Homological Methods in Group Varieties.

Homological Methods in Group Varieties

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1. Introduction

In papers by Stallings [9] and the author [10], [11] the homology theory of groups has been applied to central series, and meaningful group theoretical results have been obtained. In the present paper we show how similar homological methods can be used to obtain interesting results in arbitrary varieties of groups. It is not surprising that the methods are most successful if applied to problems involving central series.

It is the nature of a paper introducing new methods to repeat many well known results. However the approach presented here leads to an interestingly unifying point of view. Also, it is possible to simplify the proofs of many well known results.

Given a variety $\mathfrak{V}$, we first define a functor S_0- from $\mathfrak{V}$ to abelian groups. This functor is defined in terms of the integral second homology group functor $H_2(-, \mathbf{Z})$. We then prove that a surjective group homomorphism $G \twoheadrightarrow Q$ in $\mathfrak{V}$, with kernel N, gives rise to an exact sequence

$$(*) \quad S_0G \to S_0Q \to N/[G, N] \to G_{ab} \to Q_{ab} \to 0.$$

Here $[G, N]$ denotes the (normal) subgroup generated by all elements of the form $gng^{-1}n^{-1}$ with $g \in G$ and $n \in N$. G_{ab}, Q_{ab} denote the abelianized groups.

We apply this sequence to obtain Theorem 4.1, which is a generalization of a result due to Stallings [9]:

Let $\varphi: G \to H$ be a homomorphism of groups in the variety $\mathfrak{V}$. Suppose that φ induces an isomorphism $G_{ab} \cong H_{ab}$ and an epimorphism $S_0G \twoheadrightarrow S_0H$. Then, for every $n \geqslant 0$, φ induces an isomorphism $\varphi_n: G/G_n \cong H/H_n$; where $\{G_n\}$, $\{H_n\}$ denote the lower central series.

As applications of this theorem we obtain the following results:

a) Corollary 4.1.3 gives a homological characterization of the finitely generated free groups in nilpotent varieties of exponent 0 (i.e. in nilpotent varieties containing all abelian groups).

b) In Chapter 5 we reprove practically all of the known results which give sufficient conditions for a subgroup of a $\mathfrak{V}$-free group to be $\mathfrak{V}$-free. Our approach results in a substantial simplification of the proofs.

c) In Chapter 6 P. Hall's results on splitting groups in nilpotent varieties are shown to be immediate corollaries of our results in Chapter 4.

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d) Chapter 7 is devoted to the notion of deficiency and efficiency in varieties of exponent 0, in particular in the variety of all groups. The *deficiency*, $\operatorname{def} G$ of a finitely presentable group G in the variety $\mathfrak{V}$ is defined as the maximum of (number of generators − number of relators) for the finite presentations of G in $\mathfrak{V}$. If sM, for an abelian group M, denotes the minimum number of generators of M, then our result is

$$(^{**})\quad \operatorname{def} G \leqslant \operatorname{rank} G_{ab} - s(S_0 G).$$

A group in $\mathfrak{V}$ is called *(V-)efficient* if (**) is an equality. We prove (Theorem 7.3): *To a group G in $\mathfrak{V}$ there is an efficient group H in $\mathfrak{V}$ and a surjective homomorphism $\varphi: H \to G$ which induces isomorphisms $\varphi_n: H/H_n \cong G/G_n$ for all $n \geqslant 0$. $\{H_n\}$, $\{G_n\}$ denote the lower central series.*

As Corollary 7.3.1 we obtain the following result, giving a partial answer to a question of Knopfmacher [3]: *In a nilpotent variety (of exponent 0) every group is efficient.* As Corollary 7.3.2 we reprove part of Chen's result in [1].

e) In Chapter 8 we generalize a result by Magnus [5] to arbitrary varieties of exponent 0 (Theorem 8.1):

To a group G in $\mathfrak{V}$ with $n+r$ generators and r relators, whose abelianized group G_{ab} can be generated by n elements, there exists a $\mathfrak{V}$-free group F with $F/F_k \cong G/G_k$ for all $k \geqslant 0$.

