Representative Functions on Topological Groups

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1. Introduction

In this paper we shall study the relations existing between the topological properties of a completely regular topological group G and the algebraic properties of the space of all representative functions $R(G)$ over G.

In the first part we give some results which generalize those of S. Kakutani ([4] pp. 430–431) concerning compactifications of locally compact abelian groups.

For a compact group G the Tannaka duality theorem shows that the algebraic properties of $R(G)$ characterize completely those of G. Using [2], we find algebraic characterizations of the connectedness, local connectedness and arcwise connectedness of G. Similarly, we attempt to generalize, in a certain sense, the well-known result of Pontrjagin ([10] p. 32) about the covering dimension of a compact abelian group. Using these results we obtain some applications to more general topological groups.

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2. Compactifications and related questions

Let γ be the map of $R(G)$ into $R(G)\otimes_{\mathbf{C}}R(G)$ induced by the product in G. Following ([6]), one can say that, with the coproduct γ and the pointwise product, $R(G)$ is a Hopf algebra. We consider, as in [2], only Hopf subalgebras of $R(G)$ which are stable under complex conjugation.

Let $\mathscr{H}$ be a Hopf subalgebra of $R(G)$. We denote by $S(\mathscr{H})$ the set of all $\mathbf{C}$-algebra homomorphisms of $\mathscr{H}$ onto $\mathbf{C}$ which commute with complex conjugation. With the finite open topology $S(\mathscr{H})$ is a compact space ([6] p. 28). Let Γ be a non empty subset of $R(G)$; we denote by $\mathscr{H}(\Gamma)$ the least Hopf subalgebra containing Γ. It follows from ([6] p. 29–30) that $S(\mathscr{H}(\Gamma))$ is a compact group and the evaluation map φ_Γ of G into $S(\mathscr{H}(\Gamma))$ is a continuous homomorphism.

PROPOSITION 1. *The group $\varphi_\Gamma(G)$ is dense in $S(\mathscr{H}(\Gamma))$ for every $\Gamma\subset R(G)$.*

Proof. Consider $f\in R(S(\mathscr{H}(\Gamma)))$ with $f=0$ on $\varphi_\Gamma(G)$. By the Tannaka duality theorem ([6] p. 30) there exists $h \in \mathscr{H}(\Gamma)$ such that $s(h)=f(s)$ for every $s \in S(\mathscr{H}(\Gamma))$. In particular $\varphi_\Gamma(x)(h)=h(x)=0$ for every $x \in G$. This implies that $h=0$ and therefore $f=0$. Using ([7] Lemma 5.2.) we obtain $\overline{\varphi_\Gamma(G)}=S(\mathscr{H}(\Gamma))$.

COROLLARY 1. Let $\mathscr{H}$ be any Hopf subalgebra of $R(G)$. Let τ be any element of $S(\mathscr{H})$, let $f_1, \ldots, f_n$ be a finite subset of $\mathscr{H}$ and let ε be any positive number. Then there is a point $x \in G$ such that $|\tau(f_j)-f_j(x)|<\varepsilon$ $(1 \leqslant j \leqslant n)$.

Proof. By definition of the topology of $S(\mathscr{H})$ the set $\{\tau' \in S(\mathscr{H}) \mid |\tau'(f_j)-\tau(f_j)|<\varepsilon$ $1 \leqslant j \leqslant n\}$ is an open neighborhood U of τ. From prop. 1 the existence of $x \in G$ then follows with the required properties.

Remark. This result is proved for characters over a topological group in ([5]). At the end of the same paper, the authors indicate the possibility of generalization.

COROLLARY 2. Let G be an infinite maximally almost periodic group and let $f_1, \ldots, f_n \in R(G)$ and $\varepsilon>0$. Then there is an element $x \in G$ such that $x \neq e$ and $|f_j(x)-f_j(e)| < \varepsilon$ $(1 \leqslant j \leqslant n)$.

The proof is analogous (using prop. 1) to that in the locally compact abelian case ([4] p. 431).

