On the Extremality of Certain Teichmüller Mappings.

On the Extremality of Certain Teichmüller Mappings[1])

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§ 1. Introduction

Let G be the domain $G=\{z \mid y>|x|^{\alpha}\}$, where $z=x+iy$ and $\alpha \geqslant 1$, and let F be the horizontal stretching of G onto $G'=F(G)$ by the factor $K_0>1$, i.e. $F(z)=K_0 x+iy$. F is called *extremal* if the maximal dilatation K of every quasiconformal mapping f of G onto G' which has the same boundary values as F satisfies $K \geqslant K_0$. It is called *unique extremal* if $K \geqslant K_0$ and if $K=K_0$ implies $f=F$.

It is known [3] that for $\alpha=1$ F is not extremal (but there exists a unique extremal quasiconformal mapping different from F) whereas for $\alpha>1$ F is extremal [2]. In a recent paper [1] Eugen Blum proved that for $\alpha>3$ F is unique extremal. We will now give a different proof of this fact by means of a differential inequality and show that the result is also true for $\alpha=3$. Moreover by the construction of explicit mappings different from the horizontal stretching F, but with the maximal dilatation $K=K_0$, we show that for $1<\alpha<3$ uniqueness does not hold.

The extremality proof does not depend on the lower part of the domain; we may replace this part by an arbitrary Riemann surface. We would then have a (schlicht) subdomain $\{z \mid y>|x|^{\alpha}, y>y_0\}$ for some $y_0>0$ and to prove extremality of F for $\alpha>1$ the integration (see § 2 below) would have to start off at some $y_1>y_0$ instead of zero. For the proof of unique extremality this is still true if one makes the additional hypothesis that the welded-on piece of surface has finite area. The integration must then extend over this surface also rather than start at zero. The inclusion of this more general case would however just complicate the formulas with some additive constants. As the emphasis in our work is the determination of the exponent α and the method, rather than generality, we restrict the consideration to the domain G.

§ 2. The Differential Inequality

Let f be a quasiconformal mapping of G onto G' with maximal dilatation K which agrees with F on the boundary ∂G of G. The length-area method applied to the mapping f in the domain $G_y=G \cap\{\operatorname{Im} z<y\}$ yields

$$2K_0\eta^{\beta} \leqslant L(\eta)=\int_{-\eta^{\beta}}^{\eta^{\beta}} |f_z+f_{\bar{z}}|\, d\xi \tag{2.1}$$

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with $L(\eta)$ the length of the f-image of the segment

$$\gamma_\eta = \{z \mid \operatorname{Im} z = \eta, -\eta^\beta \leqslant \operatorname{Re} z \leqslant \eta^\beta\}, \beta = 1/\alpha < 1.$$

Integration over η from zero to y and subsequent application of the Schwarz inequality yield

$$Cy^{\beta+1} \leqslant \int_0^y L(\eta)\,\mathrm{d}\eta = \int_0^y \int_{-\eta^\beta}^{\eta^\beta} |f_z + f_{\bar z}|\,\mathrm{d}\xi\,\mathrm{d}\eta,\ C = \frac{2K_0}{\beta+1} \tag{2.2}$$

$$C^2 y^{2\beta+2} \leqslant \left(\int_0^y L(\eta)\,\mathrm{d}\eta\right)^2 \leqslant \iint_{G_y} J\,\mathrm{d}\xi\,\mathrm{d}\eta \cdot \iint_{G_y} \frac{|1+\kappa|^2}{1-|\kappa|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{2.3}$$

where $J(z) = |f_z|^2 - |f_{\bar z}|^2$ is the Jacobian of f and $\kappa(z)$ is its complex dilatation. We use the following estiamtes for the right hand side integrals:

$$\iint_{G_y} J\,\mathrm{d}\xi\,\mathrm{d}\eta \leqslant 2K_0 \int_0^{y+\delta(y)} \eta^\beta\,\mathrm{d}\eta = C(y+\delta(y))^{\beta+1} \tag{2.4}$$

with $\delta(y) \geqslant 0$ the maximal upper deviation of $f(\gamma_y)$ above the horiozntal $\operatorname{Im} z = y$, i.e. $\delta(y) = \sup_{z\in\gamma_y} \{\operatorname{Im} f(z) - y\}$;

