# Iterated Suspensions.

Autor(en): **Berstein, I. / Ganea, T.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **45 (1970)**

PDF erstellt am: **27.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-34665

# Iterated Suspensions

by I. BERSTEIN and T. GANEA[1])

We give conditions for a space $Y$ to have the homotopy type of a $k$-fold suspension. Our first main result is Theorem 1.4, which states that, under certain connectivity and dimension assumptions, $Y \approx \Sigma^k W$ if and only if the evaluation map $\Sigma^k \Omega^k Y \to Y$ admits a homotopy cross-section. The corresponding theorem for desuspending maps rather than spaces is 1.6. The results of the first section are used to prove a more effective criterion for the case $k=2$, answering a question raised by A. Haefliger. Specifically, Theorem 2.7 states that a $(n-1)$-connected, $(2n-1)$-dimensional finite CW-complex $Y$, $n \geqslant 3$, is homotopically equivalent to a double suspension if and only if $Sq^{n-1}$ (for $n$ odd), or $\delta Sq^{n-2}$ (for $n$ even), where $\delta$ is the integral Bockstein, vanish in $Y$. Examples are given which show that our results are best possible.

Our basic tool is Lemma 1.1 which is of an independent interest. Both 1.1 and 1.4 can be used to give new proofs for certain known results.

§ 1. The spaces we consider are supposed to have the based homotopy type of CW-complexes: all maps and homotopies respect base-points. Using [9] one can easily see that all the constructions in this paper will keep us inside this category. As usual, if $A$ is a space, $CA$ is the reduced cone over $A$ with vertex at 0. $\Sigma A$ is the reduced suspension, $PA$ is the space of paths in $A$ emanating from $*$, and $\Omega A$ is the loop-space. The smashed product of $A$ and $B$ is denoted by $A \divideontimes B = A \times B / A \vee B$ and their join by $A * B$. If $s \in [0, 1]$, $a \in A$ then the class of $(s, a)$ in $CA$ is denoted by $sa$ and in $\Sigma A$ the corresponding class is $\langle s, a \rangle$; the class of $(a, b)$ in $A \divideontimes B$ is $a \divideontimes b$ and the class of $(s, a, b)$ in $A * B$ is $(1-s)\, a \oplus sb$. Finally, the reduced diagonal $\Delta : A \to A \divideontimes A$ is defined by $\Delta(a) = a \divideontimes a$.

We start by proving

LEMMA 1.1. *If $Y$ is $(n-1)$-connected, $Z$ is 0-connected, $\pi_1(Z)$ is abelian, and if $g: Y \to \Sigma^k Z$ is $m$-connected with $m \geqslant n-1 \geqslant k \geqslant 1$, then there exists an $(n-1-k)$-connected space $X$ and an $(m+n-k)$-connected map $f: \Sigma^k X \to Y$ such that $g \circ f$ is homotopic to a $k$-fold suspension.*

[1]) Partially supported by NSF grants.

*Proof.* To simplify notation, set $A=\Sigma^{k-1}Z$. Consider the diagram

$$\begin{tikzcd}
F \arrow[r, "j"] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & A \cup CF \arrow[r] \arrow[d, "u"] & CA \cup CF \arrow[r, "\varphi"] \arrow[d, "v"] & \Sigma F \\
F \arrow[r, "j"] & A \arrow[r, "\partial\circ e"] \arrow[d, "e"] & G \arrow[r] \arrow[d, equal] & G \cup CA \arrow[r, "h"] \arrow[d, "p"] & CG \cup CA \arrow[d, "q"] \\
 & \Omega\Sigma A \arrow[r, "\partial"] & G \arrow[r, "i"] & Y \arrow[r, "d"] & Y \cup CG \arrow[r, "r"] & \Sigma A
\end{tikzcd}$$

where $G$ is the "fibre" of $g$ with "inclusion" $i$ so that

$$G=\{(y,\sigma)\in Y\times P\Sigma A \mid g(y)=\sigma(1)\} \quad\text{and}\quad i(y,\sigma)=y.$$

