Pinching Implies Strong Pinching

Autor(en): **Karcher, Hermann**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **46 (1971)**

PDF erstellt am: **27.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-35510

Pinching Implies Strong Pinching

by HERMANN KARCHER[1])

The author was led to look for bounds as in (2) by a preprint manuscript of the article by E. Ruh which precedes this paper. Denote by δ the minimum, by Δ the maximum of the sectional curvatures at $p\in M$ ($\dim M=n$) and by λ the smallest, by Λ the largest eigenvalue of the curvature operator $\varrho: M_p\wedge M_p\to M_p\wedge M_p$ (defined by $\varrho\omega(X, Y) := \operatorname{trace}_Z \omega(R(X, Y)Z, Z)$ in terms of the curvature tensor R of the Riemannian metric on M).

It is well known that $\lambda\leqslant 2\delta$ and $2\Delta\leqslant\Lambda$. We prove:

$$\textit{trace}\, \varrho = \sum_{i\neq k} \langle R(X_i, X_k)X_k, X_i\rangle = \textit{scalar curvature}. \tag{1}$$

$$\left.\begin{aligned} &\Delta+\delta-\tfrac{4}{3}(\Delta-\delta)\left[n(n-1)\left(n+\tfrac{1}{4}\right)\right]^{1/2}\leqslant\lambda\leqslant\Lambda\leqslant\\ &\qquad\Delta+\delta+\tfrac{2}{3}(\Delta-\delta)\left[n(n-1)\left(n+\tfrac{1}{4}\right)\right]^{1/2}.\end{aligned}\right\} \tag{2}$$

From (2) and Ruh's theorem [3] we have the

COROLLARY. *A complete simply connected Riemannian manifold with* $\delta/\Delta\geqslant 1-\left[\frac{1}{2}+\frac{5}{3}\left(n(n-1)\left(n+\frac{1}{4}\right)\right)^{1/2}\right]^{-1}$ *has* $\lambda/\Lambda\geqslant\frac{2}{3}$, *hence is diffeomorphic to the standard sphere*

Remark. The numerical values are worse than the ones obtained by Gromoll (explicitly computed up to $n=12$); for example for $n=7$ resp. $n=12$ we need $\delta/\Delta\geqslant 0.966$ resp. 0.985, Gromoll's figures are 0.819 resp. 0.931.

Proof. Use a normal coordinate system at $p\in M$ with associated basis X_i ($i=1,\dots,n$) of M_p. For $\omega\in M_p\wedge M_p$ put $\omega_{ik}=\omega(X_i, X_k)$, hence $\omega=\frac{1}{2}\sum_{i\neq k}\omega_{ik}X_i\wedge X_k$. The scalar product in $M_p\wedge M_p$ is given by $\langle\omega,\bar\omega\rangle=\frac{1}{2}\sum_{i\neq k}\omega_{ik}\bar\omega_{ik}$.

We have

$$\varrho(X_i\wedge X_j)(X_k, X_l)=2R_{klji}=\langle\varrho(X_i\wedge X_j), X_k\wedge X_l\rangle, \tag{3}$$

hence

$$\langle\varrho\omega,\omega\rangle=\tfrac{1}{2}\sum{}' R_{klji}\omega_{ij}\omega_{kl}. \qquad \left(\sum{}' = \sum_{k\neq l,\, i\neq j} \text{ here and below}\right) \tag{4}$$

(3) and (4) imply $\lambda\leqslant 2\delta$ and $2\Delta\leqslant\Lambda$, but (3) shows also that the eigenvectors ω^ν ($\nu=1,\dots,\frac{1}{2}n(n-1)$) of ϱ are in general not forms of rank 2 so that the reversed inequalities

[1]) This paper was prepared under the program "Sonderforschungsbereich Theoretische Mathematik" at Bonn University.

cannot be expected. (4) implies (1) immediately:

$$\begin{aligned}\operatorname{trace}\varrho &= \sum_{\nu} \langle \varrho\omega^{\nu}, \omega^{\nu}\rangle = \tfrac{1}{2}\sum{}' R_{klji} \sum_{\nu} \omega_{ij}^{\nu}\omega_{kl}^{\nu} \\ &= \tfrac{1}{2}\sum{}' R_{klji}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = \sum_{k\neq l} R_{kllk}.\end{aligned}$$

Define $R_0(X, Y)Z := \frac{1}{2}(\Delta+\delta)(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$ and $D := R - R_0$. Since D and R have the same symmetries we have from (7), (8), (9) in [2] for unit vectors in M_p:

$$|\langle D(X, Y)Y, U\rangle| \leqslant 2^{-1/2}(\Delta-\delta), \qquad (\leqslant \tfrac{1}{2}(\Delta-\delta) \text{ if } X \perp U) \tag{5}$$

$$\text{If } Z \perp X, Y \text{ then } |\langle D(X, Y)Z, U\rangle| \leqslant \tfrac{2}{3}(\Delta-\delta) \tag{6}$$

$$\|D\| = \max_{|X|=|Y|=|Z|=|U|=1} |\langle D(X, Y)Z, U\rangle| \leqslant (34/36)^{1/2}(\Delta-\delta). \tag{7}$$