We conclude this introduction with the following remark: In the formulation of many of the results of this paper, the functor $S_p -$ is not needed. This functor and hence ultimately the homology theory of groups appears here as a tool yielding purely group theoretical results.

2. Notation

By $\mathbf{Z}$ or $\mathbf{Z}_0$ we denote the additive group of the integers, by $\mathbf{Z}_p$ (p a prime) the additive group of the integers mod$p$.

In what follows let p be either a prime or $p=0$.

If U is a subgroup of the group G, we use the symbol $G \#_p U$ to denote the subgroup of G generated by all elements of the form $gug^{-1}u^{-1}v^p$ for $g \in G$ and $u, v \in U$. For convenience we shall sometimes write $[G, U]$ instead of $G \#_0 U$. To every group G and every p we define recursively a series of (normal) subgroups of G by

$$G_0^{(p)} = G; \qquad G_n^{(p)} = G \#_p G_{n-1}^{(p)}.$$

For $p=0$ this is the lower central series; for p a prime we obtain the most rapidly descending central series whose successive quotients $G_n^{(p)}/G_{(n+1)}^{(p)}$ are vector spaces over the field of p elements.

We shall use $G_\omega^{(p)}$ to denote the intersection of the $G_n^{(p)}$ for all $n \geqslant 0$. It is trivial that

G is residually nilpotent if and only if $G_\omega^{(0)}=\{e\}$. Also it is easy to see that $G_\omega^{(p)}=\{e\}$ if and only if G is residually a finite p-group.

For convenience we shall write A_pG for $G/G\#_pG$, such that $A_0G=G_{ab}$. Note that A_p- defines a functor from the category (variety) of groups to the category (variety) of abelian groups.

For any group G, denote by $H_k(G, \mathbf{Z}_p)$ the k-th homology group of G with coefficient group $\mathbf{Z}_p$. See [4] for a definition.

Hanna Neumann's book [6] will serve as a "universal" reference for group varieties.

Throughout the paper we shall use the notion of a *presentation*: Is G a group in the variety $\mathfrak{V}$, then G may be given as a quotient of a $\mathfrak{V}$-free group F, i.e., $G\cong F/R$ for some normal subgroup R. A set of elements x_α in F, $\mathfrak{V}$-freely generating F, together with a set of elements y_β in R, generating R as a normal subgroup in F, is a $\mathfrak{V}$-presentation of G. The x_α's are called *generators*, the y_β's *relators*. If there exists a presentation of G such that the set of the x_α's as well as the set of the y_β's are finite, then G is *finitely presentable* in $\mathfrak{V}$ and the corresponding presentation is called *finite*.

3. An Exact Sequence

Let $\mathfrak{V}$ be a fixed variety. For any p (p prime or $p=0$) we define the functor S_p- from the variety $\mathfrak{V}$ into the category $\mathfrak{Ab}$ of abelian groups.

DEFINITION: For a group G in $\mathfrak{V}$ choose a $\mathfrak{V}$-free presentation, i.e. a surjective homomorphism $\pi:F\to G$ with F $\mathfrak{V}$-free. Then define $S_pG=\operatorname{coker}(\pi_*:H_2(F, \mathbf{Z}_p)\to$ $\to H_2(G, \mathbf{Z}_p))$. The effect of S_p- on homomorphisms is obvious.

Of course we have to show that S_pG does not depend on the choice of the presentation $\pi:F\to G$. Let $\pi':F'\to G$ be another presentation. Then there exist f, f' such that the triangle

$$\begin{tikzcd} F \arrow[r, shift left, "f"] \arrow[dr, "\pi"'] & F' \arrow[l, shift left, "f'"] \arrow[d, "\pi'"] \\ & G \end{tikzcd}$$

commutes. It follows that

$$\begin{tikzcd} H_2(F, \mathbf{Z}_p) \arrow[r, shift left, "f_*"] \arrow[dr, "\pi_*"'] & H_2(F', \mathbf{Z}_p) \arrow[l, shift left, "f_*'"] \arrow[d, "\pi_*'"] \\ & H_2(G, \mathbf{Z}_p) \end{tikzcd}$$

is commutative, whence $\operatorname{coker}\pi_*=\operatorname{coker}\pi'_*$.