Let $d$ be the inclusion, and let $r$ be the natural extension of $g$ given by $r(s(y,\sigma))=\sigma(s)$. By the Serre theorem, $r$ is $(m+n)$-connected. Let $\partial$ and $e$ be the inclusion and the natural embedding, respectively; thus $\partial(\omega)=(*,\omega)$ and $e(a)(s)=\langle s,a\rangle$. Since $i\circ\partial=*$, we may extend $i$ to a map $p$ by setting $p(CA)=*$. Also, let $q$ be the map defined by $p$ and the identity map of $CG$. The homotopy $h_t=CG\cup CA\to\Sigma A$ given by $h_t(s(y,\sigma))=$ $=\sigma(st)$ and $h_t(sa)=\langle(1-s)(1-t)+t,\ a\rangle$ reveals that $r\circ q$ is homotopic to a map $\gamma$ which collapses $CG$ to the base-point followed by inversion of $\Sigma A$. Hence, by [10, Satz 3, p. 309], $r\circ q$ is a homotopy equivalence, and the connectivity of $r$ implies that $q$ is $(m+n-1)$-connected so that, by the five-lemma, $p$ is homology $(m+n-1)$-connected. Next, let $F$ be the "fibre" of $\partial\circ e$ with "inclusion" $j$, let $u$ be the natural extension of $\partial\circ e$, and let $v$ be defined by $u$ and the indentity map of $CA$. Since $\pi_1(A)$ is abelian if $k=1$ and zero if $k\geqslant 2$, the connectivity of $\Sigma A$ implies that $A$ is $(n-2)$-connected and so $e$ is $(2n-3)$-connected; also, $\partial$ is $(n-1)$-connected. It follows that $F$ is $(n-2)$-connected and, since $G$ is $(m-1)$-connected, the Serre theorem in the form given in [5, Prop. 2.1, p. 301] implies that $u$ is $(m+n-1)$-connected so that, by the 5-lemma, $v$ is homology $(m+n-1)$-connected. By [10, ibid.] the map $\varphi$ which collapses $CA$ to the base-point is a homotopy equivalence; let $\psi$ be its inverse. Let $h$ in the diagram denote inclusion, and let $\theta:\Sigma A\to\Sigma A$ be inversion; then, $\gamma\circ h\circ v\simeq\theta\circ\Sigma j\circ\varphi$ via the homotopy $h_t: CA\cup CF\to\Sigma A$ given by $h_t(sa)=\langle s(1-t),\ a\rangle$ and $h_t(sx)=$ $=\langle(1-s)t+1-t,\ j(x)\rangle$, $a\in A$, $x\in F$. Therefore,

$$g\circ p\circ v\circ\psi=r\circ q\circ h\circ v\circ\psi\simeq\theta\circ\gamma\circ h\circ v\circ\psi\simeq\Sigma j$$

and, since $\Sigma F$ and $Y$ are 1-connected,

$$p\circ v\circ\psi \text{ is } (m+n-1)\text{-connected.}$$

If $k=1$, we take $X=F$ and $f=p\circ v\circ\psi$. If $k\geqslant 2$, we assume the result to be true for $k-1$ and note that it may be applied to $j$. There results an $(n-2-(k-1))$-connected space $X$ and an $(m-1+n-1-(k-1))$-connected map $f':\Sigma^{k-1}X\to F$ such that $j\circ f'$ is homotopic to a $(k-1)$-fold suspension, and to conclude the proof it only remains to set $f=p\circ v\circ\psi\circ\Sigma f'$.

*Remark* 1.2. Obviously, the preceding construction is *functorial* in the sense that there is a map $\xi$ yielding homotopy-commutativity in the first square of the diagram

$$\begin{array}{ccccc} \Sigma^k X & \xrightarrow{f} & Y & \xrightarrow{g} & \Sigma^k Z \\ \downarrow{\scriptstyle \Sigma^k\xi} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \Sigma^k\zeta} \\ \Sigma^k X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & \Sigma^k Z' \end{array}$$

provided the second square homotopy-commutes for some maps $\eta$ and $\zeta$. Another equally obvious property is the fact that $f$ is a homotopy equivalence in case $g$ is one. It follows that for any 1-connected space $W$ and any map $h:\Sigma^k W\to Y$ such that $g\circ h$ is homotopic to a $k$-fold suspension, there is a map $\lambda: W\to X$ such that $f\circ\Sigma^k\lambda\simeq h$; for, in the diagram

$$\begin{array}{ccccc} \Sigma^k X' & \xrightarrow{f'} & \Sigma^k W & \xrightarrow{1} & \Sigma^k W \\ \downarrow{\scriptstyle \Sigma^k\xi} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma^k\zeta} \\ \Sigma^k W & \xrightarrow{f} & Y & \xrightarrow{g} & \Sigma^k W \end{array}$$

the second square homotopy-commutes for some $\zeta$, hence the first one homotopy commutes for some $\xi$, and $1\circ f'\simeq\Sigma^k\psi$ for some $\psi$; also, $f'$ is a homotopy equivalence and $X'$ is 1-connected since so is $W$; therefore, $\psi$ is a homotopy equivalence and its inverse $\varphi: W\to X'$ satisfies $f\circ\Sigma^k(\xi\circ\varphi)\simeq h$.

