A Note on Bases in Ordered Locally Convex Spaces

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Let E be a real Fréchet space ordered by a cone K and let the dual space E' be ordered by the dual cone $K' = \{f \in E' : f(x) \geq 0, x \in K\}$. Besides the topologies on E defined by the dual system $\langle E, E' \rangle$, e.g., the Mackey topology $\tau(E, E')$ determined by the metric on E, and the weak topology $\sigma(E, E')$, there are also topologies defined in terms of the order structure. One of these is $o(E, E')$, the topology of uniform convergence on the order bounded subsets of E'. $o(E, E')$ is defined when each x in E regarded as a linear functional on E' is an order bounded linear functional on E', that is, whenever $\{f(x) : f \in S\}$ is bounded for each order bounded set S in E'. This condition is satisfied when K is generating. If E is a barreled space, in particular, if E is a Fréchet space, with a generating cone, then $o(E, E')$ is consistent with $\langle E, E' \rangle$ as is shown in proposition 1 below. Also, if E is a locally convex lattice, then $o(E, E')$ is always consistent and is the coarsest topology finer than the weak topology for which the lattice operations are continuous. See [6, 7].

In this note, we give an answer to the question of whether each $o(E, E')$-basis for E is a $o(E, E')$-Schauder basis for E. A $\mathfrak{T}$-basis for a topological vector space $E(\mathfrak{T})$ is a sequence $\{x_i\}$ in E such that for each x in E, there is a unique sequence $\{\alpha_i\}$ of scalars such that $x = \sum_{i=1}^{\infty} \alpha_i x_i$, where the convergence of the series is with respect to the topology $\mathfrak{T}$ [4]. The uniqueness implies that each α_i may be regarded as a linear functional on E. If each α_i is $\mathfrak{T}$-continuous, then $\{x_i\}$ is called a $\mathfrak{T}$-Schauder basis for E. The weak basis theorem [2, 3, 5] for Fréchet spaces states that each $\sigma(E, E')$-basis is a $\sigma(E, E')$-Schauder basis for E. As a consequence of this fundamental theorem one gets the result that each $\sigma(E, E')$-basis for a Fréchet space E is a $\tau(E, E')$-Schauder basis for E. We show that if E is a Fréchet space ordered by a generating cone, then the analogous weak basis theorem for $o(E, E')$ is valid. Moreover, it is also shown that each lattice theoretically absolutely convergent $o(E, E')$-basis for a complete metrizable locally convex lattice E is an unconditional $\tau(E, E,)$-Schauder basis for E.

The following proposition is related to Corollary 2.4 of [7, p. 130]. It shows that if E is barreled, then the hypotheses that the cone in E is closed and E' is a full subspace of the algebraic dual E^* or of the order dual E^+ can be dispensed with.

PROPOSITION 1. *If E is a barreled space ordered by a generating cone, then $o(E, E')$ is consistent with the dual system $\langle E, E' \rangle$.*

Proof. Let S be the class of all order intervals in E' and let $\mathcal{S}$ be the saturated hull of S (i.e., the class of all scalar multiples of the $\sigma(E', E)$-closed, convex circled hulls of finite families of S). Then the topology of uniform convergence on the sets in $\bar{S}$ coincides with the topology $o(E, E')$. Since the cone K in E is generating, K' is normal for the topology $\sigma(E', E)$ in E' [7, p. 74]. From this it follows that each order interval in E', and hence each member of $\bar{S}$, is $\sigma(E', E)$ bounded. Since E is barreled, these sets are therefore equicontinuous. Thus, the sets in $\bar{S}$ are $\sigma(E', E)$-compact and the proposition is a consequence of the Mackey–Arens theorem.

PROPOSITION 2. *If $\{x_i\}$ is a $o(E, E')$-basis for a Fréchet space E ordered by a generating cone K, then $\{x_i\}$ is a $o(E, E')$-Schauder basis for E.*

Proof. The result follows immediately from proposition 1 and a generalization of the weak basis theorem that states that if $\mathfrak{T}$ is a topology consistent with $\langle E, E' \rangle$, then every $\mathfrak{T}$-basis for E is a $\mathfrak{T}$-Schauder basis for E [1, p. 508]. However, the result can also be proved directly by adapting the techniques used for proving the weak basis theorem [2, 3]. In this case, particular use must be made of the fact that convergence for $o(E, E')$ implies convergence for $\sigma(E, E')$, and that the order intervals in E' are equicontinuous sets.

PROPOSITION 3. *If $E(\mathfrak{T})$ is a Fréchet space ordered by a generating cone K, then every $o(E, E')$-basis for E is a $\mathfrak{T}$-Schauder basis for E.*

Proof. Each x in E has the weak series expansion $\sum_{i=1}^{\infty} \alpha_i x_i$, but in order to conclude that $\{x_i\}$ is actually a weak basis, we must show that the unique sequence $\{\alpha_i\}$ corresponding to x in the $o(E, E')$ expansion of x is also unique for the weak series expansion of x. To do this, we use the fact that each α_i belongs to E'. Then if $\sum_{i=1}^{\infty} \beta_i x_i = \theta$, where we assume the series converges weakly, we have $\sum_{i=1}^{\infty} \beta_i \alpha_j(x_i) = \theta$ for $j = 1, 2, \ldots$. Since $\{x_i\}$ is a $o(E, E')$-basis for E, $\alpha_j(x_i) = \delta_{ij}$. Hence $\beta_j = 0$ for each j and we conclude that the coefficients in the weak series expansion of x are unique. Thus, $\{x_i\}$ is a weak basis for E and the weak basis theorem implies that $\{x_i\}$ is a weak Schauder basis for E. From this we obtain that $\{x_i\}$ is a $\mathfrak{T}$-Schauder basis for E.

