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# Differential Structures on a Product of Spheres

R. DE SAPIO

## 1. Introduction

In this paper we give a classification under the relation of orientation preserving diffeomorphism, of all differential structures on a simply connected product of spheres  $S^k \times S^p$  of dimension greater than five. In particular, we prove the following theorem.

**THEOREM 1.** *Let  $M^n$  be a differential  $n$ -manifold of dimension greater than five that is homeomorphic to a product of spheres  $S^k \times S^p$ , where  $2 \leq k \leq p$ . Then there are homotopy spheres  $A^p$  and  $V^n$  such that  $M^n$  is diffeomorphic to  $(S^k \times A^p) + V^n$ . Furthermore, we have the following conclusions.*

(a) *If  $B^p$  and  $U^n$  are homotopy spheres such that  $M^n$  is also diffeomorphic to  $(S^k \times B^p) + U^n$ , then  $S^k \times A^p$  and  $S^k \times B^p$  are diffeomorphic. If, in addition, either  $k \equiv 2, 4, 5, 6 \pmod{8}$  or  $k \geq p-3$ , then  $V^n$  and  $U^n$  are diffeomorphic.*

(b) *If either  $k \geq p-3$  or  $p=4, 5, 6, 7$ , then there is one and only one homotopy  $n$ -sphere  $V^n$  such that  $M^n$  is diffeomorphic to  $(S^k \times S^p) + V^n$ .*

Here  $S^n$  denotes the unit  $n$ -sphere with its usual differential structure in euclidean  $(n+1)$ -space  $R^{n+1}$  and  $+$  denotes the connected sum operation. Now let  $\Theta_n$  denote the group of homotopy  $n$ -spheres and let  $\Phi_p^{k+1}$  denote the subgroup of  $\Theta_p$  consisting of these homotopy  $p$ -spheres that embed in  $R^{p+k+1}$  with a trivial normal bundle. Let  $H_{p,k}$  denote the subset of  $\Theta_p$  consisting of those homotopy  $p$ -spheres  $A^p$  such that  $S^k \times A^p$  is diffeomorphic to  $S^k \times S^p$ . It is known that if  $k \geq 2$ , then  $H_{p,k} = \Phi_p^{k+1}$ . In the next section we prove the following.

**THEOREM 2.** *Let  $A^p$  and  $B^p$  be homotopy  $p$ -spheres such that  $p > 4$  and let  $k$  be an integer greater than one. Then  $S^k \times A^p$  and  $S^k \times B^p$  are diffeomorphic if and only if  $A^p \equiv \pm B^p \pmod{H_{p,k}}$ .*

This theorem follows from Lemmas 4, 5, and 6 of the next section. Theorems 1 and 2 combine to give a classification of differential structures on  $S^k \times S^p$  in the case where either  $k \equiv 2, 4, 5, 6 \pmod{8}$  or  $k \geq p-3$ . In fact, let  $C_{p,k}$  be the set obtained from the quotient group  $\Theta_p/H_{p,k}$  by identifying  $x \in \Theta_p/H_{p,k}$  with its inverse  $-x$ . Then Theorem 1 has the following corollary.

**COROLLARY 1.1.** *The diffeomorphism classes of  $n$ -manifolds ( $n \geq 6$ ) that are homeomorphic to  $S^k \times S^p$  ( $k \leq p$ ) are in a one-to-one correspondence with the product  $C_{p,k} \times \Theta_n$ , provided that either  $k \equiv 2, 4, 5, 6 \pmod{8}$  or  $k \geq p-3$ .*

This one-to-one correspondence is given by Theorem 1. In particular, conclusion (a) of Theorem 1 asserts that the group  $\Theta_{p+k}$  acts nontrivially *via* connected sum on the differential structure of  $M^n$ , provided that either  $k \equiv 2, 4, 5, 6 \pmod{8}$  or  $k \geq p-3$ . This is not always true in the remaining case where  $k < p-3$  and  $k \equiv 0, 1, 3, 7 \pmod{8}$ , although we do show here that  $\Theta_{p+k}$  acts nontrivially on  $S^k \times S^p$  in all cases (Lemma 2 below). In a subsequent paper we shall prove the following result.

THEOREM. *Let*

$$\tau_{p,k}: \theta_p \otimes \pi_k(SO(p-1)) \rightarrow \theta_{p+k}$$

*denote the pairing of Milnor-Munkres-Novikov and let  $A^p$  and  $V^{p+k}$  be homotopy spheres. Then  $(S^k \times A^p) + V^{p+k}$  is diffeomorphic to  $S^k \times A^p$  if and only if there exists  $\alpha \in \pi_k(SO(p-1))$  such that  $V^{p+k} = \tau_{p,k}(A^n \otimes \alpha)$ . In particular, if  $k \geq p-3$ , then  $\tau_{p,k} = 0$ .*

Thus the differential structures on  $S^k \times S^p$  can be classified in terms of  $\Theta_p/H_{p,k}$  and the pairing  $\tau_{p,k}$ . We remark that the pairing  $\tau_{p,k}$  corresponds to composition in the stable homotopy groups of spheres and is sometimes nontrivial. Therefore the complete classification in the case where  $k < p-3$  and  $k \equiv 0, 1, 3, 7 \pmod{8}$  is more complicated than that given by Corollary 1.1.