Amusingly, we may derive the well-known

COROLLARY 1.3. *If $Y$ is $(n-1)$-connected and $n\geqslant k\geqslant 1$, the evaluation map $\Sigma^k\Omega^k Y\to Y$ is $(2n-k)$-connected.*

*Proof.* If $n-1\geqslant k\geqslant 1$, take $Z=*$ in Lemma 1.1 and note that $F$ may be replaced by $\Omega Y$; if $n=k$, note that any map $S^n\to Y$ lifts to a map $S^n\to\Sigma^n\Omega^n Y$ because $S^n$ is an $n$-fold suspension.

Next, if $Y$ has the homotopy type of a $k$-fold suspension, then obviously there is a homotopy cross-section of the evaluation map. A partial converse of this fact constitutes the main result of this section.

THEOREM 1.4. *If $Y$ is $(n-1)$-connected and if there is a homotopy cross-section $g: Y\to\Sigma^k\Omega^k Y$ of the evaluation map, then $Y$ has the homotopy type of a k-fold suspension provided* dim $Y\leqslant 3n-2k-1$ *and* $n-1\geqslant k\geqslant 1$.

*Proof.* By 1.3, $g$ is $(2n-k-1)$-connected. Then, 1.1 yields a 0-connected space $X$ and a $(3n-2k-1)$-connected map $f:\Sigma^k X\to Y$. Next, $H_{3n-2k-1}(Y)$ is free and $H_q(Y)=0$ for $q>3n-2k-1$. Hence, the homology decomposition result proved in [4, Theorem 2.1] yields a connected CW-complex $W$ and a map $h: W\to X$ such that $h_q: H_q(W)\to$

$\to H_q(X)$ is isomorphic for $q<3n-3k-1$, $f_{3n-2k-1}\circ(\Sigma^k h)_{3n-2k-1}$ is isomorphic, and $H_q(W)=0$ for $q>3n-3k-1$. Obviously, $f\circ\Sigma^k h:\Sigma^k W\to Y$ is a homotopy equivalence.

*Remark* 1.5. Let $\alpha_i:A_i\to\Sigma^k\Omega^k A_i$ be homotopy cross-sections of the evaluation maps, $i=1, 2$. We describe a map $\phi:A_1\to A_2$ as *primitive* if $\alpha_2\circ\phi\simeq\Sigma^k\Omega^k\phi\circ\alpha_1$; when $k=1$, this may be shown to be equivalent to the concept of primitivity used in [4]. Then, using the fact that $g\circ f$, hence $g\circ f\circ\Sigma^k h$, is homotopic to a $k$-fold suspension, it is easily proved that the homotopy equivalence $f\circ\Sigma^k h$ in the preceding proof is primitive with respect to $g$ and $\Sigma^k e$, where $e: W\to\Omega^k\Sigma^k W$ is the natural embedding.

A companion to 1.4 is given by

THEOREM 1.6. *Suppose $\phi:\Sigma^k Y_1\to\Sigma^k Y_2$ is primitive with respect to $\Sigma^k e_1$ and $\Sigma^k e_2$. Then $\phi$ is homotopic to a k-fold suspension provided $Y_2$ is $(n-1)$-connected, $Y_1$ is* 1*-connected and* dim $Y_1\leqslant 3n-2$, $n\geqslant 2$.

*Proof.* Consider the diagram

$$\begin{array}{ccccccc}
\Sigma^k Y_1 & \xrightarrow{\Sigma^k\lambda_1} & \Sigma^k X_1 & \xrightarrow{f_1} & \Sigma^k Y_1 & \xrightarrow{\Sigma^k e_1} & \Sigma^k\Omega^k\Sigma^k Y_1 \\
\downarrow{\scriptstyle \Sigma^k\mu} & & \downarrow{\scriptstyle \Sigma^k\xi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \Sigma^k\Omega^k\phi} \\
\Sigma^k Y_2 & \xrightarrow{\Sigma^k\lambda_2} & \Sigma^k X_2 & \xrightarrow{f_2} & \Sigma^k Y_2 & \xrightarrow{\Sigma^k e_2} & \Sigma^k\Omega^k\Sigma^k Y_2
\end{array}$$