PROPOSITION 2. *Let G be a topological group. Let H be a compact group. Then the following assertions are equivalent:*

(i) *There is a continuous homomorphism φ of G into H such that $\overline{\varphi(G)}=H$.*
(ii) *H is isomorphic to the compact group $S(\Gamma)$ for some Hopf subalgebra Γ of $R(G)$.*
(iii) *There is a Hopf algebra monomorphism ψ of $R(H)$ into $R(G)$.*

Proof. It is clear that (i) implies (iii) and that (ii) implies (i). Suppose that (iii) holds. The map ψ^* of $S(R(G))$ into $S(R(H))$ defined by $\psi^*(s)=s \circ \psi$ is a continuous group homomorphism. There exists a continuous group homomorphism ψ' of G into H defined by the commutativity of

$$\begin{array}{ccc} S(R(G)) & \xrightarrow{\psi^*} & S(R(H)) \\ {\scriptstyle \varphi_{R(G)}}\uparrow & & \uparrow{\scriptstyle \varphi_{R(H)}} . \\ G & \xrightarrow{\psi'} & H \end{array}$$

The relation $\overline{\psi'(G)} \neq H$ implies the existence of $f \in R(H)$ with $f \neq 0$ and $f(\psi'(x))=0$ for any $x \in G$. This contradicts the equality $f \circ \psi'=\psi(f)$. Therefore (iii) implies (i). It remains to prove that (i) implies (ii). Consider the Hopf algebra monomorphism φ^* of $R(H)$ into $R(G)$ defined by $\varphi^*(f)=f \circ \varphi$ and set $\Gamma=\varphi^*(R(H))$. To every $f \in R(H)$ there corresponds a function on $S(\Gamma)$ defined by $s(\varphi^*(f))$ for every

$s \in S(\Gamma)$. This map is a Hopf algebra isomorphism of $R(H)$ onto $R(S(\Gamma))$ and therefore H and $S(\Gamma)$ are isomorphic.

Remark. From the approximation theorem it follows that $S(R(G))$ is isomorphic to the almost periodic compactification of G ([8] p. 168).

3. Some results concerning compact groups

For a compact group G we have $\varphi(G)=S(R(G))$ (we set $\varphi_{R(G)}=\varphi$). This equality permits us to characterize the topological properties of G (as in the abelian case) using the "algebraic" properties of $R(G)$.

First we introduce some notations. If $\mathscr{H}$ is a Hopf subalgebra of $R(G)$, let $\mathscr{H}^{\perp}$ denote the closed normal subgroup of G defined by $\{h \in G \mid {}_h f=f \text{ for every } f \in \mathscr{H}\}$. Conversely, if H is a closed normal subgroup of G, let $H^{\perp}$ be the Hopf subalgebra of $R(G)$ defined by $\{f \in R(G) \mid {}_h f=f \text{ for every } h \in H\}$. In [2] the following result was proved:

THEOREM 1. *For every compact group G, $G_0^{\perp}=\{f \in R(G) \mid f$ is an algebraic element of the* **C**-*algebra $R(G)\}$, where G_0 denotes the connected component of the identity in G.*

Proof. We prove at first that the above conditions are sufficient to insure the local connectedness of a compact group G.

THEOREM 2. *A compact group G is locally connected if and only if every finite set of representative functions on G is contained in a finitely generated Hopf subalgebra $\mathscr{H}$ of $R(G)$ such that every non constant element of $R(\mathscr{H}^{\perp})$ is not algebraic.*

Proof. We prove at first that the above conditions are sufficient to insure the local connectedness of G. For every open neighborhood U of e in G there exists an $\varepsilon>0$ and there exists a sequence $\{f_j\}_{j=1}^{n} \subset R(G)$ such that the set $\{x \in G \mid |f_j(x)-f_j(e)| <\varepsilon \ 1 \leqslant j \leqslant n\}$ is contained in U. This implies that $\mathscr{H}(f_1,\ldots,f_n)^{\perp} \subset U$. By hypothesis there exists a finitely generated Hopf subalgebra $\mathscr{E}$ of $R(G)$ with $\mathscr{E} \supset \mathscr{H}(f_1,\ldots,f_n)$ and $\mathscr{E}^{\perp}$ connected. Let π be the canonical map of G onto $G/\mathscr{E}^{\perp}$. The factor group $G/\mathscr{E}^{\perp}$ is a Lie group, since $R(G/\mathscr{E}^{\perp})$ and $\mathscr{E}$ are isomorphic. Let Σ be a fundamental system of open connected neighborhoods of $\pi(e)$ in $G/\mathscr{E}^{\perp}$. It is easy to demonstrate the existence of a subset $O \in \Sigma$ with $\pi^{-1}(O) \subset U$. It suffices to prove that $\pi^{-1}(O)$ is connected. Suppose the contrary. There exist open subsets of G V_1, V_2 such that $V_1, V_2 \neq \emptyset$, $V_1 \cap V_2=\emptyset$ and $V_1 \cup V_2=\pi^{-1}(O)$. The existence of $x \in G$ with $\pi(x) \in \pi(V_1) \cap \pi(V_2)$ contradicts the connectedness of $x\mathscr{E}^{\perp}$. We therefore have $\pi(V_1) \cap \pi(V_2)=\emptyset$ and this implies that O is not connected.

