# A Note on the Fundamental Theorem of Projective Geometry

M. OJANGUREN and R. SRIDHARAN[1])

## Introduction

The aim of this note is to prove a generalisation to commutative rings of the classical fundamental theorem of projective geometry. In § 1, we introduce the notions of projective spaces and projectivities. In § 2, we prove the main theorem. The method of proof is similar to the proof of the theorem in the classical case as found for example in ARTIN [1]. The proof, as in the classical case, is elementary, but is trickier. In § 3, we give an example to show that a bijection between projective spaces of the same dimension which preserves collinear points is not necessarily a projectivity. This is in contrast to what happens in the case of projective spaces over fields.

## § 1. Projective Spaces and Projectivities

Let $A$ be a cummutative ring with 1 and let $M$ be a free $A$-module. Let $P(M)$ denote the set of all $A$-free direct summands of rank 1 of $M$. This set is called the *projective space associated to* $M$. Clearly, any element of $P(M)$ is of the form $Ae$ where $e$ is a unimodular element of $M$, i.e. there exists a linear form $g:M\to A$ with $g(e)=1$. If $(e_1,\ldots,e_n)$ is a basis for the $A$-module $M$ and $e=\sum a_i e_i$, then we note that $e$ is unimodular if and only if $\sum\limits_{1\leqslant i\leqslant n} Ae_i=A$. If the ring $A$ is such that every projective module of rank 1 is free, then $P(M)$ coincides with the usual projective space of algebraic geometry [2, p. 13].

DEFINITION. *Let $M$ and $N$ be free modules over commutative rings $A$ and $B$ respectively. A map $\alpha:P(M)\to P(N)$ is called a projectivity if $\alpha$ is bijective and for $p_1, p_2, p_3\in P(M)$, we have $\alpha p_1\subset\alpha p_2+\alpha p_3$ in $N$ if and only if $p_1\subset p_2+p_3$ in $M$.*

This definition generalises the classical notion of projectivity between projective spaces over fields.

We note that by the very definition, $\alpha^{-1}:P(N)\to P(M)$ is also a projectivity. For later purposes, we need the following

LEMMA 1. *With the notation above, if $e_1,\ldots,e_n$ is a basis of $M$ and $e\in M$ a unimodular element such that $Ae\subset\sum\limits_{1\leqslant i\leqslant k} Ae_i$, then $\alpha Ae\subset\sum\limits_{1\leqslant i\leqslant k}\alpha Ae_i$.*

[1]) The authors thank Prof. ECKMANN for having given them the opportunity to work at the Forschungsinstitut für Mathematik, ETH.

*Proof.* We prove the lemma by induction on $k$. Let $e = \sum_{1 \leqslant i \leqslant k} a_i e_i$. Then $e' = \sum_{1 \leqslant i \leqslant k-1} a_i e_i + e_k$ is unimodular and $Ae \subset Ae' + Ae_k$. By definition this implies that $\alpha Ae \subset \alpha Ae' + \alpha Ae_k$. Let $e'' = \sum_{1 \leqslant i \leqslant k-2} a_i e_i + e_k$. Since $e' \in Ae'' + Ae_{k-1}$, we again have $\alpha Ae' \subset \alpha Ae'' + \alpha Ae_{k-1}$. We thus have $\alpha Ae \subset \alpha Ae'' + \alpha Ae_{k-1} + \alpha Ae_k$. By induction,

$$\alpha Ae'' \subset \sum_{1 \leqslant i \leqslant k-2} \alpha Ae_i + \alpha Ae_k \quad \text{and hence} \quad \alpha Ae \subset \sum_{1 \leqslant i \leqslant k} \alpha Ae_i .$$

Let $A$ and $B$ be rings and $\sigma: A \to B$ a homomorphism. If $M$ and $N$ are modules over $A$ and $B$ respectively, then a map $\Phi: M \to N$ is called *$\sigma$-semilinear* if $\Phi$ is additive and $\Phi(am) = \sigma(a)\,\Phi(m)$ for all $a \in A$, $m \in M$. If $M$ and $N$ are free modules over $A$ and $B$ of the same rank and $\Phi: M \to N$ a $\sigma$-semilinear map which takes a basis $(e_1, \ldots, e_n)$ of $M$ into a basis of $N$, then if $e = \sum a_i e_i$ is a unimodular element of $M$, then $\Phi(e) = \sum \sigma(a_i)\,\Phi(e_i)$ is unimodular in $N$. For, if $\sum \lambda_i a_i = 1$, $\lambda_i \in A$, we have $\sum \sigma(\lambda_i)\,\sigma(a_i) = 1$ which implies $\Phi(e) = \sum \sigma(a_i)\,\Phi(e_i)$ is unimodular. It is clear that we have an induced map $P(\Phi): P(M) \to P(N)$ by setting for any unimodular element $e$ of $M$, $P(\Phi)(Ae) = B\Phi(e)$. We then have the following rather obvious