in which, by 1.2, the second square homotopy commutes for some $\xi$ since, by assumption, so does the third. Applying the last part of 1.2 with $h=1$, we obtain maps $\lambda_i$ such that $f_i\circ\Sigma^k\lambda_i\simeq 1$. Since, $f_2$ is $(3n+k-1)$-connected, and both $Y_2$ and $X_2$ are 1-connected, $\lambda_2$ is $(3n-2)$-connected so that, since dim $Y_1\leqslant 3n-2$, there is a map $\mu$ with $\lambda_2\circ\mu\simeq\xi\circ\lambda_1$. Obviously, $\phi\simeq\Sigma^k\mu$.

The preceding theorem yields a unicity result concerning the process of desuspending:

COROLLARY 1.7. *Suppose $Y_1$ and $Y_2$ are both $(n-1)$-connected and of dimension $\leqslant 3n-2$, $n\geqslant 2$. If $\Sigma^k Y_1$ and $\Sigma^k Y_2$ have the same "primitive" homotopy type (with respect to $\Sigma^k e_1$ and $\Sigma^k e_2$) for some $k\geqslant 1$, then $Y_1$ and $Y_2$ have the same homotopy type.*

*Remark* 1.8. The "primitivity" is obviously necessary. The condition dim $\leqslant 3n-2$ is best possible as shown by the following example. Let $Y_1$ and $Y_2$ result respectively by attaching a 17-cell to $S^6$ by means of the trivial element and a non-trivial element in the kernel of the suspension

$$Z_{72}\times Z_2=\pi_{16}(S^6)\to\pi_{17}(S^7)=Z_{24}\times Z_2.$$

Then, $Y_1\not\approx Y_2$ but $\Sigma Y_1\approx S^7\vee S^{18}\approx\Sigma Y_2$. The homotopy equivalences are certainly primitive since $S^7\vee S^{18}$ supports a single homotopy class of comultiplications, as implied by the formula on the homotopy groups of wedges of spheres [7] and the fact that $\pi_{18}(S^{13})=\pi_{18}(S^{19})=\cdots=0$.

When $k=1$, 1.4 and 1.6 yield alternative proofs of the main results in [4] without assuming any homology to be finitely generated. Let now $Y=S^3\cup e^7$ result upon attaching the 7-cell by means of an element of order 3 in $\pi_6(S^3)$. Then, $Y$ fails to have the homotopy type of a suspension but is a co-$H$-space [3, p. 444] and therefore, by [8, Chapter 7], admits a homotopy cross-section $Y\to\Sigma\Omega Y$. It follows easily that, for any $k\geqslant 1$, $M=\Sigma^{k-1}Y$ admits a homotopy cross-section $M\to\Sigma^k\Omega^k M$; also, $M$ is $(k+1)$-connected and dim $M=k+6$, i.e., one unit higher than allowed in 1.4. We prove that $M$ fails to have the homotopy type of a $k$-fold suspension, thus showing that 1.4 yields a best possible result. Suppose $M\approx\Sigma^k W$ for some $W$. Then, the Steenrod operation $P^1: H^2(W, \mathbf{Z}_3)\to H^6(W, \mathbf{Z}_3)$ vanishes since $P^1(x)=x^3$, the cup-cube, and $H^4(W, \mathbf{Z}_3)=H^5(Y, \mathbf{Z}_3)=0$; since $P^1$ commutes with suspension, this contradicts the fact [11, p. 89, Corollary 2] that $P^1: H^{k+2}(M, \mathbf{Z}_3)\to H^{k+6}(M, \mathbf{Z}_3)$ is isomorphic.

*Remark* 1.9. For $k=2$ an alternative proof of 1.4 may be given as follows. The composite

$$Y\xrightarrow{g}\Sigma^2\Omega^2 Y\xrightarrow{S}\Sigma^2\Omega^2 Y\vee\Sigma^2\Omega^2 Y\xrightarrow{p\vee p}Y\vee Y$$

where $S$ is suspension comultiplication and $p$ the evaluation map, is a homotopy-commutative comultiplication on $Y$. By [4, Th. A], $Y$ has the primitive homotopy type of $\Sigma X$ for some $X$ which, by [3, p. 443], may be assumed 1-connected, hence $(n-2)$-connected, and of dimension $\leqslant 3n-6$. Since its suspension is homotopy-commutative, $X$ is a co-$H$-space by [2, Th. 1], hence a suspension by [4, Th. A]. It would be interesting to know how this type of proof generalizes to higher values of $k$. Conversely, the presence of a homotopy-commutative comultiplication does not imply, in general, the presence of a homotopy cross-section $Y\to\Sigma^2\Omega^2 Y$; for, the space $Y=\Sigma K(Q, 1)$, where $Q$ is the group of rationals and $K(Q, 1)$ is the Eilenberg-MacLane space, has a homotopy-commutative comultiplication [6, Example 5.3] but no homotopy cross-section exists since $H_2(Y)$ is not free.