We can make some remarks that relate to the structure of the groups  $H_{p,k}$ . Let  $bP_{p+1}$  denote the subgroup of  $\Theta_p$  consisting of those homotopy  $p$ -spheres that bound parallelizable manifolds. It is known that  $bP_{p+1} \subset \Phi_p^2$  and hence, since  $\Phi_p^{k+1} = H_{p,k}$  for  $k \geq 2$ , it follows that  $bP_{p+1} \subset H_{p,k} \subset H_{p,k+1}$ , provided that  $k \geq 2$ . Theorem 1 asserts that  $H_{p,k} = \Theta_p$  if  $k \geq p-3$ , although this is not true for  $k < p-3$ . In fact, if  $\Sigma^{16}$  represents the nonzero element in  $\Theta_{16} \approx Z_2$ , then according to [5, Corollary 1.5]  $S^{12} \times \Sigma^{16}$  is not diffeomorphic to  $S^{12} \times S^{16}$ ; that is,  $H_{16,12} = 0$ . Finally, it can be shown that  $H_{p,k}/bP_{p+1}$  is isomorphic to the cokernel of the Hopf-Whitehead homomorphism  $J: \pi_p(SO(k+1)) \rightarrow \pi_{p+k+1}(S^{k+1})$ , provided that  $2k > p-1$  and  $p \neq 2^a - 2$  (c.f. [5, Th. 1.7]). This isomorphism is induced by the Pontrjagin-Thom construction and is of interest here since the groups  $bP_{p+1}$  have been determined in many cases.

We include here a result on the action of  $\Theta_{k+p}$  on the total space  $E$  of a differential  $k$ -sphere bundle over a homotopy  $p$ -sphere. Precisely,  $\pi: E \rightarrow A^p$  is a  $k$ -sphere bundle over  $A^p$  with the special orthogonal group  $SO(k+1)$  as structural group such that the homeomorphisms which specify the local product structure are diffeomorphisms, and where the fibre is  $S^k$ .

PROPOSITION 1. *Each element of  $\Theta_{k+p}$  acts nontrivially on the differential structure of the total space  $E$  of a differential  $k$ -sphere bundle over a homotopy  $p$ -sphere  $A^p$ , provided that  $k < p-1$  and  $k \equiv 2, 4, 5, 6 \pmod{8}$ .*

The proof of Theorem 1 is given in Section 3; following the proof there are some remarks on the case where  $k=1$  or  $n < 6$ . The results on the action of  $\Theta_n$  are proved in the next section. All manifolds are assumed to be smooth of class  $C^\infty$ , and oriented;

diffeomorphisms are assumed to be orientation preserving and of class  $C^\infty$ . Finally,  $D^n$  denotes the unit  $n$ -disc with the standard differential structure in euclidean  $n$ -space  $R^n$ .

## 2. Products of Homotopy Spheres

We begin with the lemmas that are needed in proving the theorems.

**LEMMA 1.** *If  $A^p$  is a homotopy  $p$ -sphere, then  $A^p \times D^k$  is diffeomorphic to  $S^p \times D^k$ , provided that  $p \neq 3$  and  $k \geq p - 2$ .*

**REMARK.** If  $k \geq p + 2 \geq 7$ , then this is a result of Mazur.

*Proof.* If  $p = 1, 2$ , then this is a classical result; if  $p = 4$ , then this is a result of HIRSCH [4, Theorem 6]. Since  $A^p$  and  $S^p$  are diffeomorphic for  $p = 5, 6$  we can assume that  $p \geq 7$ . It follows from the theorems of HAEFLIGER [3] that  $A^p$  may be differentiably embedded in  $R^{p+k}$ , provided that  $k \geq p - 2$ . Furthermore, it follows from [5, Th. 1.10] that the normal tube of this embedding is diffeomorphic to  $A^p \times D^k$ . But it is known that the normal tube of an embedded homotopy  $p$ -sphere in  $R^{p+k}$  is diffeomorphic to  $S^p \times D^k$ , provided that  $k \geq 3$ , and the lemma is proved.