We remark that for the variety of all groups $S_pG=H_2(G, \mathbf{Z}_p)$. Also, we note the trivial result:

PROPOSITION 3.1. *For any* $\mathfrak{V}$*-free group* F, $S_pF=0$.

For the reader familiar with categorical homology theory we remark that S_pG is the first homology group of G with coefficients in the functor A_p- relative to the $\mathfrak{V}$-free group cotriple. See [7].

THEOREM 3.2. *Let* $\varphi:G\to Q$ *be a surjective homomorphism in* $\mathfrak{V}$, *with kernel* N. *Then there is an exact sequence*

$$(*)\quad S_pG \xrightarrow{\varphi_*} S_pQ \to N/G\#_pN \to A_pG \xrightarrow{\varphi_*} A_pQ \to 0.$$

Proof: We recall the five term exact sequence for homology in low dimensions

$$H_2(G, \mathbf{Z}_p) \to H_2(Q, \mathbf{Z}_p) \to N_{ab}\otimes_Q \mathbf{Z}_p \to H_1(G, \mathbf{Z}_p) \to H_1(Q, \mathbf{Z}_p) \to 0.$$

See [9], [10]; or [13] for a simple and elementary proof. It is well known that $H_1(G, \mathbf{Z}_p)\cong A_pG$, and it is easy to see that $N_{ab}\otimes_Q\mathbf{Z}_p\cong N/G\#_pN$. Choose a presentation $\pi:F\to G$ of G and take the induced presentation $\varphi\pi:F\to Q$ for Q. We obtain the diagram

$$\begin{array}{ccc}
H_2(F, \mathbf{Z}_p) & \xrightarrow{\simeq} & H_2(F, \mathbf{Z}_p)\\
{\scriptstyle\pi_*}\downarrow & & \downarrow{\scriptstyle(\varphi\pi)_*}\\
H_2(G, \mathbf{Z}_p) & \xrightarrow{\varphi_*} & H_2(Q, \mathbf{Z}_p)\\
\downarrow & & \downarrow\\
S_pG & \xrightarrow{\varphi_*} & S_pQ\\
\downarrow & & \downarrow\\
0 & & 0
\end{array}$$

which immediately shows that sequence (*) is eaxct.

COROLLARY 3.2.1. *Let* $\pi:F\to G$ *be a* $\mathfrak{V}$*-free presentation, with kernel* R. *Then*

$$S_pG \cong F\#_pF\cap R/F\#_pR.$$

Proof: Consider the surjective homomorphism $\pi:F\to G$ with kernel R. Since $S_pF=0$, sequence (*) yields the result.

By Corollary 3.2.1 the group given by the formula $F\#_pF\cap R/F\#_pR$ does not depend upon the chosen presentation of G. This generalizes a result of Hopf–Baer (see [8], p. 181).

The following Lemma, which is a sort of Universal Coefficient Theorem for S_p-, will be useful.

LEMMA 3.3: *Let* $\mathfrak{V}$ *be a variety of exponent* 0. *Then*

$$S_pG \cong S_pG\otimes\mathbf{Z}_p \oplus \mathrm{Tor}_1(A_0G, \mathbf{Z}_p).$$

Proof: Let $G=F/R$, with F $\mathfrak{V}$-free. Consider the exact sequence

$$0 \to S_0G \to R/[F, R] \xrightarrow{\kappa} A_0F \to A_0G \to 0.$$

The image I of κ is as a subgroup of the free abelian group A_0F free abelian. Therefore $R/[F, R] \cong S_0G \oplus I$. On the other hand S_pG is defined by the exact sequence

$$0 \to S_pG \to R/[F, R] \otimes \mathbf{Z}_p \to A_0F \otimes \mathbf{Z}_p \to A_0G \otimes \mathbf{Z}_p \to 0.$$

Using the above decomposition of $R/[F, R]$ we obtain the claimed formula for S_pG.

4. S_pG and Central Series

The following Theorem contains the basic result of Stallings [9] for $\mathfrak{V}$ the variety of all groups.