$$\iint_{G_y} \frac{|1+\kappa|^2}{1-|\kappa|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta \leqslant \iint_{G_y} \left[K - \frac{2}{1-k^2}(k - \operatorname{Re}\kappa)\right]\mathrm{d}\xi\,\mathrm{d}\eta = \frac{2K}{\beta+1} y^{\beta+1} - I(y) \tag{2.5}$$

with $\quad k = \dfrac{K-1}{K+1} = \operatorname{ess\,sup} |\kappa(z)| \quad$ and $\quad I(y) = \dfrac{2}{1-k^2} \displaystyle\iint_{G_y} (k - \operatorname{Re}\kappa)\,\mathrm{d}\xi\,\mathrm{d}\eta.$

Evidently $I(y)$ is an increasing function as the integrand is nonnegative. Inserting (2.4) and (2.5) into (2.3) we get: *For every K-quasiconformal mapping f of G onto G' with $f|\partial G = F|\partial G$ the inequalities*

$$\left.\begin{aligned} C^2 y^{2\beta+2} &\leqslant \left(\int_0^y L(\eta)\,\mathrm{d}\eta\right)^2 \leqslant C(y+\delta(y))^{\beta+1}\left(\frac{2K}{\beta+1} y^{\beta+1} - I(y)\right) \\ &\leqslant C\frac{2K}{\beta+1} y^{\beta+1}(y+\delta(y))^{\beta+1} \end{aligned}\right\} \tag{2.6}$$

hold, and consequently

$$K_0 \leqslant K\left(1 + \frac{\delta(y)}{y}\right)^{\beta+1}. \tag{2.7}$$

To estimate $\delta(y)$ we apply the following lemma [3]: Let G be a domain of the z-plane and let γ_η, with length $l(\eta)$, denote the intersection of the straight line $\operatorname{Im} z=\eta$ with G. Let F and f be as before. Then under the assumption that $l(\eta)\leqslant M$ for $y\leqslant\eta\leqslant y+M\sqrt{KK_0}$ we have $\delta(y)\leqslant M\sqrt{KK_0}$.

For given $y\geqslant 0$ let $\bar{y}>y$ be the solution of the equation $\bar{y}=y+2\sqrt{KK_0}\bar{y}^\beta$. Then for $y\leqslant\eta\leqslant\bar{y}$ we have $l(\eta)\leqslant 2\bar{y}^\beta$ and therefore $\delta(y)\leqslant 2\sqrt{KK_0}\bar{y}^\beta$. From this we derive

$$\lim_{y\to\infty}\frac{\delta(y)}{y}\leqslant\lim_{y\to\infty}\frac{2\sqrt{KK_0}\bar{y}^\beta}{\bar{y}-2\sqrt{Kk_0}\bar{y}^\beta}=0. \tag{2.8}$$

By (2.7), therefore, $K_0\leqslant K$. This proves that F *is extremal for all* $\alpha>1$.

Assume $K=K_0$. We now conclude from the first and third member of (2.6)

$$\left(1+\frac{\delta(y)}{y}\right)^{\beta+1} I(y)\leqslant C[(y+\delta(y))^{\beta+1}-y^{\beta+1}]. \tag{2.9}$$

This gives the following *sufficient condition for the unique extremality* of F: If

$$\varliminf_{y\to\infty}[(y+\delta(y))^{\beta+1}-y^{\beta+1}]=0 \tag{2.10}$$

holds, then $\lim_{y\to\infty} I(y)=0$, and therefore $f=F$. Namely, we conclude that $k-\operatorname{Re}\kappa\equiv 0$, which implies $\kappa\equiv k$. Because of (2.8) *the sufficient condition (*2.10*) is equivalent to the condition*

$$\varliminf_{y\to\infty} y^\beta\,\delta(y)=0. \tag{2.11}$$

In order to put the inequality (2.6) into a more manageable form we introduce the lower semi-continuous function $\Delta(y)\geqslant 0$ by

