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§ 1. Introduction

Let $\mathcal{M}$ be a compact oriented 3-manifold without boundary. A knot K in $\mathcal{M}$ is said to have *period* $n>1$ if there is an orientation-preserving auto-homeomorphism g of $\mathcal{M}$ with the following properties (1.1)–(1.3):
(1.1) the set F of fixed points of g is a 1-sphere in $\mathcal{M}$,
(1.2) g is of period n, that is, g^n is the identity and no smaller power of g is the identity,
(1.3) F and K are disjoint.

If K has period n, we call K a *periodic knot* of order n. A knot can be a periodic knot of different orders. For example, the trefoil knot in the 3-sphere S^3 is a periodic knot of order 2 and 3. On the other hand, we can show that there exists a knot in S^3 which cannot have any period >1. (See §5.)

Our problem is to decide the possible period of a given knot. The first general result of this problem was obtained by Trotter [8]. He gave a necessary condition for a knot in a certain restricted class to have period n.

In this paper we shall prove, as a consequence of the main theorem,

THEOREM (*see Corollary* 1). *If K is a periodic knot of order p^r in S^3, p a prime, then the knot polynomial $\Delta(t)$ of K must satisfy*

$$\Delta(t) \equiv f(t)^{p^r}(1+t+t^2+\cdots+t^{\lambda-1})^{p^r-1} \pmod p \tag{*}$$

for some knot polynomial $f(t)$ and a positive integer λ, $(\lambda, p)=1$.

Condition (*) is of a completely different nature from Trotter's condition and it can be applied for any knot without restrictions. Using this condition, we shall be able to obtain quite satisfactory results. For example, we can give alternative proofs of the following theorems due to Trotter [8] and P. E. Conner [3].

THEOREM (*see Corollary* 5). *Any Neuwirth knot can have only finitely many distinct periods.*

THEOREM (*see Corollary* 6). *The only periods of the torus knot of type (m, n) are divisors of m or n.*

§ 2. Knot Polynomials

Let $\mathcal{M}$ be a homology 3-sphere, that is, a compact oriented 3-manifold with the same (integral) homology group $H_*(\mathscr{M})$ as S^3. Let $L=K_1\cup\cdots\cup K_\mu$ be an oriented link in $\mathscr{M}$ of μ components.

PROPOSITION 2.1. *$H_1(\mathscr{M}-L)\cong Z^\mu$, where Z^μ denotes the direct product of μ copies of Z, the infinite cyclic group.*

Proof. Since L is compact, it follows from the duality theorem [7, p. 296] that $H_2(\mathscr{M}, \mathscr{M}-L)\cong H^1(L)$. On the other hand, the homology exact sequence for $(\mathscr{M}, \mathscr{M}-L)$ yields $H_2(\mathscr{M}, \mathscr{M}-L)\cong H_1(\mathscr{M}-L)$. Since $H^1(L)\cong Z^\mu$, we obtain the required result.

Since $K_i\sim 0$ in $\mathscr{M}$, there exists an orientable 2-manifold $\mathscr{F}_i$ in $\mathscr{M}$ with $\partial\mathscr{F}_i=K_i$. Thus a meridian-longitude pair (m_i, l_i) is defined for each knot K_i in $\mathscr{M}$. Let U_i be a small open tubular neighbourhood of K_i in $\mathscr{M}$. $\partial U_i=T_i$ is a torus in $\mathscr{M}$. We may assume that m_i and l_i lie on T_i. Since K_i is oriented, we can give orientations to m_i and l_i in such a way that Link (m_i, K_i), the linking number of m_i and K_i, is one, and l_i and K_i bound an orientable band. $H_1(T_i)$ is generated by m_i and l_i, and $l_i\sim 0$ in $\mathscr{M}-U_i$.

PROPOSITION 2.2. *Let $\psi_j: H_1(T_j)\to H_1(\mathscr{M}-U_j)$ be a homomorphism induced by inclusion. Then $\psi_j(m_j)\neq 0$ in $H_1(\mathscr{M}-U_j)$ and $\{\tilde{\psi}_1(m_1),\ldots, \tilde{\psi}_\mu(m_\mu)\}$ forms a basis of $H_1(\mathscr{M}-U)$, $U=\bigcup_{i=1}^{\mu} U_i$, where $\tilde{\psi}_j(m_j)$ is a homology class in $H_1(\mathscr{M}-U)$ which is represented by $\psi_j(m_j)$.*

Proof. From Theorem 5.1 in [1], we know that ψ_j is a nonzero homomorphism. However, since $\psi_j(l_j)=0$, $\psi_j(m_j)$ cannot be 0 in $H_1(\mathscr{M}-U_j)$. Next, suppose that

$$a_1\tilde{\psi}_1(m_1)+\cdots+a_\mu\tilde{\psi}_\mu(m_\mu)=0 \tag{2.1}$$

for some integers $a_1,\ldots, a_\mu$.