THEOREM 4.1: *Let $\varphi: G \to H$ be a homomorphism of groups in $\mathfrak{V}$. Suppose that φ induces an isomorphism $A_pG \cong A_pH$ and an epimorphism $S_pG \to S_pH$. Then, for every $n \geqslant 0$, φ induces an isomorphism $\varphi_n: G/G_n^{(p)} \cong H/H_n^{(p)}$ and a monomorphism $\varphi_\omega: G/G_\omega^{(p)} \to H/H_\omega^{(p)}$.*

Proof: Our proof is essentially that of Stallings [9]. We proceed by induction: For $n=0$ the conclusion is trivial. Let $n \geqslant 1$. Consider the exact sequences

$$\begin{array}{ccccccccc}
S_pG & \to & S_pG/G_n^{(p)} & \to & G_{n-1}^{(p)}/G_n^{(p)} & \to & A_pG & \to & A_pG/G_n^{(p)} \to 0 \\
\downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_2} & & \downarrow{\scriptstyle\alpha_3} & & \downarrow{\scriptstyle\alpha_4} & & \downarrow{\scriptstyle\alpha_5} \\
S_pH & \to & S_pH/H_n^{(p)} & \to & H_{n-1}^{(p)}/H_n^{(p)} & \to & A_pH & \to & A_pH/H_n^{(p)} \to 0
\end{array}$$

and the map induced by φ. α_2, α_4, α_5 are isomorphisms, α_1 is an epimorphism. Thus by the 5-lemma α_3 is an isomorphism. The conclusion then follows by applying the 5-lemma to the diagram below:

$$\begin{array}{ccccccccc}
\{e\} & \to & G_{n-1}^{(p)}/G_n^{(p)} & \to & G/G_n^{(p)} & \to & G/G_{n-1}^{(p)} & \to & \{e\} \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
\{e\} & \to & H_{n-1}^{(p)}/H_n^{(p)} & \to & H/H_n^{(p)} & \to & H/H_{n-1}^{(p)} & \to & \{e\}.
\end{array}$$

α is an isomorphism by the above, γ is an isomorphism by induction.

The assertion about φ_ω follows trivially.

Remark: As Stallings [9] has shown for the variety of all groups, the statement about φ_ω cannot be sharpened.

COROLLARY 4.1.1: *Let $\mathfrak{V}$ be a nilpotent variety. Suppose that $\varphi: G \to H$ induces an isomorphism between the abelianized groups $A_0G \cong A_0H$ and an epimorphism $S_0G \to S_0H$. Then φ is an isomorphism.*

Proof: By the definition of nilpotency there is an $n \geqslant 0$ such that $G_n = \{e\}$ and $H_n = \{e\}$.

COROLLARY 4.1.2: *Let G be a group in $\mathfrak{V}$ with $S_p G = 0$. In the case $p = 0$ suppose that $A_0 G$ is free in $\mathfrak{V} \cap \mathfrak{Ab}$. Then there is a $\mathfrak{V}$-free group F and a homomorphism $\varphi: F \to G$ such that, for every $n \geqslant 0$, φ induces an isomorphism*

$$\varphi_n: F/F_n^{(p)} \xrightarrow{\sim} G/G_n^{(p)}.$$

Proof: Take a set $\{x_\alpha\}$ of elements in G whose images in $A_p G$ form a basis. Consider the $\mathfrak{V}$-free group F on the set $\{y_\alpha\}$ and the map $\varphi: F \to G$ defined by $\varphi y_\alpha = x_\alpha$ and apply Theorem 4.1.

COROLLARY 4.1.3: Let $\mathfrak{V}$ *be a nilpotent variety of exponent* 0. *If G is a finitely generated group in $\mathfrak{V}$ with $S_p G = 0$ for all primes p, then G is $\mathfrak{V}$-free.*

Proof: By Lemma 3.3 $S_p G = 0$ for all primes p implies $S_0 G = 0$ and $A_0 G$ is torsion free. Since $A_0 G$ is a quotient of G, it is finitely generated. Hence $A_0 G$ is free abelian, and the assertion follows from Corollary 4.1.1.