**LEMMA 2.** *Let  $A^p$  and  $B^p$  be homotopy  $p$ -spheres and let  $V^{p+k}$  be a homotopy  $(p+k)$ -sphere, where  $p+k \geq 6$ . If  $k < p - 3$  and  $k \equiv 0, 1, 3, 7 \pmod{8}$ , then assume that  $S^k \times B^p$  is diffeomorphic to  $S^k \times S^p$ . Then, if  $(S^k \times A^p) + V^{p+k}$  is diffeomorphic to  $S^k \times B^p$ , then  $V^{p+k}$  is diffeomorphic to the standard  $(p+k)$ -sphere  $S^{p+k}$  (and hence  $S^k \times A^p$  and  $S^k \times B^p$  are diffeomorphic).*

This lemma is also true for  $p+k=5$  since  $\Theta_5=0$ .

*Proof.* In the first place if  $p=k \geq 3$ , then the lemma follows from [1, Th. B]. If  $k \geq p-3$ , then Lemma 1 above implies that  $S^k \times A^p$  and  $S^k \times B^p$  are both diffeomorphic to  $S^k \times S^p$ , provided that  $p \neq 3$ . Thus if  $k=p-1$ , then Lemma 2 follows from [2, Lem. 1]. Therefore we can assume that  $k < p-1$ , which implies that  $p \geq 4$ . If  $p=4$ , then  $k=2$  and there is nothing to prove since  $\Theta_6=0$ . Therefore assume that  $p > 4$ .

Let

$$h: (S^k \times A^p) + V^{p+k} \rightarrow S^k \times B^p$$

be a diffeomorphism. It is known that there is a diffeomorphism  $f: S^{p-1} \rightarrow S^{p-1}$  such that  $A^p$  is diffeomorphic to  $D_1^p \cup_f D_2^p$ , the disjoint union of two copies  $D_1^p, D_2^p$  of the  $p$ -disc  $D^p$  identified along the boundaries *via* the diffeomorphism  $f$  (that is,  $x \in \partial D_2^p$  is identified with  $f(x) \in \partial D_1^p$  and  $D_1^p \cup_f D_2^p$  is given the orientation of  $D_2^p$ ). Similarly,  $B^p$  is diffeomorphic to  $D_1^p \cup_g D_2^p$ , where  $g: S^{p-1} \rightarrow S^{p-1}$  is a diffeomorphism. Thus we can write  $S^k \times A^p$  as a disjoint union of two copies of  $S^k \times D^p$ , in the form

$$S^k \times A^p = (S^k \times D_1^p) \cup_{i \times f} (S^k \times D_2^p), \quad (1)$$



with points identified along  $S^k \times S^{p-1}$  via the diffeomorphism  $i \times f$ , where  $i: S^k \rightarrow S^k$  is the identity map. Similarly,

$$S^k \times B^p = (S^k \times D_1^p) \cup_{i \times g} (S^k \times D_2^p). \quad (2)$$

Now let  $0 \in D_1^p$  denote the center of the  $p$ -disc  $D_1^p$ . The  $k$ -sphere  $S^k \times 0$  is embedded in  $S^k \times A^p$  and in  $S^k \times B^p$ . We can assume that the connected sum  $(S^k \times A^p) + V^{p+k}$  is made far away from the sphere  $S^k \times 0$  and hence  $S^k \times 0$  is also embedded in  $(S^k \times A^p) + V^{p+k}$ . The next step is to show that we can assume that the diffeomorphism  $h$  is the identity on the  $k$ -sphere  $S^k \times 0$ . In fact, since  $k < p-1$   $h$  maps the homotopy class of  $S^k \times 0$  in  $(S^k \times A^p) + V^{p+k}$  onto either the homotopy class of  $S^k \times 0$  in  $S^k \times B^p$  or the negative of the homotopy class of  $S^k \times 0$  in  $S^k \times B^p$ . In the latter case we can compose  $h$  with the (orientation preserving) diffeomorphism  $\varrho \times i: S^k \times B^p \rightarrow S^k \times (-B^p)$ , where  $\varrho: S^k \rightarrow S^k$  is a diffeomorphism of degree  $-1$  and  $i: B^p \rightarrow -B^p$  is the identity map ( $-B^p$  is the manifold  $B^p$  with the orientation reversed), to obtain a diffeomorphism  $(\varrho \times i) \circ h: (S^k \times A^p) + V^{p+k} \rightarrow S^k \times (-B^p)$  that maps the homotopy class of  $S^k \times 0$  in  $(S^k \times A^p) + V^{p+k}$  into the homotopy class of  $S^k \times 0$  in  $S^k \times (-B^p)$ . Thus we can assume that the restriction  $h|_{S^k \times 0}$  is homotopic to the inclusion  $S^k \times 0 \subset S^k \times B^p$ . It follows from the theorems of HAEFLIGER [3] that  $h|_{S^k \times 0}$  is diffeotopic to the inclusion  $S^k \times 0 \subset S^k \times B^p$ , and hence by application of the diffeotopy extension theorem we can assume that  $h(u, 0) = (u, 0)$  for each  $(u, 0) \in S^k \times 0$ . By the tubular neighborhood theorem we may further suppose that  $h(S^k \times D_1^p) = S^k \times D_1^p$  such that for each  $(u, v) \in S^k \times D_1^p$ ,  $h(u, v) = (u, v \cdot \alpha(u))$ , where  $\alpha: S^k \rightarrow SO(p)$  is a differentiable map and  $v \cdot \alpha(u)$  denotes the action of  $\alpha(u) \in SO(p)$  on  $v \in D_1^p$ . Now perform the spherical modification on  $(S^k \times A^p) + V^{p+k}$  that removes the  $k$ -sphere  $S^k \times 0$  with product structure  $S^k \times D_1^p$ . The following proposition implies that the result of this modification is  $V^{p+k}$ .