Fill in $U_1,\ldots, U_{i-1}, U_{i+1},\ldots, U_\mu$ in some way to obtain a manifold $\mathscr{M}-U_i$ with boundary T_i. Then the inclusion homomorphism sends the left side of (2.1) to $a_i\psi_i(m_i)$ which is 0. Since $H_1(\mathscr{M}-U_i)$ has no torsion and $\psi_i(m_i)\neq 0$, it follows that $a_i=0$. Therefore, $\tilde{\psi}_1(m_1),\ldots, \tilde{\psi}_\mu(m_\mu)$ are linearly independent. Since these generate $H_1(\mathscr{M}-U)$ by Proposition 2.1, the proof is complete.

Propositions 2.1 and 2.2 assure the existence of the Alexander polynomial, $D(t_1,\ldots, t_\mu)$, of L in $\mathscr{M}$, where t_i corresponds to the meridian m_i of K_i.

In the following, we always assume that $\mu>1$.

Now $\pi_1(T_i)$ is a free abelian group generated by m_i and l_i (or their suitable conjugates), and it has a presentation $(z_i, u_i: [z_i, u_i])$, where z_i and u_i correspond to m_i and l_i, and $[a, b]=aba^{-1}b^{-1}$. This presentation is called a *canonical* presentation of $\pi_1(T_i)$. Then we have

PROPOSITION 2.3 [2, *Theorem* 2.2]. *$G=\pi_1(\mathscr{M}-L)$ has a presentation $\mathscr{P}$ such that*

(2.2) *the deficiency of* $\mathscr{P}$ *is one,*

(2.3) $\mathscr{P}$ *contains a canonical presentation of* $\pi_i(T_i)$ *for* $i=1, 2,\dots, \mu-1$.

Therefore, we may assume that $\pi_1(\mathscr{M}-L)$ has a presentation

$$\mathscr{P}=(z_1,\dots, z_{\mu-1}, u_1,\dots, u_{\mu-1}, x_{i,j}: [z_i, u_i], 1 \leqslant i \leqslant \mu-1, r_{k,l}).$$

Now, without change of the group G, we can add to this presentation $\mathscr{P}$ a new generator z_μ representing the meridian m_μ in G and a new relator $S=z_\mu\zeta^{-1}$, where ζ is a certain word written in $z_1,\dots, z_{\mu-1}, u_1,\dots, u_{\mu-1}$ or $x_{i,j}$. The resulting presentation will be denoted by $\mathscr{P}'$. Since G/G' is generated by $z_1,\dots, z_\mu$, we can write $u_i\equiv \prod_{k=1}^{\mu} z_k^{\lambda_{i,k}}$ (mod $[G, G]$), where $\lambda_{i,i}=0$ and $\lambda_{i,j}=\text{Link}(K_i, K_j)$, and $x_{i,j}\equiv \prod_{k=1}^{\mu} z_k^{\nu_{i,j,k}}$ (mod $[G, G]$).

Introduce new generators $v_i=u_i\,(\prod_{k=1}^{\mu} z_k^{\lambda_{i,k}})^{-1}$ and $a_{i,j}=x_{i,j}(\prod_{k=1}^{\mu} z_k^{\nu_{i,j,k}})^{-1}$. Using these generators, we obtain a new presentation $\mathscr{P}''$ of G:

$$\mathscr{P}''=(z_1,\dots, z_\mu, v_1,\dots, v_{\mu-1}, a_{i,j}: w_1,\dots, w_{\mu-1}, r'_{k,l}, s'),$$

where $w_i=[z_i, v_i\,(\prod_{k=1}^{\mu} z_k^{\lambda_{i,k}})^{-1}]$, and $r'_{k,l}, s'$ are obtained from $r_{k,l}, s$ by rewriting $x_{i,j}$ in terms of $a_{q,h}, v_q, z_q$.

Let $\varphi: F=(z_1,\dots, z_\mu, v_1,\dots, v_{\mu-1}, a_{i,j}:)\to G$ and $\psi: G\to G/[G,G]=(t_1,\dots, t_\mu: [t_i, t_j], 1\leqslant i, j\leqslant \mu)$ be natural homomorphisms so that $\psi\varphi(z_i)=t_i$. We use the same symbols φ, ψ to denote the uniquely extended ring homomorphisms between integer group rings.

Now all the generators of G except $\{z_1,\dots, z_\mu\}$ belong to $[G, G]$ and $\psi\varphi(z_i)=t_i$. Therefore, the Alexander matrix M of $\mathscr{P}''$ and the associated Alexander polynomial $D(t_1,\dots, t_\mu)$ will be obtained by means of the free differential calculus [4, II]. To make our argument smooth, we assume that the first μ columns of M correspond to $z_1,\dots, z_\mu$ and the next $\mu-1$ columns to $v_1,\dots, v_{\mu-1}$ and the first $\mu-1$ rows to $w_1,\dots, w_{\mu-1}$.

PROPOSITION 2.4. *Let* M^* *be the matrix obtained from* M *by deleting the first column (corresponding to* z_1*). Then the first row of* M^* *is divisible by* $1-t_1$. *Let* $\bar{M}^*$ *be the matrix obtained from* M^* *by dividing the first row by* $1-t_1$. *Then* $D(t_1,\dots, t_\mu)\sim \sim\det\bar{M}^*$, *where* $\sim$ *means that both sides are equal up to the unit in the polynomial ring* $Z[t_1,\dots, t_\mu]$.