5. Subgroup Theorems

The Schreier Theorem says that every subgroup of an absolutely free group is free. It is well known that for an arbitrary variety $\mathfrak{V}$, the subgroups of $\mathfrak{V}$-free groups generally fail to be $\mathfrak{V}$-free. In those varieties in which the *free groups are residually nilpotent or residually finite p-groups* there exist a number of results giving sufficient conditions under which a given set of elements in a $\mathfrak{V}$-free group generates a $\mathfrak{V}$-free subgroup. By [6], page 76 examples of such varieties are a) the variety of all polynilpotent groups to a given classrow, and consequently b) the variety of all nilpotent groups of class $\leqslant k$, and c) the variety of all solvable groups of length $\leqslant l$.

THEOREM 5.1: (a) (Hall, Mostowski; see [6], p. 115) *Let $\mathfrak{V}$ be a variety in which the free groups are residually nilpotent. Let F be a $\mathfrak{V}$-free group and $\{x_\alpha\}$ a set of elements in F whose images in $A_0 F$ freely generate a direct summand. Then $\{x_\alpha\}$ freely generates a $\mathfrak{V}$-free subgroup of F.*

(b) *Let $\mathfrak{V}$ be a variety whose free groups are residually finite p-groups for a certain prime p. Let F be a $\mathfrak{V}$-free group and $\{x_\alpha\}$ a set of elements in F whose images in $A_p F$ are linearly independent. Then $\{x_\alpha\}$ freely generates a V-free subgroup of F.*

Proof: In both cases we can enlarge the set $\{x_\alpha\}$ until the images form a basis of $A_p F$ (p the given prime in (b) or $p = 0$ in (a)). We shall prove that this larger set (also denoted by $\{x_\alpha\}$) freely generates a free subgroup of F.

To do so, take F' to be the $\mathfrak{V}$-free group on the set $\{y_\alpha\}$ and define $\varphi: F' \to F$ by setting $\varphi y_\alpha = x_\alpha$. This map induces an isomorphism $A_pF' \cong A_pF$ and an epimorphism $S_pF' \to S_pF = 0$. Thus by Theorem 4.1., φ induces a monomorphism $\varphi_\omega : F'/F'^{(p)}_\omega \to \to F/F^{(p)}_\omega$. Since $F'^{(p)}_\omega = \{e\}$, $F^{(p)}_\omega = \{e\}$, we obtain that $\varphi = \varphi_\omega : F' \to F$ is a monomorphism.

COROLLARY 5.1.1: (P.Neumann, see [6], p. 117) *Let* $\mathfrak{V}$ *be a variety in which the free groups are residually finite p-groups. Let F be a* $\mathfrak{V}$*-free group and* $\{x_\alpha\}$ *a set of elements in F whose images in* A_0F *are independent. Suppose that* $F/[F, F]\ \{x_\alpha\}$ *does not contain p-torsion. Then* $\{x_\alpha\}$ *freely generate a* $\mathfrak{V}$*-free subgroup of F.*

Proof: Let $W(x_\alpha)$ denote the subgroup generated by the images of x_α in $A_0F = F/[F, F]$; then we have an exact sequence of abelian groups

$$0 \to W(x_\alpha) \to A_0F \to F/[F, F]\ \{x_\alpha\} \to 0\,.$$

Tensoring this sequence with $\mathbf{Z}_p$ and using

$$\mathrm{Tor}_1(F/[F, F]\ \{x_\alpha\}, \mathbf{Z}_p) = 0$$

we get the exact sequence

$$0 \to W\{x_\alpha\} \otimes \mathbf{Z}_p \to A_pF \to F/[F, F]\ \{x_\alpha\} \otimes \mathbf{Z}_p \to 0\,.$$

Since F is residually a finite p-group, A_0F is either free abelian or a p-group. Since the images of the x_α's are linearly independent in A_0F, they must form a basis in $W(x_\alpha) \otimes \mathbf{Z}_p$. By Theorem 5.1 (b) we are done.

Remark: By an analogous but slightly more complicated procedure the more general Theorem of P. Neumann (see [6], p. 117) can also be proved.