*Remark* 1.10. If $Y$ is a *finite* $(n-1)$-connected complex and $Y\approx\Sigma^k X$, then there exists a *finite* complex $W$ such that $Y\approx\Sigma^k W$ provided $n-2\geqslant k\geqslant 1$. For, $\Sigma X$ is $(n-k)$-connected and, since $n-k\geqslant 2$, there exists [3, p. 443] a 1-connected $CW$-complex $W$ such that $\Sigma X\approx\Sigma W$. Since $W$ has finitely generated homology, vanishing above a certain dimension, $\pi_1(W)=0$ enables us to assume that $W$ is finite. As noticed by P. J. Hilton, a slightly more complicated argument, based on [4, Th. 2.1], shows that the Remark is true even for $n=k+1$.

§ **2.** We shall first prove here two results of a rather technical nature. We describe two maps $g, h: X \to Y$ as $r$-homotopic if $g \circ f \simeq h \circ f$ for any map $f: K \to X$, where $K$ is any $r$-dimensional $CW$-complex; thus we may speak of an $r$-homotopy commutative diagram.

LEMMA 2.1. *If $X$ is $(n-2)$-connected, then the diagram*

$$\begin{tikzcd}
\Omega\Sigma X * \Omega\Sigma X \arrow[r, "j"] \arrow[d, "\theta"] & \Sigma\Omega\Sigma X \arrow[d, "\Sigma\Delta"] \\
\Sigma(\Omega\Sigma X \divideontimes \Omega\Sigma X) \arrow[r, "1+\Sigma\tau"] & \Sigma(\Omega\Sigma X \divideontimes \Omega\Sigma X)
\end{tikzcd}$$

*$(3n-3)$-homotopy commutes; here $j$ results by the Hopf construction from loop multiplication, $\Delta$ is the reduced diagonal map, $\theta$ the natural homotopy equivalence, $\tau$ is the map which interchanges factors, and "$+$". refers to track addition.*

*Proof.* One has $j((1-s)\,\alpha \oplus s\beta) = \langle s, \alpha+\beta \rangle$, and $\theta((1-s)\,\alpha \oplus s\beta) = \langle s, \alpha \divideontimes \beta \rangle$. Let $e: X \to \Omega\Sigma X$ be the natural embedding, and let $P$ be the composite

$$\Sigma(\Omega\Sigma X \divideontimes \Omega\Sigma X) \to \Sigma\Omega\Sigma X \divideontimes \Omega\Sigma X \xrightarrow{p \divideontimes 1} \Sigma X \divideontimes \Omega\Sigma X \to X \divideontimes \Sigma\Omega\Sigma X \xrightarrow{1 \divideontimes p} X \divideontimes \Sigma X$$
$$\to \Sigma(X \divideontimes X)$$

where $p$ is the evaluation map and the remaining arrows are the obvious homotopy equivalences. Define

$$h_t: X * X \to \Sigma(X \divideontimes X) \text{ by } h_t((1-s)\,a \oplus sb) = \begin{cases} \langle 4st+1-t, a \divideontimes a \rangle & \text{if } 0 \leqslant 4s \leqslant 1, \\ \langle 4s-1, a \divideontimes b \rangle & \text{if } 1 \leqslant 4s \leqslant 2, \\ \langle 4s-2, b \divideontimes a \rangle & \text{if } 2 \leqslant 4s \leqslant 3, \\ \langle (4s-3)\,t, b \divideontimes b \rangle & \text{if } 3 \leqslant 4s \leqslant 4. \end{cases}$$

Then, $h_0 \simeq P \circ (1+\Sigma\tau) \circ \theta \circ (e * e)$ and $h_1 = P \circ \Sigma\Delta \circ j \circ (e * e)$, and the result follows since $e * e$ is $(3n-3)$-connected whereas $P$ is $(3n-2)$-connected.