For the second part of the proof, we suppose that $R(G)$ does not satisfy the above conditions, and show that G is not locally connected. In this case there exists an $M \subset R(G)$ with $|M| < \infty$, such that every Hopf subalgebra $\mathscr{E}$ of $R(G)$ with $\mathscr{E} \supset M$ and $\mathscr{E}^{\perp}$ connected is not finitely generated. Let $\mathscr{H}$ be the Hopf subalgebra of $R(G)$ with the property that $\mathscr{H}^{\perp}$ is the connected component of the unit element in the subgroup $\mathscr{H}(M)^{\perp}$ (the connected component of a normal closed subgroup is itself a normal subgroup). Denoting by α the canonical map of $G/\mathscr{H}^{\perp}$ onto $G/\mathscr{H}(M)^{\perp}$, we have $\operatorname{Ker}\alpha = \mathscr{H}(M)^{\perp}/\mathscr{H}^{\perp}$. By a generalization of a wellknown theorem of Hurewicz ([9] theorem 4), $\dim \operatorname{Ker}\alpha = 0$ implies $\dim G/\mathscr{H}^{\perp} \leqslant \dim G/\mathscr{H}(M)^{\perp}$, and then $\dim S(\mathscr{H}) \leqslant \dim S(\mathscr{H}(M))$. It follows that $\dim S(\mathscr{H})$ is finite, because $S(\mathscr{H}(M))$ is a compact Lie group. By hypothesis $\mathscr{H}$ is not finitely generated. This fact implies that $S(\mathscr{H})$ is not locally connected, and therefore (since the natural map of G onto $G/\mathscr{H}^{\perp}$ is open) that G itself is not locally connected.

Remarks.

1) In this proof we have used the two following results: α) A compact group G is a Lie group if and only if the $\mathbf{C}$-algebra $R(G)$ is finitely generated; β) Every compact (or locally compact) locally connected group with a finite dimension is a Lie group.

2) The corresponding classical result ([10] p. 33) for compact abelian groups is: G is locally connected if and only if every finite number of continuous characters over G is contained in a finitely generated subgroup H of $\hat{G}$ (group of all continuous characters over G) such that $\hat{G}/H$ is torsion-free.

We denote by $\mathscr{D}(G)$ the set of all $\mathbf{C}$-derivations of the $\mathbf{C}$-algebra $R(G)$ which commute with complex conjugation and every left translation. Let $D \in \mathscr{D}(G)$. For every $f \in R(G)$ consider the finite dimensional G-module $R(f) = [\{f_x \mid x \in G\}]$. By ([7] prop. 2.5) $R(f)$ is stable under D. This implies that $\sum_{n=1}^{\infty} D^n f/n!$ defines an element $\exp Df$ of $R(f)$ and therefore of $R(G)$.

PROPOSITION 3. *For every $D \in \mathscr{D}(G)$ the map $t \mapsto \varphi^{-1}(\varphi(e)\exp tD)$ is a one-parameter subgroup of G. Conversely every one-parameter subgroup admits such a unique representation.*

Proof. Let $D \in \mathscr{D}(G)$ and $t \in \mathbf{R}$. It is easy to prove that $\exp tD(fg) = \exp tD(f)$ $\exp tD(g)$ for every $f, g \in R(G)$. It follows that $\exp tD$ is a $\mathbf{C}$-algebra endomorphism of $R(G)$. From the fact that $\exp tD$ commutes with complex conjugation it follows that $\varphi(e)\exp tD \in S(R(G))$. We have therefore that $t \mapsto \varphi^{-1}(\varphi(e)\exp tD)$ is a one-parameter subgroup of G.