COROLLARY 5.1.2: (Baumslag, see [6], p. 117). Let $\mathfrak{V}$ *be a variety in which the free groups are residually finite p-groups for infinitely many primes. Let F be a* $\mathfrak{V}$*-free group and* $\{x_\alpha\}$ *a set of elements in F whose images in* A_0F *freely* (*in* $\mathfrak{V} \cap \mathfrak{A}\mathfrak{b}$) *generate a free subgroup. Then* $\{x_\alpha\}$ *freely generate a V-free subgroup of F.*

Proof: It is easy to see that the subgroup U of F generated by $\{x_\alpha\}$ is free if and only if the subgroups generated by the *finite* subsets of $\{x_\alpha\}$ are free (see [6], p. 114).

But for every *finite* subset $\{x_\alpha\}'$ there is a prime for which F is residually a p-group such that $F/[F, F]\ \{x_\alpha\}'$ does not contain p-torsion. Corollary 5.1.1 proves the conclusion.

6. Retracts of free groups

DEFINITION: A group G in $\mathfrak{V}$ is called a *retract of a free group F* if there is a presentation $G \cong F/R$ such that the projection $p: F \to G$ has a right inverse $r: G \to F$ (i.e., $pr = 1_G$).

Remark: The notion of retracts of a free group is easily seen to be equivalent to the notion of splitting groups (see [6], p. 137).

If G is a retract of the free group F, then $S_pG \to S_pF \to S_pG$ is the identity map. Since $S_pF = 0$, $S_pG = 0$, also. Analogously A_pG is a direct summand of A_pF.

THEOREM 6.1: *Let $\mathfrak{V}$ be a variety of exponent* 0 *or a prime power. Let G be a retract of a $\mathfrak{V}$-free group. Then there is a $\mathfrak{V}$-free group F and a homomorphism $\varphi: F \to G$ such that φ induces isomorphisms $\varphi_n: F/F_n^{(p)} \cong G/G_n^{(p)}$ for all $n \geqslant 0$ and p any prime ro $p = 0$.*

Proof: Follows immediately from Corollary 4.1.2.

COROLLARY 6.1.1: *Let $\mathfrak{V}$ be a variety of exponent* 0 *or a prime power, whose free groups are residually nilpotent. Suppose G is a retract of a free group such that A_0G is $\mathfrak{V} \cap \mathfrak{Ab}$-free on a set of cardinality α. Then G contains a subgroup F which is $\mathfrak{V}$-free on a set of cardinality α.*

COROLLARY 6.1.2: (P. Hall, see [6], p. 138) *Let $\mathfrak{V}$ be a nilpotent variety of exponent* 0 *or a prime power. Then a retract of a $\mathfrak{V}$-free group is $\mathfrak{V}$-free.*

THEOREM 6.2: (P. Hall, see [6], p. 138) *Let G be a retract of a free group in a nilpotent variety of exponent $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$. Then G is of the form*

$$G = F_1(\mathfrak{V}_1) \times F_2(\mathfrak{V}_2) \times \cdots \times F_k(\mathfrak{V}_k),$$

where $\mathfrak{V}_i$ is the subvariety of $\mathfrak{V}$ consisting of all groups in $\mathfrak{V}$ of exponent dividing $p_i^{\alpha i}$.

Proof: Choose $H = F_1(\mathfrak{V}_1) \times \cdots \times F_k(\mathfrak{V}_k)$ in such a way that $A_0H \cong A_0G$. Since G is of exponent n and nilpotent, it is a direct product of groups of exponent dividing $p_i^{\alpha i}$. Therefore a map $\varphi: H \to G$ may be defined such that φ induces an isomorphism $A_0H \cong A_0G$.

7. Deficiency

In this chapter $\mathfrak{V}$ always is a variety of exponent 0.

Let the group G in $\mathfrak{V}$ be given by a finite presentation: $x_1, \ldots, x_n$ are the generators, $y_1, \ldots, y_r$ are the relators. The number $n - r$ is called the $\mathfrak{V}$-deficiency of the presentation. We define def$G$, *the $\mathfrak{V}$-deficiency of the group G* to be the maximum deficiency of the finite $\mathfrak{V}$-presentations of G.