**PROPOSITION 2.** *Let  $A^p = D_1^p \cup_f D_2^p$  be a homotopy  $p$ -sphere and let  $\varphi: S^k \times D_1^p \subset S^k \times A^p$  be the inclusion. Then, the result of the spherical modification on  $S^k \times A^p$  based on  $\varphi$  is always  $S^{k+p}$ .*

*Proof.* The result of the modification is

$$(D^{k+1} \times S^{p-1}) \cup_{i \times f} (S^k \times D_2^p), \quad (3)$$

which is clearly diffeomorphic to

$$S^{p+k} = (D^{k+1} \times S^{p-1}) \cup_i (S^k \times D^p) \quad (4)$$

(where  $i: S^k \times S^{p-1} \rightarrow S^k \times S^{p-1}$  is the identity) by virtue of the map that sends  $(u, v) \in S^k \times D^p$  into  $(u, v) \in S^k \times D_2^p$  and  $(u, v) \in D^{k+1} \times S^{p-1}$  into  $(u, f(v)) \in D^{k+1} \times S^{p-1}$  (this diffeomorphism goes from (4) to (3)). Q.E.D.

Returning to the lemma we perform the corresponding modification (under  $h$ )

on  $S^k \times B^p$  to remove the  $k$ -sphere  $S^k \times 0$  with product structure  $h(S^k \times D_1^p)$  in  $S^k \times B^p$ . From the latter modification we obtain the manifold

$$(D^{k+1} \times S^{p-1}) \cup_{\psi} (S^k \times D_2^p), \quad (5)$$

where  $\psi = (h^{-1} | S^k \times S^{p-1}) \circ (i \times g)$  (see (2)), which is clearly diffeomorphic to  $V^{p+k}$  because of the way that this modification was defined (using  $h$ ). We complete the proof of Lemma 2 by showing that (5) is diffeomorphic to  $S^{p+k}$ . This is done by constructing a diffeomorphism from (5) to (4) as follows, recalling that  $k < p-1$ . If  $k < p-3$  and  $k \equiv 2, 4, 5, 6 \pmod{8}$ , then  $\pi_k(SO(p)) = 0$  and hence we can apply Proposition 2 to conclude that (5) is diffeomorphic to  $S^{p+k}$ . If  $k = p-3$  or  $p-2$  then by Lemma 1 we can assume that  $g$  is the identity; if  $k < p-3$  and  $k \equiv 0, 1, 3, 7 \pmod{8}$ , then by hypothesis  $S^k \times B^p$  is diffeomorphic to  $S^k \times S^p$  and we can again assume that  $g$  is the identity. Thus in these cases  $\psi = h^{-1} | S^k \times S^{p-1}$  and we have the diffeomorphism that sends

$$(u, v) \in D^{k+1} \times S^{p-1} \quad \text{into} \quad (u, v) \in D^{k+1} \times S^{p-1}$$

and  $(u, v) \in S^k \times D_2^p$  into

$$h(u, v) = (u, v \cdot \alpha(u)) \in S^k \times D^p.$$

The proof of Proposition 1 is similar to the preceding and is left to the reader.

The following lemma is a weakened form of Lemma 2 but removes the special assumption which was made there in the case where  $k < p-3$  and  $k \equiv 0, 1, 3, 7 \pmod{8}$ .