Proof. By Lemma 1.1 of [2],

$$(z_1-1)^{\varphi\psi} D(t_1,\dots, t_\mu)\sim \delta\cdot\det M^*,$$

where δ denotes the g.c.d. of the fundamental ideals of $H_1(\mathscr{M}-L)$. Since the first Betti number of $H_1(\mathscr{M}-L)$ is $\mu>1$, we have $\delta=1$. This proves Proposition 2.4.

§ 3. Cyclic Coverings

Consider the n-fold cyclic covering space $\tilde{\mathscr{M}}$ of $\mathscr{M}$ branched along K_1. If $(n, \lambda_{1,j}) = \alpha_j$, each of $K_2, \ldots, K_\mu$ is covered by α_j knots $\tilde{K}_{j,1}, \ldots, \tilde{K}_{j,\alpha_j}$ in $\tilde{\mathscr{M}}$. If $\tilde{\mathscr{M}}$ is a homology 3-sphere, then, as is shown in §2, we can define the Alexander polynomial $\tilde{D}(\tilde{t}_{2,1}, \ldots, \tilde{t}_{\mu,\alpha_\mu})$, where $\tilde{t}_{i,q}$ corresponds to a meridian of $\tilde{K}_{i,q}$ in $\tilde{\mathscr{M}}$.

We wish to find a certain relation between $\tilde{D}(\tilde{t}_{2,1}, \ldots, \tilde{t}_{\mu,\alpha_\mu})$ and $D(t_1, \ldots, t_\mu)$, or more precisely speaking, the reduced Alexander polynomial $\tilde{D}(\tilde{t})$ of $\tilde{D}(\tilde{t}_{2,1}, \ldots, \tilde{t}_{\mu,\alpha_\mu})$ and $D(t_1, \ldots, t_\mu)$.

For simplicity, in the following, we assume that $\mu = 2$ and $\lambda_{1,2} \neq 0$.

Let $(n, \lambda_{1,2}) = \gamma$. Then K_2 is covered by exactly γ knots $\tilde{K}_1, \ldots, \tilde{K}_\gamma$ in $\tilde{\mathscr{M}}$. A presentation $\tilde{\mathscr{P}}$ of $\tilde{G} = \pi_1(\tilde{\mathscr{M}} - \tilde{L})$, $\tilde{L} = \tilde{K}_1 \cup \cdots \cup \tilde{K}_\gamma$, will be obtained as follows.

From the choice of the generators in the presentation $\mathscr{P}''$ of G, we see that a set of generators in $\tilde{\mathscr{P}}$ is

$$\{\tilde{z}_\nu = z_1^\nu z_2 z_1^{-\nu},\ v_{i,\nu} = z_1^\nu v_i z_1^{-\nu},\ a_{i,j,\nu} = z_1^\nu a_{i,j} z_1^{-\nu},\ 0 \leqslant \nu \leqslant n-1\}$$

and a complete set of relators is

$$\{z_1^\nu y z_1^{-\nu} = y, \quad \text{where } y \text{ is either } w_i,\ r'_{k,l}, \text{ or } s'\}.$$

Let $\tilde{\varphi}: \tilde{F} = (\tilde{z}_\nu, v_{i,\nu}, a_{i,j,\nu}:) \to \tilde{G}$ and

$$\tilde{\psi}: \tilde{G} \to \tilde{G}/[\tilde{G}, \tilde{G}] = (\tilde{t}_1, \ldots, \tilde{t}_\gamma : [\tilde{t}_i, \tilde{t}_j], \quad 1 \leqslant i, j \leqslant \gamma)$$

be natural homomorphisms, and further, let $\tilde{\tau}: \tilde{G}/[\tilde{G}, \tilde{G}] \to Z = (t:)$ be a homomorphism defined by $\tilde{\tau}(\tilde{t}_i) = t$ for all i. Then it is obvious that $\tilde{\tau}\tilde{\psi}\tilde{\varphi}(\tilde{z}_\nu) = t$ and $\tilde{\tau}\tilde{\psi}\tilde{\varphi}(v_{i,\nu}) = \tilde{\tau}\tilde{\psi}\tilde{\varphi}(a_{i,j,\nu}) = 1$.

Let N_n be the $n \times n$ matrix ring with entries in $Z[t]$, and let ω be a ring homomorphism from $Z[t_1, t_2]$ into N_n defined by

$$\omega(t_1) = P = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & 1 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \quad \text{and} \quad \omega(t_2) = tE.$$

Then the reduced Alexander matrix of $\tilde{G}$ at $\tilde{\varphi}\tilde{\psi}\tilde{\tau}$ is $M^{*\omega}$ which is denoted by $\tilde{M}$. To obtain the reduced Alexander polynomial $\tilde{D}(t)$ of $\tilde{L}$, consider the matrix $\tilde{M}^*$ which is obtained from $\tilde{M}$ by deleting the nth row and the first column. Then we see that

$$\det \tilde{M}^* = \tilde{D}(t). \tag{3.1}$$

Let $\varrho_\lambda(t) = (1 - t^\lambda)/(1 - t)$, λ being an integer. Then we can formulate a relation between $\tilde{D}(t)$ and $D(t_1, t_2)$ as follows.