Let $\lambda \in \operatorname{Hom}_{\text{cont}}(\mathbf{R}, G)$. For every $f \in R(G)$ and $t \in \mathbf{R}$ set $U_t f = f_{\lambda(t)}$. The operator U_t is unitary under the scalar-product of $R(G)$ defined by the normalized Haar measure of G. We denote by U'_t the extension of U_t to $L^2(G)$. There exists an operator D of $L^2(G)$ with iD selfadjoint and such that $\lim_{t \to 0} \|(U'_t f - f)t^{-1} - Df\|_2 = 0$ for every $f \in R(G)$. The operator $-iD$ has the spectral representation $\int_{-\infty}^{+\infty} \mu \, dE_\mu$ and U_t' is equal to $\int_{-\infty}^{+\infty} e^{i\mu t} \, dE_\mu$. For every f in $R(G)$ and $t \neq 0$ we have $(U_t f - f) \, t^{-1} \in R(f)$ and therefore $Df \in R(f)$, i.e. $D(R(G)) \subset R(G)$. It is easy to verify that the restriction of D to $R(G)$ is contained in $\mathscr{D}(G)$. As above we can define $\exp tD$. It is clear that the **C**-algebra endomorphism $\exp tD$ commutes with complex conjugation and left translations and invoking ([7] Lemma 5.4) we obtain that $\exp tD$ is a unitary operator of $R(G)$. For every f of $R(G)$ we have $\lim_{t \to 0} \| (\exp tDf - f) \, t^{-1} - Df \|_2 = 0$. Let U_t'' be the extension of $\exp tD$ to $L^2(G)$. As above there exists an operator D' of $L^2(G)$ with iD' self-adjoint and $\lim_{t \to 0} \| (U_t'' h - h) \, t^{-1} - D'h \|_2 = 0$ for every $h \in R(G)$. We have therefore $D = D'$ and $U_t'' = U_t'$ i.e. $\exp tDf = f_{\lambda(t)}$ for every $f \in R(G)$.

COROLLARY. For a compact Lie group G, the Lie algebra g of G is isomorphic to $\mathscr{D}(G)$.

Remarks.

1) Proposition 3 gives a characterisation of the Lie algebra of a compact group. The corollary has been already proved for more general Lie groups than compact Lie groups ([7] Theorem 11.1).

2) For the second part of the proof of proposition 3 Professor G. Hochschild has suggested a method which avoids the use of operator theory in $L^2(G)$. If V is any finite dimensional right-submodule of $R(G)$ the map $t \mapsto U_t$ (where $U_t f = f_{\lambda(t)}$) defines a continuous homomorphism of **R** into the full linear group of V. This homomorphism is therefore of the form $t \mapsto \exp tD_V$, where D_V is some linear endomorphism of V. Since $R(G)$ is the union of such V's, the D_V's match up to give a linear endomorphism D of $R(G)$ with the required properties.

We set for $\Gamma \subset R(G)$ and $M \subset \mathscr{D}(G)$:

(i) $\operatorname{Ann}(\Gamma) = \{ D \in \mathscr{D}(G) \mid Df = 0 \text{ for every } f \in \Gamma \}$,

(ii) $\mathscr{H}_l(\Gamma) =$ the least subalgebra of $R(G)$ invariant under the left-translations and the complex conjugation containing Γ.

(iii) $\operatorname{Ann}(M) = \{ f \in R(G) \mid Df = 0 \text{ for every } D \in M \}$.

It is easy to see that $\operatorname{Ann}(\Gamma)$ is a Lie subalgebra of $\mathscr{D}(G)$, and that $\operatorname{Ann}(M) = \mathscr{H}_l(\operatorname{Ann}(M))$.

PROPOSITION 4. *For every subset Γ of $R(G)$, we have $\mathscr{H}_l(\Gamma \cup \mathscr{A}) = \operatorname{Ann}(\operatorname{Ann}(\Gamma))$, where $\mathscr{A}$ is the subset of all algebraic elements of $R(G)$.*

Proof. Denote by $\lambda(D)$ the element of $\operatorname{Hom}_{\text{cont}}(\mathbf{R}, G)$ corresponding to $D \in \mathscr{D}(G)$. From $f \in \lambda(D)(\mathbf{R})_r^\perp$ [1]) it follows that $\exp tDf = f$ for every $t \in \mathbf{R}$ i.e. $f \in \operatorname{Ker} D$ and