COROLLARY 4. *Every $o(E, E')$-basis for a complete metrizable locally convex lattice $E(\mathfrak{T})$ is a $\mathfrak{T}$-Schauder basis for E.*

Proof. Since E is a locally convex lattice, the cone in E is obviously generating so that the corollary follows from proposition 3.

If E is a locally convex lattice, then a θ-neighborhood basis for $o(E, E')$ is given by polars of order intervals in E' of the form $[-f, f]$ where f is in K'. Thus $o(E, E')$ is generated by the family $\{P_f : f \in K'\}$ of seminorms defined by

$$P_f(x) = \sup\{|g(x)| : -f \leqslant g \leqslant f\}$$

and these seminorms have the simple form

$$P_f(x) = f(|x|), \quad f \in K', \quad x \in E$$

where $|x|$ is the lattice theoretic absolute value of x in E. To see this, first note that the inequality $P_f(x) \leqslant f(|x|)$ is evident since $-f \leqslant g \leqslant f$ implies that $g(x) \leqslant f(|x|)$. To obtain the reverse inequality, we may use the fact that the canonical mapping $\varphi: E \to E''$ is a lattice isomorphism of E onto a sublattice of E''. Then for f in K' we have

$$f(|x|) = |\varphi(x)|(f) = \sup\{\varphi(x)(g) : |g| \leqslant f\} = \sup\{g(x) : -f \leqslant g \leqslant f\}$$

See [9, p. 212]. From this we have $f(|x|) \leqslant P_f(x)$ and hence $P_f(x) = f(|x|)$ for f in K' and x in E.

Using the concept of lattice theoretical absolute convergence of a series introduced by Pietsch [8], one gets a new type of $o(E, E')$-basis. A $o(E, E')$-basis $\{x_i\}$ for E is called *lattice theoretically absolutely convergent* if for each x in E the sequence $\{\sum_{i=1}^{n} |\alpha_i x_i|\}$ is majorized in E, where $\sum_{i=1}^{\infty} \alpha_i x_i$ is the basis expansion of x. Since E' is a lattice ideal in the order dual E^+ of E and since E^+ coincides with the order bound dual E^b [7, 9], it follows that $\{x_i\}$ is a lattice theoretically absolutely convergent basis for E if and only if it is a $o(E, E')$-basis such that $\sum_{i=1}^{\infty} P_f(\alpha_i x_i)$ is finite for x in E, f in K' [8, p. 17] (i.e., a $o(E, E')$- absolutely convergent basis for E).

PROPOSITION 5. *If $\{x_i\}$ is a lattice theoretically absolutely convergent $o(E, E')$-basis for a complete metrizable locally convex lattice $E(\mathfrak{T})$, then $\{x_i\}$ is an unconditional $\mathfrak{T}$-basis for E.*

Proof. Let Σ denote the collection of all finite subsets of the set of positive integers. For each $\sigma \in \Sigma$, define the continuous linear transformation $T_\sigma: E \to E$ by $T_\sigma x = \sum_{i \in \sigma} \alpha_i x_i$ where $\{\alpha_i\}$ is the sequence of coefficients corresponding to x. We have for all $\sigma \in \Sigma$ and $f \in E'$ that

$$|f(T_\sigma x)| \leqslant \sum_{i \in \sigma} |f(\alpha_i x_i)| \leqslant \sum_{i \in \sigma} |f|(|\alpha_i x_i|) \leqslant \sum_{i=1}^{\infty} P_{|f|}(\alpha_i x_i)$$

which is finite. Thus, for each x in E, the family $\{T_\sigma x : \sigma \in \Sigma\}$ is weakly bounded, and hence bounded in $E(\mathfrak{T})$. It then follows that the family $\{T_\sigma : \sigma \in \Sigma\}$ is equicontinuous. Let U be any θ-neigborhood in E. Then there is a circled θ-neighborhood V such that $V + V \subset U$. Choose a θ-neighborhood W in E such that $T_\sigma(W) \subset V$ for all σ. Since $\{x_i\}$ is a $\mathfrak{T}$-basis for E by proposition 3, there exists an integer n such that $x - T_{\sigma_n} x$ is in $V \cap W$ where $\sigma_n = \{1, 2, 3, \ldots, n\}$. Then $T_\sigma(x - T_{\sigma_n} x)$ is in V for all $\sigma \in \Sigma$. Hence, for all $\sigma \supset \sigma_n$ we have

$$x - T_\sigma x = x - T_{\sigma_n} x - T_\sigma(x - T_{\sigma_n} x) \in V + V \subset U.$$

This shows that $\sum_{i=1}^{\infty} \alpha_i x_i$ is unconditionally convergent to x and the proof is complete.

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