PROPOSITION 3.1. *$\varrho_\lambda(t)\tilde{D}(t)\sim\prod_{i=0}^{n-1} D(t, \xi^i)$, where ξ is a primitive nth root of unity and $\lambda=\lambda_{1,2}$.*

Proof. From Proposition 2.4, it follows that

$$\det\tilde{M}^{*\omega} \sim \prod_{i=0}^{n-1} D(t, \xi^i).$$

By using (3.1), it only remains to show that

$$\varrho_\lambda(t)\det\tilde{M}^* \sim \det\tilde{M}^{*\omega}. \tag{3.2}$$

Consider $\tilde{M}^*$ and $\tilde{M}^{*\omega}$. Since $w_1=[z_1, v_1 z_2^\lambda]$, it follows that

$$\left(\frac{\partial w_1}{\partial z_2}\right)^{\varphi\psi} = (t_1-1)\varrho_\lambda(t_2), \quad \left(\frac{\partial w_1}{\partial v_1}\right)^{\varphi\psi} = t_1-1, \quad \left(\frac{\partial w_1}{\partial v_2}\right)^{\varphi\psi} = \left(\frac{\partial w_1}{\partial a_{i,j}}\right)^{\varphi\psi} = 0.$$

Hence, the entries of the first row of M^* are

$$((t_1-1)\varrho_\lambda(t_2), t_1-1, 0, \dots, 0).$$

Therefore, the first n rows of $M^{*\omega}$ are represented by

$$((P-E)\varrho_\lambda(t)E, P-E, 0, \dots, 0).$$

Hence, the first $n-1$ rows of $\tilde{M}^*$ are represented by

$$M' = \left[\begin{array}{cccc|cccc|c} & \varrho & & & -1 & 1 & & & \\ -\varrho & \varrho & & & & -1 & 1 & & \\ & \ddots & \ddots & & & & \ddots & \ddots & 0 \\ & & \ddots & \ddots & & & & \ddots & \\ & & -\varrho & \varrho & & & -1 & 1 & \end{array}\right], \qquad \varrho=\varrho_\lambda(t).$$

On the other hand, the first n rows of $\tilde{M}^{*\omega}$ are represented by

$$(\varrho_\lambda(t)E, E, 0, \dots, 0).$$

The differences between two matrices $\tilde{M}^*$ and $\tilde{M}^{*\omega}$ are the following.

(3.3) (1) *The number of rows (and columns) of $\tilde{M}^*$ is less than that of $\tilde{M}^{*\omega}$ by exactly one;*

(2) *If N denotes the minor matrix of $\tilde{M}^*$ obtained by deleting M', then $\tilde{M}^*$ and $\tilde{M}^{*\omega}$ are of the form*

$$\tilde{M}^* = \begin{pmatrix} M' \\ N \end{pmatrix} \quad \textit{and} \quad \tilde{M}^{*\omega} = \left[\begin{array}{c|ccc} \varrho_\lambda(t)E & E & 0\dots0 \\ \hline * & & N & \\ & & & \end{array}\right].$$

Now, add the 2nd,..., nth columns of $\tilde{M}^{*\omega}$ to the first to obtain the new matrix $\hat{M}$.

(3.4) *Only the first n entries of the first column of* $\hat{M}$ *are non-zero and each is* $\varrho_\lambda(t)$.

Proof. It is obvious that each of the first n entries of the first column is $\varrho_\lambda(t)$. Therefore, it remains to show that the other entries are 0. By the fundamental formula [4, I], for any relator r of $\mathscr{P}''$,

$$\frac{\partial r}{\partial z_1}(z_1-1)+\frac{\partial r}{\partial z_2}(z_2-1)+\sum_{i=1}^{2}\frac{\partial r}{\partial v_i}(v_i-1)+\sum_{i,j}\frac{\partial r}{\partial a_{i,j}}(a_{i,j}-1)=r-1. \tag{3.5}$$

Evaluate (3.5) at $\varphi\psi\omega$. Since $z_2^{\varphi\psi\omega}=tE$ and $r^{\varphi\psi\omega}=E$, it follows that (3.5) becomes

$$\left(\frac{\partial r}{\partial z_1}\right)^{\varphi\psi\omega}(P-E)+\left(\frac{\partial r}{\partial z_2}\right)^{\varphi\psi\omega}(tE-E)=0.$$

Let $R=(d_{ij})$, $1\leqslant i,j\leqslant n$, where $d_{ii}=d_{i1}=1$ for $1\leqslant i\leqslant n$, and $d_{kl}=0$ otherwise. Then the first column of $(P-E)R$ is zero, and hence that of $(\partial r/\partial z_2)^{\varphi\psi\omega}\,(1-t)R$ is zero. Therefore, the first column of $(\partial r/\partial z_2)^{\varphi\psi\omega}R$ must be zero. This proves (3.4).