**LEMMA 3.** *Let  $A^p$  and  $B^p$  be homotopy  $p$ -spheres such that for some integer  $k$ ,  $A^p \times S^k$  and  $B^p \times S^k$  are diffeomorphic up to a point. Then  $A^p \times S^m$  and  $B^p \times S^m$  are diffeomorphic for all  $m \geq \max(k, 2)$ .*

*Proof.* If  $k \geq p-3$ , then the lemma is a trivial consequence of Lemma 1. Thus we can assume that  $k < p-3$ . If  $h: A^p \times S^k \rightarrow B^p \times S^k$  is a diffeomorphism up to a point, then we can compose  $h$  with the inclusion  $B^p \times S^k \subset B^p \times D^{m+1}$  and we obtain a differentiable embedding  $A^p \rightarrow B^p \times D^{m+1}$  with a trivial normal bundle. We show that the embedding  $A^p \rightarrow B^p \times D^{m+1}$  is also a homotopy equivalence, by an elementary argument. In fact, let  $y \in A^p$  and  $z \in B^p$  such that the  $k$ -spheres  $y \times S^k$  and  $z \times S^k$  do not contain the singularity of  $h$ . Now  $k < p-3$  and hence by standard arguments (theorems of HAEFLIGER and diffeotopy extension) we can assume that  $h$  maps  $y \times S^k$  diffeomorphically onto  $z \times S^k$ . It follows that the induced homomorphism

$$h_*: \pi_p(A^p \times S^k) = \pi_p(A^p) + \pi_p(S^k) \rightarrow \pi_p(B^p \times S^k) = \pi_p(B^p) + \pi_p(S^k)$$

maps  $\pi_p(S^k)$  isomorphically onto  $\pi_p(S^k)$  and hence  $h_*$  maps the generator of  $\pi_p(A^p)$  into a generator of  $\pi_p(B^p)$  plus an element of  $\pi_p(S^k)$ . Consequently the composition  $A^p \times S^k \rightarrow B^p \times S^k \subset B^p \times D^{m+1}$  maps the generator of  $\pi_p(A^p)$  into a generator of  $\pi_p(B^p \times D^{m+1})$  and it follows that the embedding  $A^p \rightarrow B^p \times D^{m+1}$  is a homotopy

equivalence with a trivial normal bundle. We can apply [7, Th. 4.1] to conclude that  $A^p \times D^{m+1}$  is diffeomorphic to  $B^p \times D^{m+1}$ , provided that  $m \geq \max(k, 2)$ .

The remainder of the present section is devoted to the study of the diffeomorphism classes of manifolds of the form  $S^k \times A^p$ , where  $A^p$  is an arbitrary homotopy  $p$ -sphere. Let  $p \geq 4$  and  $k \geq 2$  be a given pair of integers. Then two homotopy  $p$ -spheres  $A^p$  and  $B^p$  are called *k-equivalent* if and only if  $S^k \times A^p$  and  $S^k \times B^p$  are diffeomorphic. Thus the group  $\Theta_p$  is divided into  $k$ -equivalence classes. It is clear that the  $k$ -equivalence class of an element  $A^p \in \Theta_p$  contains its inverse  $-A^p$  in the group  $\Theta_p$ . Lemma 4 below asserts that the  $k$ -equivalence class of  $S^p$  is a subgroup of  $\Theta_p$ . This subgroup is denoted by  $H_{p,k}$ .

LEMMA 4. *The set  $H_{p,k}$  of those homotopy  $p$ -spheres  $A^p$  such that  $S^k \times A^p$  is diffeomorphic to  $S^k \times S^p$  forms a subgroup of  $\Theta_p$ , provided that  $p \neq 3$  and  $k \geq 2$ .*

This lemma follows from the next lemma. Lemma 5 implies that any  $k$ -equivalence class is the union of cosets of the subgroup  $H_{p,k}$  of the group  $\Theta_p$ .

LEMMA 5. *Let  $A^p$  and  $B^p$  be homotopy  $p$ -spheres such that  $A^p \in H_{p,k}$ . Then  $S^k \times (A^p + B^p)$  is diffeomorphic to  $S^k \times B^p$ .*

*Proof.* Since  $A^p \in H_{p,k}$  it follows that  $A^p$  may be embedded in the interior of a  $(p+k+1)$ -disc in  $D^{k+1} \times B^p$  with a trivial normal bundle. But  $B^p$  is embedded in  $D^{k+1} \times B^p$  in the obvious way with a trivial normal bundle, and hence we can form the connected sum  $A^p + B^p$  in  $D^{k+1} \times B^p$  so that  $A^p + B^p$  has a trivial normal bundle. Furthermore, the resulting embedding  $A^p + B^p \rightarrow D^{k+1} \times B^p$  is a homotopy equivalence and hence by [7, Th. 4.1]  $D^{k+1} \times (A^p + B^p)$  is diffeomorphic to  $D^{k+1} \times B^p$ , provided that  $k \geq 2$ . Q.E.D.

In general it does not seem likely that each  $k$ -equivalence class contains exactly one coset of  $H_{p,k}$ . If this is the case, then the  $k$ -equivalence classes are in a one-to-one correspondence with the quotient group  $\Theta_p/H_{p,k}$ ; in particular, this would imply that  $\Theta_p/H_{p,k}$  consists entirely of elements of order two. The next lemma is the best that we can do in this direction.