We maintain the notation and prove

LEMMA 2.2. *Suppose $X$ is $(n-2)$-connected and* $\dim X \leqslant 3n-6$. *Then, $\Sigma X$ has the homotopy type a double suspension if and only if there is a map $\xi: \Sigma X \to \Omega\Sigma X * \Omega\Sigma X$ with $(1+\Sigma\tau) \circ \theta \circ \xi \simeq \Sigma(\Delta \circ e)$, where $e: X \to \Omega\Sigma X$ is the natural embedding.*

*Proof.* Let $Y = \Sigma X$ and introduce the diagram

$$\begin{tikzcd}
\Sigma^2\Omega^2 Y \arrow[r, "\Sigma q"] & \Sigma\Omega Y \arrow[r, "\Sigma r"] \arrow[d, shift right, "p"'] \arrow[dr, "\Sigma\Delta"] & \Sigma Q \arrow[d, "\Sigma h"] \\
\Omega Y * \Omega Y \arrow[ur, "j"] & Y \arrow[l, "\xi"'] \arrow[u, shift right, "\Gamma"'] & \Sigma(\Omega Y \divideontimes \Omega Y)
\end{tikzcd}$$

where $p$ is the evaluation map, $q$ is the evaluation map for $\Omega Y$, and $Q$ the "cofibre" of $q$ with "projection" $r$. By [5, Th. 2.3], there is a $(3n-4)$-connected map $h$ with

$h \circ r \simeq \Delta$. It follows easily that an arbitrary map $\Gamma$ compresses to $\Sigma^2\Omega^2 Y$ if and only if $\Sigma\Delta \circ \Gamma \simeq 0$. Also, a compression of $\Gamma$ is a homotopy cross-section of $p \circ \Sigma q$ if and only if $\Gamma$ is one of $p$. The "fibre" of $p$ has the homotopy type of the join $\Omega Y * \Omega Y$, and the "inclusion" $j$ results by the Hopf construction from loop multiplication [1], [5, p. 303]. Therefore, any homotopy cross-section of $p$ is of the form $\Gamma = \Sigma e - j \circ \xi$, where "$-$" indicates track subtraction on $\Sigma X$. By 2.1, $\Sigma\Delta \circ \Gamma = \Sigma(\Delta \circ e) - (1 + \Sigma\tau) \circ \theta \circ \xi$ and the result now follows from 1.4.

In order to transform the condition in 2.2 into a cohomological one we need.

LEMMA 2.3. *Suppose that $X$ is $(n-2)$-connected, $\pi = \pi_{n-1}(X)$, and that $T: \pi \otimes \pi \to \pi \otimes \pi$ is given by $T(a \otimes b) = (-1)^{n-1} b \otimes a$. If $u \in H^{n-1}(X, \pi)$, $U \in H^{2n-2}(X \circledast X, \pi \otimes \pi)$ are the fundamental classes, then*

i) $T_*(U) = \tau^*(U)$ *and*

ii) $T_*(u^2) = \tau^*(u^2) = u^2$.

*Proof.* Since $\tau$ permutes the factors in $X \circledast X$, *i)* follows directly from the definition of $U$; ii) follows from i) since $u^2 = \Delta^*(U)$, $\Delta^*$ commutes with $T_*$, and $\Delta^* = \Delta^* \circ \tau^*$.

LEMMA 2.4. *Let $X$ be as above,* $\dim X \leqslant 2n-2$, *and let $\pi = \pi_{n-1}(X)$ be finitely generated abelian. Suppose further that*

i) $Sq^{n-1} \mid H^{n-1}(X, \mathbf{Z}_2) = 0$ *if $n$ is odd;*

ii) $\delta Sq^{n-2} \mid H^{n-1}(X, \mathbf{Z}_2) = 0$ *if $n$ is even, where*

$\delta: H^{2n-3}(X, \mathbf{Z}_2) \to H^{2n-2}(X, \mathbf{Z})$ *is the integral Bockstein coboundary operator.*

*Then the class $u^2 \in H^{2n-2}(X, \pi \otimes \pi)$ lies in the image of* $1 + T_*$.

*Proof.* Let $\pi = \Sigma_i A_i$ $i = 1, \ldots, m$, where $A_i$ are infinite or primary cyclic groups. Then $\pi \otimes \pi = \Sigma_{i<j} A_{ij} \oplus \Sigma_k B_k$, where $A_{ij} = (A_i \otimes A_j) \oplus (A_j \otimes A_i)$, $B_k = A_k \otimes A_k$. Clearly $A_{ij}$ and $B_k$ are invariant under $T$.