Now we know that $\hat{M}$ is of the form:

$$\hat{M}=\left[\begin{array}{c|ccc|c|c}
\varrho & & & & & \\
\varrho & \varrho & & & E & 0 \\
\cdot & & \ddots & & & \\
\cdot & & & & & \\
\varrho & & & \varrho & & \\
\hline
0 & & & N & &
\end{array}\right].$$

Subtract the second row from the first, and then the third from the second, and so fourth, consecutively, to obtain $\check{M}$. Then the first $n-1$ rows except the first column of $\check{M}$ are identical with the first $n-1$ rows of $\tilde{M}^*$ multiplied by (-1). Thus

$$\varrho_\lambda(t)\det\tilde{M}^*\sim\det\check{M}=\det\hat{M}=\det\bar{M}^{*\omega}.$$

This proves (3.2), and hence Proposition 3.1.

§ 4. Main Theorem

PROPOSITION 4.1. *Let* $D(t_1,t_2)$ *and* $D(t_1)$ *be the Alexander polynomials of a link* $L=K_1\cup K_2$ *and a knot* K_1, *respectively, in a homology* 3-*sphere* $\mathscr{M}$. *Then*

$$D(t_1,1)\sim\varrho_\lambda(t_1)\,D(t_1), \tag{4.1}$$

where $\lambda=\mathrm{Link}(K_1,K_2)$, *and* $D(1)=\pm1$.

Proof. Since it is well known that $D(1)=\pm1$, we shall prove (4.1). However, in order to use the same notations as is used in §2, we shall prove the following equivalent formula (4.2) instead of (4.1).

$$D(1, t_2) \sim \varrho_\lambda(t_2)\, D(t_2), \tag{4.2}$$

where $D(t_2)$ is the Alexander polynomial of K_2.

Now let $\mathscr{P}'' = (z_1, z_2, v_1, a_{i,j} : w_1, r'_{k,l}, s')$ be a presentation of $\pi_1(\mathscr{M} - L)$ given in §2 with $\mu = 2$. The deficiency of $\mathscr{P}''$ is one. Then, a presentation $\mathscr{P}_1$ of $\pi_1(\mathscr{M} - K_2)$ is obtained from $\mathscr{P}''$ by adding one relator $z_1 = 1$ to $\mathscr{P}''$. Let M be the Alexander matrix of $\mathscr{P}''$. Then the Alexander matrix M_1 of $\mathscr{P}_1$ is obtained as M^τ with one extra row $(1, 0, 0, \dots, 0)$, where τ denotes the *retraction* homomorphism $G/G' \to Z = (t_2 :)$ defined by $\tau(t_1) = 1$ and $\tau(t_2) = t_2$. Note that the entries of the first row of M are $(0, 0, t_1 - 1, 0, \dots, 0)$. Now, the Alexander polynomial $D(t_1, t_2)$ of L is, by Proposition 2.4,

$$D(t_1, t_2) \sim \det \begin{bmatrix} \dfrac{\partial r'_{i,j}}{\partial z_2} & \dfrac{\partial r'_{i,j}}{\partial a_{k,l}} \\[2ex] \dfrac{\partial s'}{\partial z_2} & \dfrac{\partial s'}{\partial a_{k,l}} \end{bmatrix}^{\varphi\psi} .$$

On the other hand, since the deficiency of $\mathscr{P}_1$ is 0 and K_2 is a knot, it follows that $D(t_2)$ is the determinant of a matrix obtained from M_1 by deleting one row and one column corresponding to w_1 and z_2, respectively. Therefore,

$$D(t_2) \sim \det \begin{bmatrix} \dfrac{\partial r'_{i,j}}{\partial z_1} & \dfrac{\partial r'_{i,j}}{\partial v_1} & \dfrac{\partial r'_{i,j}}{\partial a_{k,l}} \\[2ex] \dfrac{\partial s'}{\partial z_1} & \dfrac{\partial s'}{\partial v_1} & \dfrac{\partial s'}{\partial a_{k,l}} \\[2ex] 1 & 0 & 0 \end{bmatrix}^{\varphi\psi\tau} \sim \det \begin{bmatrix} \dfrac{\partial r'_{i,j}}{\partial v_1} & \dfrac{\partial r'_{i,j}}{\partial a_{k,l}} \\[2ex] \dfrac{\partial s'}{\partial v_1} & \dfrac{\partial s'}{\partial a_{k,l}} \end{bmatrix}^{\varphi\psi\tau} .$$

Using the fundamental formula, we see that if W is either $r'_{i,j}$ or s', then

$$\left(\frac{\partial W}{\partial v_1}\right)^{\varphi\psi\tau} \varrho_\lambda(t_2) = -\left(\frac{\partial W}{\partial z_2}\right)^{\varphi\psi\tau} .$$

Hence, we have

$$D(1, t_2) \sim \det \begin{bmatrix} \dfrac{\partial r'_{i,j}}{\partial z_2} & \dfrac{\partial r'_{i,j}}{\partial a_{k,l}} \\[2ex] \dfrac{\partial s'}{\partial z_2} & \dfrac{\partial s'}{\partial a_{k,l}} \end{bmatrix}^{\varphi\psi\tau} = -\varrho_\lambda(t_2) \det \begin{bmatrix} \dfrac{\partial r'_{i,j}}{\partial v_1} & \dfrac{\partial r'_{i,j}}{\partial a_{k,l}} \\[2ex] \dfrac{\partial s'}{\partial v_1} & \dfrac{\partial s'}{\partial a_{k,l}} \end{bmatrix}^{\varphi\psi\tau} \sim \varrho_\lambda(t_2)\, D(t_2).$$

This proves (4.2).