[1]) For every subset H of G, $H_r{}^\perp$ denotes the set $\{ f \in R(G) \mid f_x = f \text{ for every } x \in H \}$ and for any subalgebra Γ of $R(G)$ with $\mathscr{H}_l(\Gamma) = \Gamma$ $\Gamma_r{}^\perp$ is the closed subgroup $\{ x \in G \mid f_x = f \text{ for every } f \in \Gamma \}$.

conversely, we have therefore $\lambda(D)(\mathbf{R})_r^\perp=\operatorname{Ker}D$. Using the fact that every one-parameter subgroup is contained in G_0 we obtain $\operatorname{Ker}D\supset\mathscr{A}$ and in particular $\operatorname{Ann}(\operatorname{Ann}(\Gamma))\supset\mathscr{H}_l(\Gamma\cup\mathscr{A})$. It is easy to verify that $\operatorname{Ann}(\Gamma)=\{D\in\mathscr{D}(G)\mid\lambda(D)(\mathbf{R})\subset\mathscr{H}_l(\Gamma\cup\mathscr{A})_r^\perp\}$. Since the closed subgroup $\mathscr{H}_l(\Gamma\cup\mathscr{A})_r^\perp$ is connected, we have $\mathscr{H}_l(\Gamma\cup\mathscr{A})=\{\lambda(D)(\mathbf{R})\mid D\in\mathscr{D}(G),\lambda(D)(\mathbf{R})\subset\mathscr{H}_l(\Gamma\cup\mathscr{A})_r^\perp\}_r^\perp$ and therefore $\mathscr{H}_l(\Gamma\cup\mathscr{A})=\{\lambda(D)(\mathbf{R})\mid D\in\operatorname{Ann}(\Gamma)\}_r^\perp=\bigcap\{\operatorname{Ker}D\mid D\in\operatorname{Ann}(\Gamma)\}=\operatorname{Ann}(\operatorname{Ann}(\Gamma))$.

Remarks.

1) For $\Gamma=\emptyset$ we obtain $\mathscr{A}=\operatorname{Ann}(\mathscr{D}(G))$ which gives another characterisation of the set of all algebraic elements of $R(G)$.

2) The group G is solenoïdal if and only if there is $D\in\mathscr{D}(G)$ with $\operatorname{Ker}D=\mathbf{C}\cdot 1_G$.

3) There is a bijective map between the closed subgroups of G_0 and the Lie subalgebras M of $\mathscr{D}(G)$ such that $M=\operatorname{Ann}(\operatorname{Ann}(M))$. That is, to every closed subgroup H of G_0 we associate $M=\operatorname{Ann}(H_r^\perp)$. The subgroup H is normal in G if and only if M is an ideal of $\mathscr{D}(G)$.

THEOREM 3. *A compact group G is arcwise connected if and only if for every $x\in G$ there is an element D of $\mathscr{D}(G)$ such that the following diagram commutes:*

$$\begin{tikzcd} R(G) \arrow[rr, "\varphi(x)"] \arrow[dr, "\exp D"'] & & \mathbf{C} \\ & R(G) \arrow[ur, "\varphi(e)"'] & \end{tikzcd}$$

LEMMA. *A compact group is arcwise connected if and only if it is the union of all one-parameter subgroups.*

Proof. By ([11] Theorem 1) every arc beginning at the unit element is homotopic to the restriction to $[0,1]$ of a one-parameter subgroup.

Proof of theorem 3. Suppose first that G is arcwise connected. In this case for every $x\in G$ there exists $\lambda\in\operatorname{Hom}_{\text{cont}}(\mathbf{R},G)$ and $a\in\mathbf{R}$ with $\lambda(a)=x$. There exists $D\in\mathscr{D}(G)$ such that $\lambda(at)=\varphi^{-1}(\varphi(e)\exp tD)$ and therefore $\varphi(e)\exp D=\varphi(x)$.

Conversely suppose that for every $x\in G$ there exists $D\in\mathscr{D}(G)$ such that $\varphi(x)=\varphi(e)\exp D$. If we set $\lambda(t)=\varphi^{-1}(\varphi(e)\exp tD)$ we obtain $\lambda\in\operatorname{Hom}_{\text{cont}}(\mathbf{R},G)$ and $\lambda(1)=x$.

Remarks.

1) The classical result for compact abelian groups ([3]) is: G is arcwise connected if and only if for every $x\in G$ there exists $\lambda\in\operatorname{Hom}(\hat{G},\mathbf{R})$ such that

$$\begin{tikzcd} \hat{G} \arrow[r, "x"] \arrow[dr, "\lambda"'] & S^1 \\ & \mathbf{R} \arrow[u, "e^{2\pi i}"'] \end{tikzcd}$$

commutes.