LEMMA 6. *If  $A^p$  and  $B^p$  are homotopy  $p$ -spheres such that  $S^k \times A^p$  and  $S^k \times B^p$  are diffeomorphic, then either  $S^k \times (A^p + B^p)$  or  $S^k \times (A^p + (-B^p))$  is diffeomorphic to  $S^k \times S^p$ , provided that  $k \geq 2$ .*

*Proof.* If  $k \geq p-3$ , then the lemma is a consequence of Lemma 1. Thus we can assume that  $k < p-3$ . The hypothesis implies that  $A^p$  may be embedded in  $D^{k+1} \times B^p$  with a trivial normal bundle. Furthermore, it follows from an argument given in the proof of Lemma 3 that the embedding  $A^p \rightarrow D^{k+1} \times B^p$  is a homotopy equivalence. Let us assume that the embedding maps the orientation class of  $A^p$  onto the orientation class of  $B^p$  in  $D^{k+1} \times B^p$  (otherwise we replace  $B^p$  by  $-B^p$ ). Now  $-B^p$  is embedded in  $D^{k+1} \times B^p$  in the obvious way with a trivial normal bundle and we can

assume that  $A^p$  and  $-B^p$  are disjoint in  $D^{k+1} \times B^p$ . Thus we can form the connected sum  $A^p + (-B^p)$  in  $D^{k+1} \times B^p$  such that the resulting embedding  $A^p + (-B^p) \rightarrow D^{k+1} \times B^p$  has a trivial normal bundle and is homotopically trivial. Now the engulfing result of [9, Chapter 7] applies to conclude that  $A^p + (-B^p)$  is embedded in the interior of a  $(p+k+1)$ -disc in  $D^{k+1} \times B^p$ . But the normal tube of a homotopy  $p$ -sphere embedded in the interior of a  $(p+k+1)$ -disc is diffeomorphic to  $D^{k+1} \times S^p$ , provided that  $k \geq 2$  and  $p \neq 3, 4$ , and hence it follows that  $D^{k+1} \times (A^p + (-B^p))$  is diffeomorphic to  $D^{k+1} \times S^p$  (if  $p=3$  or  $4$ , then apply [4, Th. 6]).

Lemma 6 implies that the  $k$ -equivalence class of an element  $A^p \in \Theta_p$  is exactly the union of the cosets  $A^p + H_{p,k}$  and  $-A^p + H_{p,k}$ . That is, each  $k$ -equivalence class consists of at most two cosets. Furthermore, a  $k$ -equivalence class consists of exactly one coset if and only if it contains an element  $A^p$  such that  $A^p + H_{p,k}$  is of order two in the group  $\Theta_p/H_{p,k}$ . This completes the proof of Theorem 2.

### 3. Classification

*Proof of Theorem 1.* Since  $M^n$  is homeomorphic to  $S^k \times S^p$ , where  $n \geq 6$  and  $p \geq k \geq 2$ , it follows that  $M^n$  is simply connected and  $H_3(M^n; \mathbb{Z})$  has no 2-torsion. Therefore the ‘‘Hauptvermutung’’ of D. SULLIVAN [8] implies that there is a combinatorial equivalence

$$h: M^n \rightarrow S^k \times S^p, \quad (5)$$

where the combinatorial structures are compatible with the differential structures and  $S^k \times S^p$  has the usual combinatorial structure. We now apply the obstruction theory of MUNKRES [6]. The application is particularly simple since we are dealing with a product of spheres. We note that the combinatorial equivalence  $h$  is a diffeomorphism mod the  $n-1$  skeleton  $L_{n-1}$  of  $M^n$ ; suppose that  $h$  is a diffeomorphism mod the  $n-q$  skeleton  $L_{n-q}$  of  $M^n$ , where  $1 \leq q < k-1$ . The obstruction to an approximation  $g: M^n \rightarrow S^p \times S^k$  of  $h$  such that  $g$  is a diffeomorphism mod the  $n-q-1$  skeleton  $L_{n-q-1}$ , is a simplicial  $(n-q)$ -cycle  $\lambda_{n-q}h$  of  $L_{n-q}$  with coefficients in the group  $\Gamma^q$  (see [6, § 3];  $g$  is called a smoothing of  $h$ ), where  $\Gamma^m$  is the group of diffeomorphisms of  $S^{m-1}$  modulo those diffeomorphisms that are extendable to diffeomorphisms of  $D^m$ . If  $\lambda_{n-q}h=0$ , then the smoothing  $g$  exists according to [6, § 4]. Since  $\Gamma^1=0$  it follows that the smoothing  $g$  does exist if  $q=1$ . Furthermore, if  $\lambda_{n-q-1}g$  is homologous to zero in  $H_{n-q-1}(L_{n-q}; \Gamma^{q+1})$ , then it follows from [6, § 5] that there is a smoothing  $f$  of  $h$  such that  $\lambda_{n-q-1}f=0$ . But  $H_{n-q-1}(L_{n-q}; \Gamma^{q+1}) \approx H_{n-q-1}(M^n; \Gamma^{q+1})=0$  for  $1 \leq q < k-1$  and hence by induction there exists a map  $g: M^n \rightarrow S^k \times S^p$  that is a diffeomorphism mod the  $n-k$  skeleton of  $M^n$ . Thus the first obstruction to deforming  $h$  into a diffeomorphism is a well defined homology class  $\gamma h$  (called the obstruction class) in  $H_p(M^n; \Gamma^k)$ . We first consider the case where  $k < p$ ; then  $H_p(M^n; \Gamma^k) \approx \Gamma^k$  and hence we can consider the obstruction class  $\gamma h$  to be an element of  $\Gamma^k$ . Let