PROPOSITION 1: *With the same notation as above, for any $p_1, p_2, p_3 \in P(M)$ with $p_1 \subset p_2 + p_3$, $P(\Phi)\,p_1 \subset P(\Phi)\,p_2 + P(\Phi)\,p_3$. If $\sigma$ is an isomorphism, then $P(\Phi)$ is a projectivity.*

## § 2 The Theorem

Our object in this section is to prove the following theorem which generalises to commutative rings the classical "Fundamental theorem of projective geometry".

THEOREM. *Let $M$ and $N$ be free modules of finite rank $\geqslant 3$ over commutative rings $A$ and $B$ respectively. If $\alpha: P(M) \to P(N)$ is a projectivity, then there exists an isomorphism $\sigma: A \to B$ and a $\sigma$-semilinear isomorphism $\Phi: M \to N$ such that $\alpha = P(\Phi)$. If $\sigma_i: A \to B$, $i = 1, 2$, are isomorphisms and $\Phi_i: M \to N$ are $\sigma_i$-semilinear isomorphisms such that $P(\Phi_1) = P(\Phi_2)$, then there exists a $b \in B$ such that $\Phi_1 = b \cdot \Phi_2$ and $\sigma_1 = \sigma_2$.*

*Proof.* Let $e_1, \ldots, e_n$ be a basis for $M$ and let $\alpha Ae_i = Bf_i$, $1 \leqslant i \leqslant n$. We assert that $f_1, \ldots, f_n$ generate the $B$-module $N$. Since any element of $N$ is a linear combination of elements of a basis for $N$, it is enough to check that any unimodular element $f \in N$ is a linear combination of $f_1, \ldots, f_n$. If $e \in M$ is a unimodular element with $\alpha Ae = Bf$ and $e = \sum_{1 \leqslant i \leqslant n} a_i e_i$, we have $Ae \subset \sum_{1 \leqslant i \leqslant n} Ae_i$ and by lemma 1, we get $Bf \subset \sum_{1 \leqslant i \leqslant n} Bf_i$.

This proves that $f_1, \ldots, f_n$ generate $N$. Since $B$ is a commutative ring, this implies that rank $N \leqslant n$. Since $\alpha^{-1}$ is also a projectivity, it follows that rank $M =$ rank $N$ and $f_1, \ldots, f_n$ is a basis for $N$.

Let $\alpha A e_1 = Bf_1$ and $\alpha A e_2 = Bg_2$. Now $e_1+e_2$ is unimodular and $A(e_1+e_2)\subset \subset Ae_1+Ae_2$ which implies that $\alpha A(e_1+e_2)\subset Bf_1+Bg_2$. Hence $\alpha A(e_1+e_2)=B(b_1f_1+ +b_2g_2)$. Since $Ae_2\subset Ae_1+A(e_1+e_2)$ we have $Bg_2\subset Bf_1+B(b_1f_1+b_2g_2)$. Thus $g_2=bf_1+c(b_1f_1+b_2g_2)$. Since $f_1$, $g_2$ are independent, it follows that $cb_2=1$, i.e. $b_2$ is a unit in $B$. Similarly $b_1$ is also a unit. Writing $f_2=b_1^{-1}b_2g_2$, we see that $f_2$ is unimodular, $Bf_2=Bg_2$ and $\alpha A(e_1+e_2)=B(f_1+f_2)$. Doing this for any $i>1$, we get a basis $f_1, f_2, \ldots, f_n$ of $N$ such that

$$\begin{aligned} \alpha A e_i &= B f_i & 1 \leqslant i \leqslant n \\ \alpha A(e_1 + e_i) &= B(f_1 + f_i) & 2 \leqslant i \leqslant n. \end{aligned} \tag{1}$$