$$L(y)=2Ky^\beta+\Delta(y). \tag{2.12}$$

The inequality between the second and the last member of (2.6) then becomes

$$\int_0^y \Delta(\eta)\,d\eta\leqslant Cy^{\beta+1}\left[\left(1+\frac{\delta(y)}{y}\right)^{(\beta+1)/2}-1\right]\leqslant C_1 y^\beta\,\delta(y), \tag{2.13}$$

where the last step holds for all sufficiently large y, say $y\geqslant y_0$, and a constant $C_1>C(\beta+1)/2$. From this we deduce that

$$\lim_{y\to\infty}\frac{1}{y^{\beta+1}}\int_0^y \Delta(\eta)\,d\eta=0 \tag{2.14}$$

and therefore

$$\varliminf_{y\to\infty} \frac{\Delta(y)}{y^\beta} = 0. \tag{2.15}$$

For a given value of $\delta(y)$, $L(y)$ is minimized when $f(\gamma_y)$ is the upper part of an isosceles triangle; that is,

$$\delta(y)^2 \leqslant \left(\frac{L(y)}{2}\right)^2 - (Ky^\beta)^2 = Ky^\beta\,\Delta(y) + \frac{\Delta(y)^2}{4}. \tag{2.16}$$

This allows us to transform (2.13) into an inequality involving only the function Δ:

$$\int_0^y \Delta(\eta)\,\mathrm{d}\eta \leqslant C_2\left[y^{3\beta}\,\Delta(y)\right]^{1/2}\left[1+\frac{\Delta(y)}{4Ky^\beta}\right]^{1/2}, \tag{2.17}$$

with $C_2 = C_1\sqrt{K}$. Setting $u(y) = \int_0^y \Delta(\eta)\,d\eta$ we get the differential inequality

$$u(y) \leqslant C_2\left[y^{3\beta}u'(y)\right]^{1/2}\left[1+\frac{u'(y)}{4Ky^\beta}\right]^{1/2} \tag{2.18}$$

which we solve for u':

$$u' \geqslant Ay^\beta\{\sqrt{1+u^2B^{-1}y^{-4\beta}}-1\},\ y\geqslant y_0, \tag{2.19}$$

with $A=2K$, $B=KC_2^2=K^2C_1^2$. This holds for every quasiconformal mapping f of G onto G' with maximal dilatation $K=K_0$ which is equal to F on ∂G. Dividing both sides of (2.19) by y^β and using (2.15) gives

$$\varliminf_{y\to\infty} y^{-2\beta}\,u(y) = 0. \tag{2.20}$$

§ 3. Solution of the Differential Inequality (2.19)

We first want to show that the relation (2.20) can be strengthened to

$$\lim_{y\to\infty} y^{-2\beta}\,u(y) = 0. \tag{3.1}$$

To this end we derive from (2.19) a differential inequality for $h(y) = y^{-2\beta}u(y)$. We get

$$u' = 2\beta y^{2\beta-1}h + y^{2\beta}h' \geqslant Ay^\beta\{\sqrt{1+h^2B^{-1}}-1\} \tag{3.2}$$

and therefore

$$2\beta y^{\beta-1}\,h + y^\beta h' \geqslant A\{\sqrt{1+h^2B^{-1}}-1\} \tag{3.3}$$

Assume that for some $M>0$ $\varlimsup_{y\to\infty} h(y) \geqslant 3M$. Then because of (2.20) there is a sequence of points $y_n \to \infty$ such that $M \leqslant h(y_n) \leqslant 2M$ and $h'(y_n) \leqslant 0$. Consequently

$$4\beta M y_n^{\beta-1} \geqslant A\{\sqrt{1+M^2B^{-1}}-1\} \tag{3.4}$$

which is impossible, since the left hand side goes to zero with $y_n \to \infty$ whereas the right hand side is a positive constant.