Remark. In forthcoming papers on the differentiable pinching problem Sugimoto and Shiohama have used $\|D\| \leqslant k$ (with normalization $\Delta+\delta=2$) as a pinching condition. $\|D\| \leqslant k$ clearly implies for the sectional curvatures $1-k \leqslant K \leqslant 1+k$. The converse is not true since for the complex projective space $\|D\| = \frac{4}{3}k$. However $1-k \leqslant K \leqslant 1+k$ and (7) imply $\|D\| \leqslant (34/9)^{1/2}\, k$.
We rewrite (4) as

$$\langle \varrho\omega, \omega\rangle = \Delta + \delta + \tfrac{1}{2}\sum{}' D_{klji}\omega_{ij}\omega_{kl} \tag{8}$$

and apply Schwarz' inequality (note $|\omega|=1$):

$$|\tfrac{1}{2}\sum{}' D_{klji}\omega_{ij}\omega_{kl}| \leqslant (\sum{}' D_{klji}^2)^{1/2}. \tag{9}$$

(Schwarz' inequality can be applied in various ways to (8) leading to different expressions for the [ ]-bracket in (2); in our computations they were all of the order $n^{3/2}$ or worse. If (2) is a poor estimate the loss probably occurs in (9) since the following estimates seem fairly sharp to us.)

$$\sum{}' D_{klji}^2 = \sum_{k,\, l=j,\, i} D_{klji}^2 + \sum_{k,\, l=i,\, j} D_{klji}^2 + \sum_{\substack{k,\, l,\, i,\, j \\ l \neq i,\, j}} D_{klji}^2 \tag{10}$$

We have from (5) with l, i fixed and $a_k := D_{kjji}(\sum_k D_{kjji}^2)^{-1/2}$

$$2^{-1/2}(\Delta-\delta) \geqslant \left|\sum_k a_k D_{kjji}\right| = \left(\sum_k D_{kjji}^2\right)^{1/2}. \tag{11}$$

If $l \neq i, j$ we have from (6) with $a_k := D_{klji}(\sum_k D_{klji}^2)^{-1/2}$

$$\tfrac{2}{3}(\Delta-\delta) \geqslant \left|\sum_k a_k D_{klji}\right| = \left(\sum_k D_{klji}^2\right)^{1/2}. \tag{12}$$

Insert (11) and (12) in (10) to get

$$\sum{}' D_{klji}^2 \leqslant \sum_{j \neq i} (\Delta - \delta)^2 + \sum_{l \neq i \neq j \neq l} \tfrac{4}{9} (\Delta - \delta)^2 = \tfrac{4}{9} (\Delta - \delta)^2 n (n-1) (n + \tfrac{1}{4}). \quad (13)$$

(8), (9) and (13) prove (2).

Remark. The method can also be used to improve Berger's estimate [1] for the Eulercharacteristic:

$$\chi(M^{2m}) = \frac{(-1)^m}{2^{3m} \pi^m m!} \int_M (\varepsilon_{i_1 \ldots i_{2m}} \cdot \varepsilon_{j_1 \ldots j_{2m}} \cdot R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2m-1} i_{2m} j_{2m-1} j_{2m}})\, dV$$
$$\leqslant 2^{-m} (2m)! (\Delta/\delta)^m.$$

Berger estimates the integrand by $((2m)!)^2 \Delta^m$. We use first $m \cdot \prod_{i=1}^m a_i \leqslant \sum |a_i|^m \leqslant \leqslant \max |a_i|^{m-2} \sum a_i^2$, then Berger's $|R_{kllj}| \leqslant \frac{1}{2}(\Delta - \delta)$, $|R_{klji}| \leqslant \frac{2}{3}(\Delta - \delta)$ – see (5), (6) – and the analogues of (11), (12) for R instead of D to estimate the integrand by

$$((2m-2)!)^2 \Delta^m 2 \cdot 2m (2m-1) \left[1 + \left(1 - \frac{\delta}{\Delta}\right)^m ((\tfrac{1}{4})^m + (m-1)(\tfrac{4}{9})^m) \right].$$

We obtain $(m \geqslant 3)$

$$\chi(M^{2m}) \leqslant 2.4 \cdot 2^{-m} (2m-2)! (\Delta/\delta)^m.$$

In Dimension 4 Chern's coordinate choice [1] simplifies the integrand to $32(R_{1212}R_{3434} + R_{1313}R_{2424} + R_{1414}R_{2323} + R_{1234}^2 + R_{2314}^2 + R_{3124}^2)$ which Berger estimates by $32(3\Delta'^2 + \frac{8}{9}(\Delta' - \delta')^2)$. Here Δ' and δ' are minimum and maximum at each point. However $R_{1212} = \delta'$ in Chern's coordinates. This improves the estimate to $32(\Delta'\delta' + 2\Delta'^2 + \frac{8}{9}(\Delta' - \delta')^2) \leqslant 32 \cdot 3\Delta'^2$ and gives $\chi(M^4) \leqslant (\Delta/\delta)^2$ instead of $\leqslant (\Delta/\delta)^2 + \frac{8}{27}((\Delta/\delta) - 1)^2$. The same estimate improves known lower bounds for the volume in terms of the second Betti number:

$$2 + b_2 = \chi(M^4) \leqslant (4\pi^2)^{-1} \cdot 3\Delta^2 \cdot \mathrm{vol}(M^4).$$

BIBLIOGRAPHY

[1] M. Berger, *On the Characteristic of Positively Pinched Riemannian Manifolds*, Proc. Nat. Ac. Science *48* (1962), 1915–1917.

[2] H. Karcher, *A Short Proof of Berger's Curvature Tensor Estimates*, Proc. Amer. Math. Soc. Nr. 4, 1970.

[3] E. Ruh, *Curvature and Differentiable Structure on Spheres*, Comment Math. Helv. 66(1971) fasc. 2.

Received October 20, 1970