It is clear as before that for any $a\in A$ $\alpha A(e_1+ae_2)=B(b_1f_1+b_2f_2)$ with $b_1$ a unit of $B$. Thus we can write

$$\alpha A(e_1 + a e_2) = B(f_1 + \sigma(a) f_2), \tag{2}$$

where $\sigma: A\to B$ is a well defined map. Clearly

$$\sigma(0) = 0 \quad \text{and} \quad \sigma(1) = 1. \tag{3}$$

For any fixed $i>2$, we can similarly define $\tau: A\to B$ by

$$\alpha A(e_1 + a e_i) = B(f_1 + \tau(a) f_i) \tag{4}$$

and we have

$$\tau(0) = 0 \quad \text{and} \quad \tau(1) = 1. \tag{5}$$

Since $e_1+ae_2+a'e_i\in A(e_1+ae_2)+Ae_i$, we have $\alpha A(e_1+ae_2+a'e_i)\subset B(f_1+\sigma(a)f_2) +Bf_i$. Hence $\alpha A(e_1+ae_2+a'e_i)=B(b(f_1+\sigma(a)f_2)+b'f_i)$. Similarly, $\alpha A(e_1+ae_2+ +a'e_i)=B(c(f_1+\tau(a')f_i)+c'f_2)$.
Combining the above equations, we find that

$$\alpha A(e_1 + a e_2 + a' e_i) = B(f_1 + \sigma(a) f_2 + \tau(a') f_i). \tag{6}$$

Since $ae_2+e_i\in A(e_1+ae_2+e_i)+Ae_1$, using (6) and (5) we have $\alpha A(ae_2+e_i)= =B(b(f_1+\sigma(a)f_2+f_i)+cf_1)$. Since $\alpha A(ae_2+e_i)\subset Bf_2+Bf_i$, we get $b+c=0$ and this proves

$$A(a e_2 + e_i) = B(\sigma(a) f_2 + f_i). \tag{7}$$

Now using (6) and (5), we have for $a, a'\in A$, $\alpha A(e_1+(a+a')e_2+e_i)=B(f_1+\sigma(a+ +a')f_2+f_i)$. But $\alpha A(e_1+(a+a')e_2+e_i)\subset \alpha A(e_1+ae_2)+\alpha A(a'e_2+e_i)$. Using (7), we therefore have $\alpha A(e_1+(a+a')e_2+e_i)\subset B(f_1+\sigma(a)f_2)+B(\sigma(a')f_2+f_i)$. Using the above, we see that for $a, a'\in A$, we have

$$\sigma(a + a') = \sigma(a) + \sigma(a'). \tag{8}$$

Now for $a, a'\in A$, we have, using (6), that $\alpha A(e_1+aa'e_2+ae_i)=B(f_1+\sigma(aa')f_2+$

$+\tau(a)f_i)$. On the other hand, $\alpha A(e_1+aa'e_2+ae_i)\subset\alpha Ae_1+\alpha A(a'e_2+e_i)$ which implies that $\alpha A(e_1+aa'e_2+ae_i)=B(bf_1+b'(\sigma(a')f_2+f_i))$. Comparing coefficients, we find that $\sigma(aa')=\tau(a)\,\sigma(a')$. Setting $a'=1$, we get

$$\sigma(a)=\tau(a) \quad \text{for all} \quad a\in A \tag{9}$$

and

$$\sigma(a\,a')=\sigma(a)\,\sigma(a') \quad \text{for } a,\, a'\in A. \tag{10}$$

Thus, the map $\sigma: A\to B$ defined by (2) is a homomorphism. Replacing $\alpha$ by $\alpha^{-1}$, we can define a homomorphism $\sigma': B\to A$ satisfying

$$\alpha^{-1}B(f_1+b\,f_2)=A(e_1+\sigma'(b)\,e_2)$$

and clearly $\sigma$ and $\sigma'$ are inverses of each other. Thus $\sigma: A\to B$ is an isomorphism.

We now show that, for $a_2,\ldots, a_n\in A$, we have

$$\alpha A(e_1+a_2e_2+\cdots+a_ne_n)=B(f_1+\sigma(a_2)\,f_2+\cdots+\sigma(a_n)\,f_n). \tag{11}$$

We can assume by induction that

$$\alpha A(e_1+a_2e_2+\cdots+a_{n-1}e_{n-1})=B(f_1+\sigma(a_2)\,f_2+\cdots+\sigma(a_{n-1})\,f_{n-1}).$$

Since

$$\alpha A(e_1+a_2e_2+\cdots+a_ne_n)\subset\alpha A(e_1+a_2e_2+\cdots+a_{n-1}e_{n-1})+\alpha A\,e_n,$$

we have

$$\alpha A(e_1+a_2e_2+\cdots+a_ne_n)=B(b(f_1+\sigma(a_2)\,f_2+\cdots+\sigma(a_{n-1})\,f_{n-1})+b'f_n).$$

On the other hand, we also have

$$\alpha A(e_1+a_2e_2+\cdots+a_ne_n)\subset\alpha A(e_1+a_ne_n)+\alpha A\,e_2+\cdots+\alpha A\,e_{n-1}.$$

Comparing coefficients we find that $b'=b\sigma(a_n)$ and this proves (11).