We now go back to the original inequality (2.19). Since $u^2y^{-4\beta}$ is bounded for $y \geqslant y_0$, there is a positive number a such that

$$\sqrt{1+u^2B^{-1}y^{-4\beta}}-1 > au^2B^{-1}y^{-4\beta}$$

The inequality (2.19) therefore becomes

$$u' \geqslant b\frac{u^2}{y^{3\beta}},\ y \geqslant y_0, \tag{3.5}$$

with $b = Aa/B$. The right hand side is non-negative. Thus $u' \geqslant 0$. Suppose $u(y_1) > 0$ for some $y_1 \geqslant y_0$. Then $u(y) \geqslant u(y_1) > 0$ for $y \geqslant y_1$, and we can integrate (3.5) after separation of variables to get

$$\frac{1}{u(y_1)} \geqslant \frac{1}{u(y_1)} - \frac{1}{u(y)} \geqslant b\int_{y_1}^{y} \frac{dt}{t^{3\beta}}. \tag{3.6}$$

This is clearly impossible for $\beta \leqslant \frac{1}{3}$ since the right hand side is unbounded as $y \to \infty$. Therefore $u(y) = \int_0^y \Delta(\eta)\, d\eta = 0$ for $y \geqslant y_0$. As Δ is lower semicontinuous, we must have $\Delta(y) \equiv 0$. Therefore by (2.16), $\delta(y) \equiv 0$ $(y \geqslant 0)$. Thix implies uniqueness, by (2.11).

We have thus proved uniqueness for $\alpha \geqslant 3$. For $\alpha < 3$, i.e. $\beta > \frac{1}{3}$, the differential inequality (2.19) with the side condition (2.20) has positive solutions and therefore does not allow us to prove uniqueness. It is in fact easily verified that $u(y) = cy^{3\beta-1}$ satisfies both (2.19) and (2.20) for sufficiently small $c > 0$.

§ 4. Construction of a Mapping, $1 < \alpha < 2$

For the case $1 < \alpha < 2$ we now proceed to construct a $K = K_0$ - quasiconformal mapping $f \neq F$ of G onto G' which is equal to F on ∂G. This proves non-uniqueness of the extremal mapping for the exponents $\alpha \in (1, 2)$. [2]).

First let $\alpha > 1$ be arbitrary, i.e. $\beta = 1/\alpha \in (0, 1)$, and $K > 1$. We define the mapping $w = f(z) = u + iv$ for $y \geqslant 0$, $0 \leqslant x \leqslant y^\beta$ by

$$\left.\begin{aligned} u(x, y) &= Kx \\ v(x, y) &= y + (1 - y^{-\beta}x)\,\delta(y), \end{aligned}\right\} \tag{4.1}$$

[2]) In § 6 we construct a class of mappings for $\alpha \in (1, 3)$, but the definition is more complicated than the mappings of §§ 4, 5.

where $\delta(y)$ is assumed to be continuous and piecewise continuously differentiable, $\delta(0)=0$. For fixed y this is simply a linear mapping of the horizontal segment $0 \leqslant \operatorname{Re} z \leqslant y^\beta$, $\operatorname{Im} z = y$, onto the segment with the same point (Ky^β, y) of δG and the point $(0, y+\delta(y))$ as endpoints. We then extend this mapping to the left half of G, i.e. for $-y^\beta \leqslant x \leqslant 0$, $y \geqslant 0$, by symmetry. In order that f be topological it is necessary and sufficient that $y+\delta(y)$ is strictly increasing. The mapping f is K-quasiconformal, i.e. $D \leqslant K$, if and only if

$$D + 1/D = \frac{u_x^2 + u_y^2 + v_x^2 + v_y^2}{u_x v_y - u_y v_x} \leqslant K + 1/K. \tag{4.2}$$

We have $u_x = K$, $u_y = 0$, $v_x = -y^{-\beta}\delta$, $v_y = 1 + (1 - y^{-\beta}x)\,\delta' + \beta x y^{-\beta-1}\delta = 1 + (1-\theta)\delta' + \beta\theta y^{-1}\delta$, with $\theta = x \cdot y^{-\beta}$, $0 \leqslant \theta \leqslant 1$, and therefore

$$u_x v_y - u_y v_x = K[1 + (1-\theta)\,\delta' + \beta\theta y^{-1}\,\delta], \tag{4.3}$$

$$\left.\begin{aligned} u_x^2 + u_y^2 + v_x^2 + v_y^2 = K^2 + y^{-2\beta}\delta^2 + 1 + (1-\theta)^2\delta'^2 + \beta^2\theta^2 y^{-2}\delta^2 \\ + 2(1-\theta)\,\delta' + 2\beta y^{-1}\,\delta + 2\beta\theta(1-\theta)\,y^{-1}\,\delta\delta'. \end{aligned}\right\} \tag{4.4}$$