$\varphi: S^{k-1} \rightarrow S^{k-1}$  be a diffeomorphism that represents  $\gamma h$  and let  $N(\gamma h)$  denote the homotopy  $k$ -sphere  $D_1^k \cup_{\varphi} D_2^k$ . There is the combinatorial equivalence  $j: S^k \rightarrow N(\gamma h)$  of degree  $+1$ , defined by writing  $S^k = D_1^k \cup_i D_2^k$  and letting  $j$  be the identity on  $D_1^k$  and the radial extension of  $\varphi^{-1}$  on  $D_2^k$ , and hence we have a combinatorial equivalence  $j \times i: S^k \times S^p \rightarrow N(\gamma h) \times S^p$ , where  $i$  is the identity. It follows from [6, Def. 3.4] that the first obstruction  $\gamma(j \times i)$  to deforming  $j \times i$  to a diffeomorphism is  $-\gamma h$ . Furthermore, by [6, 3.8] the first obstruction to deforming the composition  $(j \times i) \circ h: M^n \rightarrow N(\gamma h) \times S^p$  into a diffeomorphism is

$$\begin{aligned} \gamma((j \times i) \circ h) &= \gamma(j \times i) + \gamma h \\ &= -\gamma h + \gamma h \\ &= 0 \end{aligned}$$

and hence we can assume that there is a map

$$h': M^n \rightarrow N(\gamma h) \times S^p,$$

that is a diffeomorphism mod the  $k$ -skeleton of  $M^n$ . By Lemma 1  $N(\gamma h) \times S^p$  is diffeomorphic to  $S^k \times S^p$  since  $p > k$  (if  $k=3$ , then  $N(\gamma h)$  is diffeomorphic to  $S^3$  since  $\Gamma^3=0$ , as is well known) and hence we have a map (also denoted by  $h'$ )  $h': M^n \rightarrow S^k \times S^p$  that is a diffeomorphism mod the  $k$ -skeleton of  $M^n$ . The first obstruction to deforming  $h'$  into a diffeomorphism is a class  $\gamma h' \in H_k(M^n; \Gamma^p) \approx \Gamma^p$ . Let  $\psi: S^{p-1} \rightarrow S^{p-1}$  be a diffeomorphism that represents  $\gamma h'$ , let  $N(\gamma h') = D_1^p \cup_{\psi} D_2^p$ , and let  $j': S^p \rightarrow N(\gamma h')$  be the combinatorial equivalence of degree  $+1$  as defined above for  $N(\gamma h')$ . Then we have the combinatorial equivalence  $i \times j': S^k \times S^p \rightarrow S^k \times N(\gamma h')$  and the first obstruction to smoothing  $(i \times j') \circ h': M^n \rightarrow S^k \times N(\gamma h')$  is  $\gamma(i \times j') + \gamma h'$ , which is zero since  $\gamma(i \times j') = -\gamma h'$ . It follows that there is a map  $h'': M^n \rightarrow S^k \times N(\gamma, h')$  that is a diffeomorphism mod a point of  $M^n$ . Under these circumstances it is known that there is a homotopy  $n$ -sphere  $V^n$  such that  $M^n$  is diffeomorphic to  $(S^k \times N(\gamma, h')) + V^n$ .

Now suppose that  $k=p$ . The first obstruction to deforming the combinatorial equivalence (5) into a diffeomorphism is a class  $\gamma h \in H_k(M^n; \Gamma^k) \approx \Gamma^k \oplus \Gamma^k$ ; write  $\gamma h = \gamma^1 \oplus \gamma^2$ , where  $\gamma^1, \gamma^2 \in \Gamma^k$ , and let  $\varphi_1, \varphi_2: S^{k-1} \rightarrow S^{k-1}$  be diffeomorphisms that represent  $\gamma^1, \gamma^2$ . As before we have the homotopy spheres  $N(\gamma^1) = D_1^k \cup_{\varphi_1} D_2^k$ ,  $N(\gamma^2) = D_1^k \cup_{\varphi_2} D_2^k$  and the combinatorial equivalences  $j_1: S^k \rightarrow N(\gamma^1)$ ,  $j_2: S^k \rightarrow N(\gamma^2)$ . The first obstruction to smoothing  $j_1 \times j_2: S^k \times S^k \rightarrow N(\gamma^1) \times N(\gamma^2)$  is the class  $\gamma(j_1 \times j_2) = (-\gamma^1) \oplus (-\gamma^2)$  and hence the first obstruction to deforming the composition