If M is an abelian group, then we denote by sM the minimum number of generators of M.

The following is a generalization of a Theorem by Epstein [2] and a Theorem by Knopfmacher [3].

THEOREM 7.1: (a) $\operatorname{def} G \leqslant \operatorname{rank} A_0G - s\, S_0G$
(b) $\operatorname{def} G \leqslant \dim A_pG - \dim S_pG$.

Proof: We only prove the first inequality, the proof of the second being similar.

Let $G = F/R$ be a presentation with n generators and r relators. Then we consider the exact sequence

$$0 \to S_0G \to R/[F, R] \xrightarrow{\kappa} A_0F \to A_0G \to 0.$$

The image I of κ is as a subgroup of a free abelian group free abelian. Therefore $R/[F, R]$ is isomorphic to $S_0G \oplus I$. Since $R/[F, R]$ is easily seen to be generated by the r relators we get the inequality

$$\begin{aligned} r \geqslant s(R/[F, R]) &= s(S_0G) + \operatorname{rank} I = s(S_0G) + \operatorname{rank} A_0F - \operatorname{rank} A_0G \\ &= s(S_0G) + n - \operatorname{rank} A_0G. \end{aligned}$$

DEFINITION: A group G for which inequality (a) ((b)) becomes an equality is called *efficient* (*p-efficient*) in $\mathfrak{V}$.

PROPOSITION 7.2: *We have* $\operatorname{rank} A_0G - s(S_0G) \leqslant \dim A_pG - \dim S_pG$ *and there always exists a prime p for which we have equality.*

Proof: We have $A_pG = A_0G \otimes \mathbf{Z}_p$ and by Lemma 3.4

$$S_pG = S_0G \otimes \mathbf{Z}_p \oplus \operatorname{Tor}_1(A_0G, \mathbf{Z}_p).$$

We therefore obtain:

$$\begin{aligned} \dim(A_pG) - \dim(S_pG) &= \dim(A_0G \otimes \mathbf{Z}_p) - \dim \operatorname{Tor}_1(A_0G, \mathbf{Z}_p) - \dim(S_0G) \otimes \mathbf{Z}_p \\ &= \operatorname{rank} A_0G - \dim(S_0G \otimes \mathbf{Z}_p) \geqslant \operatorname{rank} A_0G - s(S_0G). \end{aligned}$$

Moreover S_0G can be written as $\mathbf{Z} \oplus \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \oplus \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \cdots \oplus \mathbf{Z}_{n_k}$

with n_i/n_{i+1} for $1 \leqslant i \leqslant k$. Consequently we have equality if $k=0$ or if $k \neq 0$ and p/n_1.

It is well known that abelian groups are efficient in the variety of all groups [2], and that one relator groups are efficient in the variety of all groups [2]. Also Swan [12] gave an example of a group which is *not* efficient in the variety of all groups.

THEOREM 7.3: *Given a finitely presentable group in* $\mathfrak{B}$. *Then there is an* (p-)*efficient group* H *in* $\mathfrak{B}$ *and a surjective homomorphism* $\varphi: H \to G$ *which induces isomorphisms* $\varphi l: H/H_l^{(p)} \cong G/G_l^{(p)}$ *for all* $l \geqslant 0$.

Proof: We give the proof for $p=0$; the proof for p a prime being similar.

Consider a finite $\mathfrak{B}$-presentation $G \cong F/R$ and the corresponding exact sequence

$$0 \to S_0 G \to R/[F, R] \xrightarrow{\kappa} A_0 F \to A_0 G \to 0.$$

Denote the image of κ by I. Suppose the generators of this presentation are $x_1, \ldots, x_n$. Then we define the group H by the following presentation: the generators are $x_1, \ldots, x_n$, the relators $y_1, \ldots, y_j, z_1, \ldots, z_k$ are chosen in such a way that

(i) $y_1, \ldots, y_j, z_1, \ldots, z_k$ are elements in R;

(ii) the images of $y_1, \ldots, y_j$ in I form a basis of I;

(iii) the images of $z_1, \ldots, z_k$ in $R/[F, R]$ form a minimal set of generators of $S_0 G$ (i.e., such that k is minimal).