If $a_2,\ldots, a_n\in A$ are such that $a_2e_2+\cdots+a_ne_n\in M$ is unimodular, we have

$$\alpha A(a_2e_2+\cdots+a_ne_n)\subset A(e_1+a_2e_2+\cdots+a_ne_n)+\alpha A\,e_1.$$

Using (11) we have

$$\alpha A(a_2e_2+\cdots+a_ne_n)=B(b(f_1+\sigma(a_2)\,f_2+\cdots+\sigma(a_n)\,f_n)+b'f_1).$$

We also have

$$\alpha A(a_2e_2+\cdots+a_ne_n)\subset B\,f_2+\cdots+B\,f_n.$$

Combining these two facts, we get

$$\alpha A(a_2e_2+\cdots+a_ne_n)=B(\sigma(a_2)\,f_2+\cdots+\sigma(a_n)\,f_n). \tag{12}$$

We now assert that for any $a_1,\ldots, a_{i-1}, a_{i+1},\ldots, a_n\in A$ and $i=2,\ldots, n$,

$$\begin{aligned}\alpha A(e_i+a_1e_1+\cdots+a_{i-1}e_{i-1}+a_{i+1}e_{i+1}+\cdots+a_ne_n)\\ =B(f_i+\sigma(a_1)\,f_1+\cdots+\sigma(a_n)\,f_n).\end{aligned} \tag{13}$$

To prove (13), we first observe, using (1) and (12) that $\alpha A(e_i+e_j)=B(f_i+f_j)$ for any $j\neq i$. Fixing an $i$ and replacing $e_1$ by $e_i$, we can repeat the previous arguments to get an isomorphism $\varrho: A\to B$ such that for $a_1,\ldots, a_{i-1}, a_{i+1},\ldots, a_n\in A$, we have the following equation:

$$\begin{aligned}\alpha A(e_i + a_1 e_1 + \cdots + a_{i-1} e_{i-1} + a_{i+1} e_{i+1} + \cdots + a_n e_n)\\ = B(f_i + \varrho(a_1) f_1 + \cdots + \varrho(a_n) f_n).\end{aligned} \tag{14}$$

instead of (11).

Taking in (14) $a_1=0$ and comparing this equation with (12), we find that $\sigma=\varrho$. Now (14) gives (13).

Let $e=\sum_{1\leqslant i\leqslant n} a_i e_i\in M$ be a unimodular element. We now show that

$$\alpha A(a_1 e_1 + \cdots + a_n e_n) = B(\sigma(a_1) f_1 + \cdots + \sigma(a_n) f_n). \tag{15}$$

Since for $i=1, 2, 3$, we have $\alpha Ae\subset\alpha Ae_i+\alpha A(e_i+\cdots+\widehat{a_i e_i}+\cdots)$ (where $\wedge$ indicates that the corresponding term is omitted), we can write $\alpha Ae=Bf$ where

$$\begin{aligned} f &= b_1\sigma(a_1) f_1 + c_1\sigma(a_2) f_2 + c_1\sigma(a_3) f_3 + \cdots\\ &= c_2\sigma(a_1) f_1 + b_2\sigma(a_2) f_2 + c_2\sigma(a_3) f_3 + \cdots\\ &= c_3\sigma(a_1) f_1 + c_3\sigma(a_2) f_2 + b_3\sigma(a_3) f_3 + \cdots. \end{aligned}$$

Comparing coefficients, we find

$$\left.\begin{array}{l} b_1\sigma(a_1)\sigma(a_2) = c_3\sigma(a_1)\sigma(a_2) = c_1\sigma(a_1)\sigma(a_2)\\ \text{and for every } i\geqslant 3, \text{ we have}\\ b_1\sigma(a_1)\sigma(a_i) = c_2\sigma(a_1)\sigma(a_i) = c_1\sigma(a_1)\sigma(a_i). \end{array}\right\} \tag{16}$$

Since $e=\sum a_i e_i$ is unimodular, it follows that $\sum\sigma(a_i) f_i$ is unimodular and hence there exist $k_1,\ldots, k_n\in B$ such that $\sum\sigma(a_i) k_i=1$. Set

$$d = b_1\sigma(a_1) k_1 + c_1\sigma(a_2) k_2 + \cdots + c_1\sigma(a_n) k_n.$$

Using the equations (16), we easily verify that $d\sigma(a_1)=b_1\sigma(a_1)$ and $d\sigma(a_i)=c_1\sigma(a_i)$ for $i\geqslant 2$. Then $d$ is a unit and (15) is proved.