$$(j_1 \times j_2) \circ h: M^n \rightarrow N(\gamma^1) \times N(\gamma^2)$$

is

$$\gamma(j_1 \times j_2) \circ h = \gamma(j_1 \times j_2) + \gamma h = 0.$$

It follows that there is a map  $h': M^n \rightarrow N(\gamma^1) \times N(\gamma^2)$  that is a diffeomorphism mod a point. Now a result of WALL applies to conclude that  $N(\gamma^1) \times N(\gamma^2)$  and  $S^k \times S^k$  are

diffeomorphic up to a point (in fact, see [1, Th. B]) and hence there is a homotopy  $2k$ -sphere  $V^n$  such that  $M^n$  is diffeomorphic to  $(S^k \times S^k) + V^n$ .

The proof of Theorem 1 is now completed by applying Lemmas 1 and 2.

In this theorem it is assumed that  $2 \leq k \leq p$  and  $n = p + k \geq 6$ . We conclude with some remarks on the excluded cases.

CASE 1.  $k = 1, n = p + k \geq 6$ . The Hauptvermutung of [8] does not apply in this case and so consider a differential  $n$ -manifold  $M^n$  that is combinatorially equivalent to  $S^1 \times S^p$ . We can apply the obstruction theory of [6] as was done in the proof of Theorem 1 to conclude that  $M^n$  is diffeomorphic to  $(S^1 \times A^p) + V^n$ , where  $A^p$  and  $V^n$  are homotopy spheres that are combinatorially equivalent to the standard spheres. Then by application of Lemma 2 we have the following theorem: *If  $M^n$  is a differential  $n$ -manifold that is combinatorially equivalent to  $S^1 \times S^p$ , where  $n = p + 1 \geq 6$ , then there are homotopy spheres  $A^p$  and  $V^n$  such that  $M^n$  is diffeomorphic to  $(S^1 \times A^p) + V^n$ . If  $S^1 \times A^p$  is diffeomorphic to  $S^1 \times S^p$ , then  $V^n$  is uniquely determined by  $M^n$ .* On the other hand, if we assume only that there is a homeomorphism  $h$  between  $M^n$  and  $S^1 \times S^p$ , then according to [8] there is an integer  $q$  such that the homeomorphism  $h \times \text{identity}$  between  $M^n \times R^q$  and  $S^1 \times S^p \times R^q$  is homotopic to a combinatorial equivalence ( $R^q$  is euclidean  $q$ -space). One can try to smooth the combinatorial equivalence between  $M^n \times R^q$  and  $S^1 \times S^p \times R^q$  by applying [6].

CASE 2.  $p + k \leq 6$ . Since  $\Gamma^q = 0$  for  $q \leq 6$ , it follows from MUNKRES' obstruction theory that combinatorial equivalences can be smoothed to diffeomorphisms for manifolds of dimension  $\leq 6$ . Thus if  $M^n$  is combinatorially equivalent to  $S^k \times S^p$ , where  $p + k \leq 6$ , then  $M^n$  is diffeomorphic to  $S^k \times S^p$ .

#### REFERENCES

- [1] R. DE SAPIO, *Embedding  $\pi$ -manifolds*, Ann. of Math. 82 (1965), 213–224.
- [2] —, *Actions of  $\theta_{2k+1}$* , Michigan Math. J. 14 (1967), 97–100.
- [3] A. HAEFLIGER, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. 36 (1961), 47–82.
- [4] M. W. HIRSCH, *On homotopy spheres of low dimension*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), (Princeton University Press, Princeton 1965), pp. 199–204.
- [5] W. C. HSIANG, J. LEVINE, and R. H. SZARBA, *On the normal bundle of a homotopy sphere embedded in euclidean space*, Topology 3 (1965), 173–181.
- [6] J. MUNKRES, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. 72 (1960), 521–554.
- [7] S. SMALE, *On the structure of manifolds*, Amer. Math. J. 84 (1962), 387–399.
- [8] D. SULLIVAN, *On the Hauptvermutung for manifolds*, Bull. Amer. Math. Soc. 73 (1967), 598–600.
- [9] E. C. ZEEMAN, *Seminar on combinatorial topology* (mimeographed notes), Inst. Hautes Études Sci. Publ. Math. (1963, revised 1965).

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