Let $\kappa_{ij}: \pi \otimes \pi \to A_{ij}$ be the projections and $u_{ij} = \kappa_{ij*}(u^2)$. Then it follows from 2.3 ii), that $(1 - T_*) u_{ij} = 0$. The sequence

$$H^{2n-2}(X, A_{ij}) \xrightarrow{1+T_*} H^{2n-2}(X, A_{ij}) \xrightarrow{1-T_*} H^{2n-2}(X, A_{ij})$$

is easily seen to be exact, so that

$$u_{ij} \in \operatorname{Im}(1 + T_*). \tag{2.5}$$

Similarly, let $\kappa_k: \pi \to A_k$ be the projection; then if $u_k = \kappa_{k*}(u)$ we have $(\kappa_k \otimes \kappa_k)_*(u^2) = u_k^2 \in H^{2n-2}(X, B_k)$.

a) *$n$ is odd.* Then $T \mid B_k =$ identity, and $1 + T_*$ is multiplication by 2 on $H^{2n-2}(X, B_k)$ Let $\lambda$ be the composition $A_k \to A_k/2A_k \to \mathbf{Z}_2$ (the second map is either an isomorphism or zero) and let $\mu$ be the composition $B_k \xrightarrow{\lambda \otimes \lambda} \mathbf{Z}_2 \otimes \mathbf{Z}_2 \xrightarrow{\approx} \mathbf{Z}_2$. Then the sequence $B_k \xrightarrow{2} B_k \xrightarrow{\mu} \mathbf{Z}_2$ is exact. By hypothesis, $\mu_*(u_k^2) = (\lambda_* u_k)^2 = Sq^{n-1}(\lambda_k u_k) = 0$, so that $\dim X \leqslant 2n-2$, and the exactness of the cohomology sequences induced by $\mu$ and by multiplication by 2, imply

$$u_k^2 = \operatorname{Im} 2 = \operatorname{Im}(1 + T_*). \tag{2.6}$$

b) *n is even.* This time $T \mid B_k$ is multiplication by $-1$ so that $\operatorname{Im}(1+T_*)=0$.

If $\delta$ is the integral Bockstein coboundary, corresponding to the sequence of coefficients $0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}_2 \to 0$ and if $\delta_q$ is the similar coboundary for the sequence $0 \to \mathbf{Z}_{2^q} \to \mathbf{Z}_{2^{q+1}} \to \mathbf{Z}_2 \to 0$ then $\delta_q = \varrho_{q*} \circ \delta$, where $\varrho_q$ is reduction mod $2^q$. Therefore ii) implies

$$\text{(ii}')\ \delta_q Sq^{n-2} \mid H^{n-1}(X, \mathbf{Z}_2) = 0 \quad \text{for all } q.$$

If $A_k = B_k = \mathbf{Z}$ or if $A_k = B_k = \mathbf{Z}_{2^q}$, the identities $u_k^2 = \delta Sq^{n-2}(\varrho_{1*} u_k)$ or $u_k^2 = \delta_q Sq^{n-2}(\varrho_{1*} u_k)$ hold universally (it is enough to verify then for $X = K(\mathbf{Z}, n-1)$ and $X = K(\mathbf{Z}_{2^q}, n-1)$, where they follow from known calculations). Those identities, together with ii) and ii′), and the anticommutativity of cup-product imply $u_k^2 = 0$, i.e., (2.6) is true for even $n$ too.

The conclusion of Lemma 2.4 now follows from (2.5) and (2.6) since $u^2 = \sum_{i<j} u_{ij} + \sum_k u_k^2$.

We are finally able to prove the main result of this section, which is

THEOREM 2.7. *Suppose that $Y$ is an $(n-1)$-connected space,* $\dim Y \leqslant 2n-1$, $n \geqslant 3$, *and suppose that $\pi = \pi_n(Y)$ is finitely generated. Then $Y$ is homotopically equivalent to a double suspension if and only if*

i) $Sq^{n-1} \mid H^n(Y, \mathbf{Z}_2) = 0$ *for n odd, or*

ii) $\delta Sq^{n-2} \mid H^n(Y, \mathbf{Z}_2) = 0$ for *n even.*

*Proof.* Necessity is immediate: if $Y \approx \Sigma^2 W$ both $Sq^{n-1}$ and $\delta Sq^{n-2}$ vanish in $W$ for dimensional reasons and therefore, by stability of these operations, they vanish also in $Y$.