PROPOSITION 4.2. *Let $f(x, y)$ be an integer polynomial of two indeterminates x, y. Let $n=p^r m$, $(p, m)=1$, p a prime, $r>0$. Let ξ and η denote primitive nth and mth roots of unity. Then*

$$\prod_{i=0}^{n-1} f(x, \xi^i) \equiv \left[\prod_{j=0}^{m-1} f(x, \eta^j)\right]^{p^r} \quad (\operatorname{mod} p). \tag{4.3}$$

Proof. Since $n=p^r m$, ξ^{p^r} is a primitive mth root of unity. Thus, η may be written as ξ^{p^r}, and hence, $\eta^j=\xi^{p^r j}$, $1\leqslant j\leqslant m$. Let $\mathscr{I}_i$ denote the ideal generated by $1-\xi^{im}$, $1\leqslant i\leqslant p^r-1$, in the ring $Z[x, \xi]$. Note that ξ^m is a primitive p^rth root of unity. Since $(p, m)=1$, there exist integers α, β such that $\alpha p^r+\beta m=1$. Then, $\xi=\eta^\alpha \xi^{\beta m}$, and hence $\xi\equiv\eta^\alpha$ $(\operatorname{mod}\mathscr{I}_\beta)$. Similarly, $\xi^k\equiv\eta^{k\alpha}$ $(\operatorname{mod}\mathscr{I}_{k\beta})$, $1\leqslant k\leqslant m$. Since $(\alpha, m)=1$, $k\alpha\equiv l\alpha$ $(\operatorname{mod} m)$ implies $k\equiv l$ $(\operatorname{mod} m)$. Therefore,

$$\prod_{i=1}^{m} f(x, \xi^i) \equiv \prod_{j=1}^{m} f(x, \eta^j) \quad (\operatorname{mod}\mathscr{I}),$$

where $\mathscr{I}$ is the ideal contained in all $\mathscr{I}_\beta, \ldots, \mathscr{I}_{k\beta}$. Similarly, for any q, $0\leqslant q\leqslant n-1$, $\xi^q\equiv\eta^r$ $(\operatorname{mod}\mathscr{I}_s)$ for some integers r and s. Thus, we obtain

$$\prod_{i=0}^{n-1} f(x, \xi^i) \equiv \left[\prod_{j=0}^{m-1} f(x, \eta^j)\right]^{p^r} \quad (\operatorname{mod}\tilde{\mathscr{I}}), \tag{4.4}$$

where $\tilde{\mathscr{I}}$ is the ideal contained in $\mathscr{I}_1, \ldots, \mathscr{I}_{p^r-1}$.

Since both sides of (4.4) are integer polynomials, $\tilde{\mathscr{I}}$ must be in Z. Therefore, $\tilde{\mathscr{I}}$ is some power of p, and hence, we obtain the required formula (4.3).

By combining Propositions 3.1, 4.1, and 4.2, we obtain the main theorem.

THEOREM 1. *Let $L=K_1\cup K_2$ be a link of two components in a homology 3-sphere $\mathscr{M}$. Suppose that the n-fold cyclic covering space $\tilde{\mathscr{M}}$ of $\mathscr{M}$ branched along K_2 is again a homology 3-sphere. Then the knot K_1 is covered by a link $\tilde{L}$ in $\tilde{\mathscr{M}}$. Let $D(t_1, t_2)$ and $D(t_1)$ be the Alexander polynomials of L and K_1 in $\mathscr{M}$, respectively and let $\tilde{D}(t)$ be the reduced Alexander polynomial of $\tilde{L}$ in $\tilde{\mathscr{M}}$. Let $\lambda=$Link $(K_1, K_2)\neq 0$, and let ξ be a primitive nth root of unity. Then*

$$\varrho_\lambda(t)\,\tilde{D}(t) \sim \prod_{i=0}^{n-1} D(t, \xi^i). \tag{4.5}$$

If $n=p^r m$, p a prime, $(p, m)=1$, $r>0$, then

$$\varrho_\lambda(t)\,\tilde{D}(t) \equiv \left[\prod_{j=0}^{m-1} D(t, \eta^j)\right]^{p^r} \quad (\operatorname{mod} p), \tag{4.6}$$

where η denotes a primitive mth root of unity. In particular, if $m=1$, then

$$\tilde{D}(t) \equiv D(t)^{p^r}\,\varrho_\lambda(t)^{p^r-1} \quad (\operatorname{mod} p). \tag{4.7}$$

Proof. (4.7) follows from the fact that $D(t, 1) \sim \varrho_\lambda(t)\, D(t)$ and $D(t) \not\equiv 0 \pmod{p}$, since $D(1) = \pm 1$.