2) It is not necessary to give conditions which imply the local arcwise connectedness of G because a compact connected group is locally arcwise connected if and only if it is arcwise connected ([11]).

The dimension of a compact abelian group is equal by ([10] p. 32) to the rank of its character group. The next theorem is to be considered as a possible generalization to the non abelian case.

THEOREM 4. *The dimension of a compact group G is equal to the dimension of the real vector space $\mathscr{D}(G)$.*

Proof. There exists an inverse system $(G_\alpha, u_{\alpha\beta})$ consisting of compact Lie groups G_α and continuous epimorphisms $u_{\beta\alpha}: G_\beta \to G_\alpha (\alpha < \beta)$ such that $G \cong \overleftarrow{\lim}(G_\alpha, u_{\alpha\beta})$. We denote by π_α the projection of G onto G_α; by R_α, the Hopf subalgebra of $R(G)$, $(\operatorname{Ker}\pi_\alpha)^\perp$; by $\mathscr{D}_\alpha$ the set of all **C**-derivations of R_α which commute with complex conjugation and all left translations and finally by $i_{\alpha\beta}$ $(\alpha < \beta)$ the natural injection of R_α into R_β. It follows from ([2]) that $R(G)$ and $\overrightarrow{\lim}(R_\alpha, i_{\alpha\beta})$ are isomorphic. The restriction $R_{\beta\alpha}$ $(\alpha < \beta)$ of an element of $\mathscr{D}_\beta$ to R_α belongs to $\mathscr{D}_\alpha$. The differential $\dot{u_{\beta\alpha}}$ of $u_{\beta\alpha}$ is a linear map of the Lie algebra g_β of G_β onto g_α. It is easy to verify that the projective systems $(\mathscr{D}(G), \mathrm{id})$, $(\mathscr{D}_\alpha, \mathrm{Res}_{\alpha\beta})$ and $(g_\alpha, \dot{u_{\alpha\beta}})$ are isomorphic. From $\dim \mathscr{D}_\alpha = \dim G_\alpha$ (corollary of prop. 3), $\dim G = \sup_\alpha \dim G_\alpha$ and $\dim \mathscr{D}(G) = \sup_\alpha \dim \mathscr{D}_\alpha$ the theorem follows.

4. Applications

For non-compact groups the relations between the properties of G and those of $R(G)$ are more complicated.

If the **C**-algebra $R(G)$ of a locally compact maximally almost periodic group G is finitely generated, then G is a Lie group. The condition is not necessary. However, if G is a Lie group such that G/G_0 is finite then $R(G)$ is finitely generated if and only if the factor group of G modulo the closure of the commutator of G_0 is compact ([7] theorem 11.1).

PROPOSITION 5. *If a topological group G is connected, then every non constant representative function over G is non algebraic. If every representative function over a maximally almost periodic group is algebraic then the group is totally disconnected.*

Proof. The connectedness of G implies the same property for $S(R(G))$. From Theorem 1 the first part of proposition 5 follows. The proof of the second part is completely analogous.

THEOREM 5. *Every locally countably compact torsion group with a maximally almost periodic connected component of the identity is totally disconnected.*

Proof. Suppose that G is a compact torsion group. For every $f \in R(G)$ consider $R(f)$ and the corresponding continuous finite dimensional representation ϱ_f; $\varrho_f(G)$ is a compact torsion Lie group and therefore is a finite group. It follows $\operatorname{Ker} \varrho_f \supset G_0$ i.e. $f \in G_0^{\perp}$. Using theorem 1, we have that G is totally disconnected. For the general case consider, the continuous map $\alpha_n : G \to G$ defined by $\alpha_n(x) = x^n$ for every positive integer n. By assumption we have $G = \bigcup_{n=1}^{\infty} \operatorname{Ker} \alpha_n$, the category theorem of Baire implies the existence of n_0 such that $\operatorname{Ker} \alpha_{n_0}$ is open and therefore $\operatorname{Ker} \alpha_{n_0} \supset G_0$. From this it follows that $S(R(G_0))$ is a torsion group. Using the first part of the proof, theorem 1 and proposition 5 we have the desired result.

Remark. This theorem generalizes a result proved by Braconnier ([1] p. 51) for the case of a locally compact abelian group.

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