In order to get necessary conditions for δ we look at the points $x=0$ and $x=y^\beta$. For $x=0$, i.e. $\theta=0$ (or rather $\theta=0+$, as we take the limits from the right hand side) we get

$$D + 1/D = \frac{K^2 + 1 + y^{-2\beta}\,\delta^2 + \delta'^2 + 2\,\delta'}{K(1+\delta')} \tag{4.5}$$

and therefore

$$x = 0+ : D \leqslant K \Leftrightarrow (K^2 - 1)\,\delta' \geqslant \delta'^2 + y^{-2\beta}\,\delta^2. \tag{4.6}$$

This gives us the necessary conditions

$$0 \leqslant \delta' \leqslant K^2 - 1,\ \delta y^{-\beta} \leqslant K^2 - 1. \tag{4.7}$$

As $\delta(0)=0$, we must have $\delta(y) \geqslant 0$.

For $x = y^\beta$, i.e. $\theta = 1$, we get similarly

$$D + 1/D = \frac{K^2 + 1 + y^{-2\beta}\,\delta^2 + \beta^2 y^{-2}\,\delta^2 + 2\beta y^{-1}\,\delta}{K[1 + \beta y^{-1}\,\delta]} \tag{4.8}$$

and therefore

$$x = y^\beta : D \leqslant K \Leftrightarrow (K^2 - 1)\,\beta \geqslant \beta^2 y^{-1}\,\delta + y^{1-2\beta}\,\delta. \tag{4.9}$$

This gives us the following necessary condition (stronger than the last inequality in (4.7) for $y \geqslant 1$)

$$\delta(y)\,y^{1-2\beta} \leqslant (K^2 - 1)\,\beta \tag{4.10}$$

which, in view of the first inequality of (4.7), excludes $\beta < \frac{1}{2}$. An extremal mapping of the proposed form (4.1) can therefore at most be constructed for the exponents $\frac{1}{2} \leqslant \beta < 1$, i.e. $1 < \alpha < 2$.

From now on assume $\frac{1}{2} < \beta < 1$. We want to show that a mapping can in fact be constructed with a δ-function of the form $\delta(y) = By^{1-t}$. We first consider the points $y \geqslant 1$. Because of (4.10) we must have $1 - 2\beta + 1 - t = 2(1-\beta) - t \leqslant 0$, and because of (4.7) we must have $\delta'(y) = B(1-t)\,y^{-t} \geqslant 0$. Therefore the only possible values for t are

$$2(1-\beta) \leqslant t \leqslant 1\,. \tag{4.11}$$

But $t = 1$ is excluded, for then $\delta' = 0$ and thus $\delta = 0$ by (4.6). This shows that the case $\beta = \frac{1}{2}$ cannot be handled in this way.

The expressions (4.3) and (4.4) now become

$$u_x v_y - u_y v_x = K\{1 + By^{-t}[(1-\theta)(1-t) + \beta\theta] \tag{4.12}$$

$$\left.\begin{aligned} u_x^2 + u_y^2 + v_x^2 + v_y^2 = K^2 + 1 + B^2 y^{-2t}[(1-\theta)^2(1-t)^2 \\ + \beta^2\theta^2 + 2\beta\theta(1-\theta)(1-t) + y^{2-2\beta}] \\ + 2By^{-t}[(1-t)(1-\theta) + \beta\theta] \end{aligned}\right\} \tag{4.13}$$

The condition (4.2) becomes

$$\left.\begin{aligned} (K^2-1)[(1-\theta)(1-t) + \beta\theta] \geqslant By^{-t}[(1-\theta)^2(1-t)^2 + \beta^2\theta^2 \\ + 2\beta\theta(1-\theta)(1-t)] + By^{2-2\beta-t} \end{aligned}\right\} \tag{4.14}$$

We put $t = 2(1-\beta)$. Then the left hand side of (4.14) becomes

$$(K^2-1)[(2\beta-1) + \theta(1-\beta)] \geqslant (K^2-1)(2\beta-1)\,. \tag{4.15}$$

On the other hand the right side of (4.14) becomes, for $y \geqslant 1$,

$$By^{-2(1-\beta)}[(1-\theta)^2(2\beta-1)^2 + \beta^2\theta^2 + 2\beta\theta(1-\theta)(2\beta-1)] + B \leqslant B\cdot\text{const.} \tag{4.16}$$

It is therefore clear that the inequality (4.2) will be satisfied for $y \geqslant 1$ provided the constant $B > 0$ is taken sufficiently small.