Trivially there is a surjective map $\varphi: H \to G$. Also φ induces clearly an isomorphism $A_0 H \cong A_0 G$ and an epimorphism $S_0 H \to S_0 G$. The assertion about the lower central series then follows from Theorem 4.1. It remains to check that H is efficient. We have $j+k=s(S_0 G)+n-\text{rank } A_0 G$ and therefore

$$\begin{aligned} \operatorname{rank} A_0 G - s(S_0 G) &= n - (j + k) \\ &\leqslant \operatorname{def} H \leqslant \operatorname{rank} A_0 H - s(S_0 H) \leqslant \operatorname{rank} A_0 G - s(S_0 G); \end{aligned}$$

the last inequality since $S_0 H \to S_0 G$ is an epimorphism. It follows that H is efficient.

The following Corollary gives a partial answer to a question of Knopfmacher [3].

COROLLARY 7.3.1: *In a nilpotent variety (of exponent* 0*) every group is efficient.*

The following is a simple proof of a result by Chen [1].

COROLLARY 7.3.2: *Let* G *be a group, with* $n+r$ *generators and* $r+k$ *relators, where* $n=sA_0 G$. *Then there exists to every* $d \geqslant 0$ *a group* H *with* n *generators and* k *relators with* $H/H_d \cong G/G_d$.

Proof: The emphasis is on the fact that k relators suffice for a presentation of H.

The deficiency of G/G_d in the variety $\mathfrak{B}$ of all nilpotent groups of nilpotency class $\leqslant d$ clearly is at least the deficiency of G in the variety of all groups which in turn is at least $n+r-r-k=n-k$. Corollary 7.3.1 shows that there is a presentation of G/G_d in $\mathfrak{B}$ with n generators and k relators.[2]) This presentation of G/G_d in $\mathfrak{B}$ can

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[2]) Note that there is a presentation of G/G_d by $n=sA_0 G$ generators.

easily be "lifted" to a presentation of H in the variety of all groups by taking counter-images of the given relators in the absolutely free group on the same generators.

8. The Magnus Theorem

In this Chapter $\mathfrak{V}$ always is a variety of exponent 0.

THEOREM 8.1: *Let G be a group in $\mathfrak{V}$ with $n+r$ generators and r relators, such that A_pG for p a prime or $p=0$ is generated by n elements. Then there is a $\mathfrak{V}$-free group F and a map $\varphi: F \to G$ which induces isomorphisms $\varphi_k: F/F_k^{(p)} \cong G/G_k^{(p)}$ for all $k \geqslant 0$.*

Proof: We only give the proof for $p=0$, the proof for p a prime being similar.

It follows from Theorem 7.1 that $s(S_0G)=0$, i.e. $S_0G=0$; therefore $\operatorname{rank} A_0G=n$ and A_0G is free abelian. Take the $\mathfrak{V}$-free group F on n generators and consider a map $\varphi: F \to G$ which induces an isomorphism $A_0F \cong A_0G$. Since φ induces an epimorphism $S_0F \to S_0G=0$, the conclusion follows by Theorem 4.1.

Remark: If the hypothesis of the above theorem holds for $p=0$, then the conclusion holds not only for $p=0$ but also for all primes.

The following is a generalization of a Theorem by Magnus [5].

COROLLARY 8.1.1: *Let $\mathfrak{V}$ be a variety in which the free groups are residually nilpotent. Let G be a group which has both a $\mathfrak{V}$-presentation with $n+r$ generators and r relators and a $\mathfrak{V}$-presentation with n generators. Then G is a $\mathfrak{V}$-free group with n generators.*

COROLLARY 8.1.2: *Let $\mathfrak{V}$ be a nilpotent variety. Suppose G is a group which has a $\mathfrak{V}$-presentation with $n+r$ generators and r relators, and whose abelianized group A_0G is generated by n elements. Then G is a $\mathfrak{V}$-free group with n generators.*

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