To prove sufficiency we first notice that $Y \approx \Sigma X$ for some $X$ (this is well known and also an immediate consequence of 1.3 and 1.4). Let $V \in H^{2n-2}(\Omega\Sigma X \circledast \Omega\Sigma X, \pi \otimes \pi)$ be the fundamental class. If $e: X \to \Omega\Sigma X$ is the natural embedding, then, since $(e \circledast e) \circ \Delta = \Delta \circ e$ and $(e \circledast e)^*(V) = U \in H^{2n-2}(X \circledast X, \pi \otimes \pi)$, i) or ii) imply by 2.4 that $(1+T_*)(w) = u^2 = \Delta^*(U) = (\Delta \circ e)^*(V)$ for some $w \in H^{2n-2}(X, \pi \otimes \pi)$. If we denote by $\Sigma w \in H^{2n-1}(\Sigma X, \pi \otimes \pi)$ and by $\Sigma V = H^{2n-1}(\Sigma(\Omega\Sigma X \circledast \Omega\Sigma X), \pi \otimes \pi)$ the images of $w$ and $V$ under suspension, we obtain

$$(1 + T_*)(\Sigma w) = (\Sigma(\Delta \circ e))^* (\Sigma V).$$

Since in 2.1 and 2.2 the map $\theta$ is a homotopy equivalence and $\Sigma V$ is a fundamental class, the assumption that $\dim \Sigma X \leqslant 2n-1$ implies the existence of a $\xi: \Sigma X \to \Omega\Sigma X * \Omega\Sigma X$

such that $(\theta\circ\xi)^*(\Sigma V)=\Sigma w$. We now get

$$(\theta\circ\xi)^*(1+T_*)(\Sigma V)=(\Sigma(\Delta\circ e))^*(\Sigma V).$$

It follows from 2.3 i) that $T_*(\Sigma V)=(\Sigma\tau)^*(\Sigma V)$ so that

$$(\theta\circ\xi)^*(1+\Sigma\tau)^*(\Sigma V)=(\Sigma(\Delta\circ e))^*(\Sigma V).$$

Again since $\dim\Sigma X\leqslant 2n-1$ and $\Sigma V$ is a fundamental class, we obtain $(1+\Sigma\tau)\circ\theta\circ\xi\simeq\Sigma(\Delta\circ e)$ which, by 2.2, yields the needed result.

*Remark* 2.8. The restriction $n\geqslant 3$ in 2.7 is necessary. Indeed if $n=2$, a 3-dimensional 1-connected CW-complex is a double suspension if and only if it has the homotopy type of a wedge of 2-spheres and 3-spheres. On the other hand, $Y=S^2\cup_3 e^3$ (the attaching map is of degree 3) clearly satisfies condition 2.7 ii).

REFERENCES


[1] W. D. Barcus and J. P. Meyer, *The suspension of a loop-space*, Amer. J. Math. *80* (1958), 895–920.
[2] I. Berstein and T. Ganea, *On the homotype-commutativity of suspensions*, Illinois J. Math. *6* (1962), 336–340.
[3] I. Berstein and P. J. Hilton, *Category and generalized Hopf invariant*, Illinois J. Math. *4* (1960), 437–452.
[4] ——, *On suspensions and comultiplications*, Topology, *2* (1963), 73–82.
[5] T. Ganea, *A generalization of the homology and homotopy suspensions*, Comment. Math. Helvetici *39* (1965), 295–322.
[6] T. Ganea, P. J. Hilton and F. P. Peterson, *On the homotopy commutativity of loop-spaces and suspensions*, Topology *1* (1962), 133–141.
[7] P. J. Hilton, *On the homotopy groups of wedges of spheres*, J. London Math. Soc. *30* (1955), 154–172.
[8] ——, *Homotopy theory and duality* (Gordon and Breach, New York 1965).
[9] J. W. Milnor, *On spaces having the homotopy type of CW-complexes*, Trans. Amer. Math. Soc. *90* (1959), 272–280.
[10] D. Puppe, *Homotopiemengen und ihre induzierten Abbildungen I*, Math. Z. *69* (1958), 299–344.
[11] N. E. Steenrod and D. B. A. Epstein, *Cohomology operations* (Princeton University Press, Princeton 1962).


*Cornell University, University of Washington*

Received Novembei 28, 1969