§ 5. Applications

Let K be a periodic knot of order n in a homology 3-sphere $\tilde{\mathcal{M}}$ and let g be a periodic auto-homeomorphism of $\tilde{\mathcal{M}}$. Let F be the set of fixed points of g. Let p be a prime.

PROPOSITION 5.1. *The orbit space* $\mathcal{M} = \tilde{\mathcal{M}}/g$ *is also a homology* 3-*sphere.*

Proof. Since $\mathcal{M}$ is a compact orientable closed 3-manifold, it is enough to show that $H_1(\mathcal{M}) = 0$. Since F is a 1-sphere, $F \sim 0$ in $\tilde{\mathcal{M}}$, and hence the image F_0 of F under the collapse is also homologous to zero in $\mathcal{M}$. Now it is obvious that $\tilde{\mathcal{M}}$ is an n-fold cyclic covering space of $\mathcal{M}$ branched along F_0. Further, it is easy to verify that there is a surjective homomorphism $f: \pi_1(\tilde{\mathcal{M}}) \to \pi_1(\mathcal{M})$. f induces the surjective homomorphism $f_*: H_1(\tilde{\mathcal{M}}) \to H_1(\mathcal{M})$. Since $H_1(\tilde{\mathcal{M}}) = 0$, it follows that $H_1(\mathcal{M}) = 0$.

Now, with a periodic knot K in $\tilde{\mathcal{M}}$ we can associate a link $K \cup F$ in $\tilde{\mathcal{M}}$. Since the knots K and F are mapped onto the knots K_0 and F_0, respectively, in the orbit space $\mathcal{M}$ under the collapse, we obtain a link $L_0 = K_0 \cup F_0$ in $\mathcal{M}$. Since $\mathcal{M}$ is a homology 3-sphere, the Alexander polynomial $D(t_1, t_2)$ of L_0 is defined. $\tilde{\mathcal{M}}$ is an n-fold cyclic covering space of $\mathcal{M}$ branched along F_0, and since K_0 is covered by K in $\tilde{\mathcal{M}}$, it follows that Link $(K_0, F_0) = \lambda \neq 0$ and $(\lambda, n) = 1$. Thus, Theorem 1 implies

THEOREM 2. *Suppose that K is a periodic knot in $\tilde{\mathcal{M}}$ of order p^r. Let $D(t)$ and $\tilde{D}(t)$ be the Alexander polynomials of K_0 in $\mathcal{M}$ and K in $\tilde{\mathcal{M}}$, respectively. Then*

$$\tilde{D}(t) \equiv \varrho_\lambda(t)^{p^r - 1} D(t)^{p^r} \pmod{p}.$$

If we consider a periodic knot in a simply connected 3-manifold S, then the orbit space is also simply connected. Therefore, $D(t)$ and $\tilde{D}(t)$ in Theorem 2 become the knot polynomials (see [1]), and we have the following result.

COROLLARY 1. *Suppose that K is a periodic knot of order $n = p^r$ in S. Then the knot polynomial $\Delta(t)$ of K must satisfy the following:*

$$\Delta(t) \equiv \varrho_\lambda(t)^{n-1} D(t)^n \pmod{p},$$

for some positive integer λ, $(\lambda, p) = 1$, and a certain knot polynomial $D(t)$.

COROLLARY 2. *Under the same assumption as in Corollary* 1, *if $\Delta(t)$ is not a* *product of other knot polynomials in* $Z[t]$, *then for some positive integer* λ, $(\lambda, p)=1$,

$$\Delta(t) \equiv \varrho_\lambda(t)^{n-1} \pmod p, \tag{5.1}$$

and hence, for any integer s,

$$\Delta(s) \equiv 0 \quad or \quad \pm 1 \pmod p. \tag{5.2}$$

Proof. From (4.5), (4.7), and Corollary 1, it follows that

$$\varrho_\lambda(t)\,\Delta(t) \sim \prod_{i=0}^{n-1} D(t, \xi^i) \equiv \varrho_\lambda(t)^n\, D(t)^n \pmod p. \tag{5.3}$$

Since $D(t, 1) \sim \varrho_\lambda(t)\, D(t)$ by (4.1), and $\varrho_\lambda(t) \not\equiv 0 \pmod p$, we see that

$$\Delta(t) \sim D(t) \prod_{i=1}^{n-1} D(t, \xi^i) \equiv \varrho_\lambda(t)^{n-1}\, D(t)^n \pmod p. \tag{5.4}$$

Since $D(t)$ and $\prod_{i=1}^{n-1} D(t, \xi^i)$ are knot polynomials, either $\Delta(t) \sim D(t)$ or $D(t) \sim 1$, If $\Delta(t) \sim D(t)$, then $\prod_{i=1}^{n-1} D(t, \xi^i) \sim 1$. Since $D(t) \not\equiv 0 \pmod p$, it follows from (5.4) that

$$1 \sim \prod_{i=1}^{n-1} D(t, \xi^i) \equiv \varrho_\lambda(t)^{n-1}\, D(t)^{n-1} \pmod p.$$

This is possible only when both $D(t)$ and $\varrho_\lambda(t)$ are congruent to ± 1 modulo p. Thus $\Delta(t) \equiv 1 \pmod p$. This is a special case of (5.1). If $D(t) \sim 1$, then from (5.4) we have

$$\Delta(t) \equiv \varrho_\lambda(t)^{n-1} \pmod p.$$

This proves (5.1). (5.2) follows from Fermat's Theorem.