We still have to define δ for $0 \leqslant y \leqslant 1$. We try $\delta(y) = By$. Putting $t = 0$ in (4.14) we get for the left hand side

$$(K^2-1)[(1-\theta) + \beta\theta] = (K^2-1)[1 - \theta(1-\beta)] \geqslant (K^2-1)\,\beta \tag{4.17}$$

and for the right hand side, $0 \leqslant y \leqslant 1$,

$$\left.\begin{aligned} B[(1-\theta)^2 + \beta^2\theta^2 + 2\beta\theta(1-\theta)] + By^{2(1-\beta)} \\ = B[1 - \theta(1-\beta)]^2 + By^{2(1-\beta)} \leqslant 2B\,. \end{aligned}\right\} \tag{4.18}$$

The condition $D \leqslant K$ is therefore satisfied in $0 \leqslant y \leqslant 1$ as soon as $0 < B \leqslant \frac{1}{2}(K^2-1)\,\beta$.

In view of the foregoing let us therefore set

$$\delta(y)=\begin{cases} By & 0\leqslant y\leqslant 1 \\ By^{2\beta-1} & 1\leqslant y \end{cases} \tag{4.19}$$

The condition $D(z)\leqslant K$ will be satisfied a.e. in G providing $B>0$ is chosen sufficiently small. We thus have a one-parameter family of extremal mappings $f\neq F$.

§ 5. Construction of a Mapping, $\alpha=2$.

To prove non-uniqueness of the extremal mapping for the exponent $\alpha=2$ we define $w=f(z)=u+iv$ as follows for $y\geqslant 0$, $x^2\leqslant y$.

$$\left.\begin{aligned} u(x,y) &= Kx \\ v(x,y) &= y+\delta(y)-x^2y^{-1}\,\delta(y) \end{aligned}\right\} \tag{5.1}$$

where

$$\delta(y)=\begin{cases} By, & 0\leqslant y\leqslant 1 \\ B, & y\geqslant 1 \end{cases}.$$

Here the horizontal segment Im $z=y$ is mapped onto a parabola with endpoints $(\pm K\sqrt{y}, y)$; that is, f has the proper boundary values. Since f maps vertical lines in G onto vertical lines and $v(x, y)$ is a strictly increasing function of y, f is topological.

We find that the condition (4.2) is expressible as follows:

$$(1+4x^2)\,B\leqslant K^2-1,\; x^2\leqslant y\leqslant 1 \tag{5.2}$$

and

$$(4+x^2y^{-2})\,B\leqslant K^2-1,\; x^2\leqslant y,\; y\geqslant 1\,. \tag{5.3}$$

Hence, if we choose

$$0<B\leqslant\frac{K^2-1}{5}$$

f will be K-quasiconformal, and therefore extremal.

§ 6. Construction of a Mapping, $1<\alpha<3$

For the general case we define $f(z)=u+iv$ as follows. For $0\leqslant x\leqslant y^{\beta}$, $y\geqslant 1$,

$$\left.\begin{aligned} u(x,y) &= K\left[x+\frac{B(1-\beta)}{2}\left(x^2y^{\beta-2}-xy^{2\beta-2}\right)\right]. \\ v(x,y) &= y+\left[y^{2\beta-1}-\frac{1}{K}\,y^{\beta-1}\,u(x,y)\right]B\,. \end{aligned}\right\} \tag{6.1}$$

For $0 \leqslant x \leqslant y^\beta$, $0 \leqslant y \leqslant 1$, we define

$$\left.\begin{aligned} u(x, y) &= K\left[x + \frac{B(1-\beta)}{2}(x^2 y - x y^{\beta+1})\right] \\ v(x, y) &= y + \left(y - \frac{1}{K} y^{1-\beta} u\right) B . \end{aligned}\right\} \tag{6.2}$$

Furthermore, let $u(-x, y) = -u(x, y)$, $v(-x, y) = v(x, y)$. B will again be a sufficiently small positive constant.