Let $\Phi: M\to N$ be the $\sigma$-semilinear isomorphism $M\to N$ defined by $\Phi(e_i)=f_i$. The equation (15) shows that $\alpha=P(\Phi)$. The proof of the second statement of the theorem is the same as in the classical case which can be found for instance in E. ARTIN [1, chap. II].

## § 3 A Counter-Example

If $M$, $N$ are finite dimensional vector spaces of the same rank over fields $A$ and $B$ respectively and if $\alpha: P(M)\to P(N)$ is a bijection which is such that for any $p_1, p_2, p_3,\in P(M)$ with $p_1\subset p_2+p_3$, we have $\alpha p_1\subset\alpha p_2+\alpha p_3$, it can be proved (see for instance Artin [1, chap. II]) that $\alpha$ is a projectivity. We now give an example to show that this need not be the case if $A$ and $B$ are arbitrary rings.

Let $K$ be a field; let $A=K\langle x\rangle$ be the ring of formal power series in $x$ and $B$ the quotient field of $A$. The canonical inclusion $\sigma: A \to B$ induces a $\sigma$-semilinear map $A^3 \to B^3$ which in turn gives rise to a map $P(\sigma): P(A^3) \to P(B^3)$.

PROPOSITION 2.*) *The map $P(\sigma)$ is a bijection such that for any $p_1, p_2, p_3 \in P(A^3)$ with $p_1 \subset p_2 + p_3$, we have $P(\sigma)\, p_1 \subset P(\sigma)\, p_2 + P(\sigma)\, p_3$. However $P(\sigma)$ is not a projectivity.*

*Proof*. Let $(a_1, a_2, a_3)$, $(a'_1, a'_2, a'_3)$ be unimodular elements of $A^3$ which represent the same element of $P(B^3)$. We then have $a, a' \in A, a \neq 0, a' \neq 0$ such that $a'(a'_1, a'_2, a'_3) = a(a_1, a_2, a_3)$, i.e. $a'a'_i = aa_i\, 1 \leqslant i \leqslant 3$. If $\sum\limits_{1 \leqslant i \leqslant 3} a_i k_i = 1$, we have $a'\lambda = a$ with $\lambda = \sum a_i k_i A$. Similarly, $a\mu = a$ for some $\mu \in A$. This implies that $a$ and $a'$ differ by a unit of $A$ and hence $A(a_1, a_2, a_3) = A(a'_1, a'_2, a'_3)$. This proves that $P(\sigma)$ is injective. Given any element of $P(B^3)$, we can write it in the form $Be$ where $e \in A^3$. Dividing if necessary by a suitable power of $x$, we may assume that at least one coordinate of $e$ has a nonzero constant term and hence is a unit in $A$. Therefore we may assume that $e$ is a unimodular element of $A^3$ and this proves that $P(\sigma)$ is surjective. If $p_1, p_2, p_3 \in P(A^3)$ are such that $p_1 \subset p_2 + p_3$, it is trivial to check that $P(\sigma)\, p_1 \subset P(\sigma)\, p_2 + P(\sigma)\, p_3$. Now, $P(\sigma)\ A(1, 0, 0) = B(1, 0, 0) = B(x, 0, 0) \subset P(\sigma)\ A(x, 1, 0) + P(\sigma)\ A(0, 1, 0)$. However, $(1, 0, 0) \notin A(x, 1, 0) + A(0, 1, 0)$. This shows that $P(\sigma)$ is not a projectivity.

## REFERENCES

[1] ARTIN, E., *Geometric Algebra* (Interscience, New York 1957).
[2] GABRIEL, P., Séminaire Heidelberg-Strasbourg 1965/66, Exposé 1.

*Forschungsinstitut für Mathematik, E.T.H. Zürich,*
*Tata Institute of Fundamental Research, Bombay.*

Received July 1, 1968

*) (Added in proof.) This proposition and its proof are valid equally for any unique factorisation domain.