *Suppose S is a d-simplex in E^d and P is a $(d - 1)$-polytope in S that has at most $d + 1$ vertices. If P is not a $(d - 1)$-face of S, then some $(d - 1)$-face of S has greater $(d - 1)$-measure than P.*

Proof. For each hyperplane H in E^d whose intersection with S is $(d - 1)$-dimensional, let $P(H)$ be a $(d - 1)$-polytope of maximum $(d - 1)$-measure among those $(d - 1)$-polytopes that are the convex hull of $d + 1$ or fewer vertices of the $(d - 1)$-polytope $S \cap H$. By compactness, there is a hyperplane H_0 such that $\mu_{d-1}(P \cap H)$ is a maximum when $H = H_0$, and of course this maximum value is positive. Let $v_1, \ldots, v_k$ denote the vertices of S that are in H_0 and let $u_{k+1}, \ldots, u_{d+1}$ denote the remaining vertices of S.

Clearly the points v_i are vertices of $S \cap H_0$, and in fact they are vertices of $P(H_0)$. For suppose v_i is in H_0 but not in $P(H_0)$ and let F_i denote the $(d - 1)$-face of S that is opposite to v_i. Since $S \cap H_0$ is the convex hull of the union $(F_i \cap H_0) \cup \{v_i\}$, F_i includes all the vertices of $S \cap H_0$ other than v_i and consequently $F_i \supset P(H_0)$. But then $P(H_0)$ lies in the $(d - 2)$-flat $F_i \cap H_0$ and $\mu_{d-1}(P(H_0)) = 0$, a contradiction showing that, indeed, each v_i is a vertex of $P(H_0)$, or $S \cap H_0$ is F_i and the result is proved.

We want to show that $P(H_0)$ is a $(d - 1)$-face of S, and for this it suffices to show that $P(H_0)$ includes at least d vertices of S. We suppose, on the contrary, that $k < d$, and let $w_{k+1}, \ldots, w_l$ denote the remaining vertices of $P(H_0)$, where $l = d$ or $l = d + 1$. In each case we shall produce a hyperplane H_z for which $\mu_{d-1}(P(H_z)) > \mu_{d-1}(P(H_0))$, a contradiction completing the proof.

First case: $l = d$. In this case the $(d - 1)$-polytope $S \cap H_0$ has exactly d vertices and hence is identical with $P(H_0)$, for otherwise, by adding to the vertices of $P(H_0)$ a new vertex from those of $S \cap H_0$ not in $P(H_0)$, we could generate a $(d - 1)$-polytope in $S \cap H_0$ that has at most $d + 1$ vertices and that properly contains $P(H_0)$, contradicting the definition of $P(H_0)$. Let

$$K \cdot = \text{con}\,\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_{d-1}\}$$

and let $M \cdot = \operatorname{aff} K$, a $(d-2)$-flat in E^d. Let $M^\perp$ be a 2-flat orthogonal to M in E^d, intersecting M at a point q, and let π denote the transformation that projects E^d orthogonally onto $M^\perp$.

With $k < d$, w_d is a vertex of $P(H_0)$ but not of S. Hence w_d is a point of an open segment Z in S, and Z is not parallel to M, for if it were then Z would lie in H_0 and hence in $S \cap H_0 = P(H_0)$. For each $z \in Z$, let

$$H_z \cdot = \operatorname{aff}(M \cup \{z\}),$$

a hyperplane in E^d, and let $K_z \cdot = \operatorname{con}(K \cup \{z\})$. Then K_z is a $(d-1)$-polytope in $S \cap H_z$ and K_z has d vertices. Since $\pi K = \{q\}$, the altitude of the pyramid K_z is equal to $\|\pi(z) - q\|$, and consequently

$$\mu_{d-1}(P(H_z)) \geq \mu_{d-1}(K_z) = \frac{\|\pi(z) - q\|}{d-1} \mu_{d-2}(K).$$

Since Z is not parallel to M, πZ is an open segment in $M^\perp$ and it is possible to choose $z \in Z$ such that $\|\pi(z) - q\| > \|\pi(w_d) - q\|$. But then $\mu_{d-1}(P(H_z) > \mu_{d-1}(P(H_0))$, a contradiction completing the discussion of the first case.

Second case: $l = d + 1$. Note that every point of S lies on a segment joining a point of the simplex $T' \cdot = \operatorname{con}\{v_1, \ldots, v_k\}$ to a point of the simplex $T'' \cdot = \operatorname{con}\{u_{k+1}, \ldots, u_{d+1}\}$. Since $T' \subset H_0$, every vertex of $S \cap H_0$ that is not in T' must be in T''. In particular,

$$T'' \supset Q \cdot = \operatorname{con}\{w_{k+1}, \ldots, w_{d+1}\}.$$

But $P(H_0) = \operatorname{con}(T' \cup Q)$, and since the sets T' and Q lie respectively in the skew flats $\operatorname{aff} T'$ and $\operatorname{aff} T''$ it follows that

$$\dim P(H_0) = \dim T' + \dim Q + 1,$$

whence

$$\dim Q = d - k - 1.$$

Since Q has $d - k + 1$ vertices, there must be two vertices of Q that are separated in $\operatorname{aff} Q$ by a $(d-k-2)$-flat F which is determined by the remaining $d - k - 1$ vertices of Q. Indeed, it suffices to choose $d - k$ vertices of K that form a simplex and to note that the remaining vertex of Q must lie outside one of the closed halfspaces (in $\operatorname{aff} Q$) that contain the simplex and have bounding hyperplane determined by a face of the simplex.

We may assume that

$$F = \operatorname{aff}\{w_{k+1}, \ldots, w_{d-1}\},$$

separating w_d from w_{d+1} in $\operatorname{aff} Q$, whence the $(d-2)$-flat

$$M \cdot = \operatorname{aff}\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_{d-1}\}$$

separates w_d from w_{d+1} in H_0. Let $K, M^\perp, q, \pi, Z$, and H_z be as in the preceding case, and let Z' be an open segment in S that includes w_{d+1} and is not parallel to M. Since M separates w_d from w_{d+1} in H_0, since $\pi M = \{q\}$, and since πH_0 is a line in $M^{\perp}$, the point q must lie in the open segment $]\pi(w_d), \pi(w_{d+1})[$. Since Z and Z' are not parallel to M, the sets πZ and $\pi Z'$ are open segments in $M^{\perp}$, and it is possible to choose $z \in Z$ and $z' \in Z'$ such that $q \in]\pi(z), \pi(z')[$ and

$$\| \pi(z) - \pi(z') \| > \| \pi(w_d) - \pi(w_{d+1}) \| .$$

But then $z' \in H_z$, and since the set $K_z \cdot = \operatorname{con}(K \cup \{z, z'\})$ has $d+1$ vertices and is a double pyramid over K, we have

$$\mu_{d-1}(P(H_z)) \geq \mu_{d-1}(K_z) = \frac{\| \pi(z) - \pi(z') \|}{d-1} \mu_{d-2}(K) >$$

$$> \frac{\| \pi(w_d) - \pi(w_{d+1}) \|}{d-1} \mu_{d-2}(K) = \mu_{d-1}(P(H_0)) ,$$

a contradiction completing the proof.

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