It is clear form the definition of v as a function of u that f maps horizontal segments onto precisely the same images as in § 4. Since $u_x > 0$ when B is sufficiently small, (see below) the mapping is a homeomorphism.

For $y \geqslant 1$ we find the partial derivatives to be as follows:

$$u_x = K[1 + \varphi_1(x, y) B], \ \varphi_1(x, y) = \frac{1-\beta}{2}(2xy^{\beta-2} - y^{2\beta-2})$$

$$u_y = \varphi_2(x, y) B, \ \varphi_2(x, y) = \frac{K(1-\beta)}{2}[(\beta-2) x^2 y^{\beta-3} - (2\beta-2) x y^{2\beta-3}]$$

$$v_x = -y^{\beta-1} B - y^{\beta-1} \varphi_1 B^2$$

$$v_y = 1 + \varphi_3(x, y) B + \varphi_4(x, y) B^2, \ \varphi_3(x, y) = (2\beta-1) y^{2\beta-2} - (\beta-1) x y^{\beta-2}$$

$$\varphi_4(x, y) = \frac{(1-\beta)^2}{2} y^{\beta-2}(x^2 y^{\beta-2} - x y^{2\beta-2})$$

$$- \frac{1-\beta}{2}[(\beta-2) x^2 y^{2\beta-4} - (2\beta-2) x y^{3\beta-4}].$$

The condition (4.2) becomes, as $B \to 0$,

$$\frac{K^2[1 + 2\varphi_1 B] + 1 + 2\varphi_3 B + 0(B^2)}{K[1 + (\varphi_1 + \varphi_3) B + 0(B^2)]} \leqslant \frac{K^2+1}{K}. \tag{6.3}$$

It is easy to verify that

$$|\varphi_n(x, y)| \leqslant C y^{2\beta-2}, \ n = 1, 2, 3, 4, \ (0 \leqslant x \leqslant y^\beta, \ y \geqslant 1),$$

for some constant C. The $0(B^2)$ terms in (6.3) are therefore bounded by

$$\text{const} \cdot B^2 y^{2\beta-2}.$$

Simplifying (6.3) we therefore obtain

$$(K^2 - 1)(\varphi_3 - \varphi_1) \geqslant 0(B) \tag{6.4}$$

where

$$0(B) \leqslant \text{const} \cdot B y^{2\beta-2}$$

To satisfy (6.3) for sufficiently small B it is therefore sufficient if $K>1$, and if

$$\frac{\varphi_1 - \varphi_1}{y^{2\beta-2}}$$

is positive, and bounded away from 0. In fact, we find

$$\frac{\varphi_3 - \varphi_1}{y^{2\beta-2}} = \frac{3\beta - 1}{2} > 0.$$

We next consider $0 \leqslant y \leqslant 1$. We find

$$u_x = K[1 + \psi_1 B], \psi_1 = \frac{1-\beta}{2}(2xy - y^{\beta+1})$$

$$u_y = \psi_2 B,$$

$$v_x = - y^{1-\beta} B - y^{1-\beta} \psi_1 B^2,$$

$$v_y = 1 + \psi_3 B + \psi_4 B^2, \psi_3 = 1 - (1-\beta)\, x y^{-\beta}.$$

The functions ψ_n are all bounded for $0 \leqslant x \leqslant y^\beta \leqslant 1$. To satisfy (4.2) for sufficiently small B it is, by analogy with (6.3), (6.4), therefore sufficient that $\psi_3 - \psi_1$ is positive and bounded away from 0. We find, in fact,

$$\begin{aligned}\psi_3 - \psi_1 &= 1 - (1-\beta)\, x y^{-\beta} - (1-\beta)\, xy + \frac{(1-\beta)}{2} y^{\beta+1}\\ &\geqslant 1 - (1-\beta) - (1-\beta)\, y^{\beta+1} + \frac{1-\beta}{2} y^{\beta+1} = \beta - \frac{1-\beta}{2} y^{\beta+1}\\ &\geqslant \beta - \frac{1-\beta}{2} = \frac{3\beta-1}{2} > 0.\end{aligned}$$

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