COROLLARY 3. *If K is a periodic knot of order* p^r *in S, then the degree of* $\Delta(t)$ *is not less than* $p^r - 1$, *unless* $\Delta(t) \equiv 1 \pmod p$.

Proof. Let Φ_p be a natural ring homomorphism: $Z[t] \to Z_p[t]$, $Z_p = Z/pZ$. Let $d_p(f)$ denote the reduced degree of a polynomial $f(t)$ in $Z_p[t]$. That is, if $f(t) = a_k t^k + \cdots + a_l t^l$, $a_k \neq 0$, $a_l \neq 0$, in $Z_p[t]$, then we define $d_p(f) = l - k$. The reduced degree of an integer polynomial $f(t)$ will be denoted by $d_0(f)$. Then, obviously, for any integer polynomial $f(t)$, $d_p(\Phi_p f) \leqslant d_0(f)$. Now, Corollary 1 shows that $d_p(\Phi_p \Delta) \geqslant (n-1)|\lambda| + n d_p(\Phi_p D) \geqslant n-1$, since $\lambda \neq 0$ or $d_p(\Phi_p D) > 0$ unless $\Delta(t) \equiv 1 \pmod p$.

COROLLARY 4. *Any knot in S has only finitely many distinct prime periods unless* $\Delta(t) = 1$.

COROLLARY 5. *If, for any prime q,* $\Delta(t) \not\equiv 1 \pmod q$, *then K can have only finitely many distinct periods. In particular, a fibre knot (or Neuwirth knot) can have only finitely many distinct periods.*

Proof. Let d be the degree of $\Delta(t)$. Then it follows from Corollary 3 that the possible prime power periods of K are the powers p^α such that $p^\alpha \leqslant d+1$. Let $p_1,\dots,p_m$ be all the primes such that $p_i^{\alpha_i} \leqslant d+1$, where we assume that α_i is the maximal exponent satisfying the above condition. Then the possible period of K must be of the form $p_1^{\beta_1}\cdots p_m^{\beta_m}$, $0 \leqslant \beta_i \leqslant \alpha_i$. In fact, if, for some β_i, $\beta_i > \alpha_i$, then K would have period $p_i^{\alpha_i+1}$. Therefore, the number of possible periods of K is finite, that is, at most $\prod_{i=1}^m (\beta_i+1)$. If K is a fibre knot, then $|\Delta(0)|=1$ and $d\neq 0$. Thus, $\Delta(t) \not\equiv 1 (\operatorname{mod} q)$ for any prime q. Hence, it has only finitely many distinct periods.

Remark. The latter assertion of Corollary 5 follows from Theorem (1.2) of [8] if $\Delta(t)$ has no repeated roots.

COROLLARY 6. *Let $K_{m,n}$ be the torus knot of type (m, n). Then any period of $K_{m,n}$ is a divisor of m or of n.*

Proof. It is obvious that any divisor of m or of n can be a period of $K_{m,n}$. Therefore, it remains to show that these divisors are only periods of $K_{m,n}$. To do this, it suffices to show the following.

(5.5) *Let $m=p^r a$ and $n=q^s b$, where p and q are distinct primes, $(p, a)=1$, $(q, b)=1$. Then*

(i) *p^{r+1} cannot be a period of $K_{m,n}$,*

(ii) *$p^\alpha q^\beta$, $\alpha, \beta>0$, cannot be a period of $K_{m,n}$.*

Proof of (5.5). Without loss of generality, we may assume that $m, n>1$. Now we know that the knot polynomial $\Delta_{m,n}(t)$ of $K_{m,n}$ is given by

$$\Delta_{m,n}(t) = \frac{1-t}{1-t^n}\,\frac{1-t^{mn}}{1-t^m}\,. \tag{5.6}$$

(i) Suppose that $K_{m,n}$ has period p^{r+1}. Then, by Corollary 1, we see that for some λ, $(\lambda, p)=1$, and $f(t)\in Z[t]$,

$$\varrho_\lambda(t)\,\Delta_{m,n}(t) \equiv f(t)^{p^{r+1}} \quad (\operatorname{mod} p). \tag{5.7}$$

Since $\varrho_\lambda(t)=(1-t^\lambda)/(1-t)$ and $1-t^{p^r s}\equiv(1-t^{rs})^{p^r}$ (mod p), an easy calculation yields

$$(1-t^\lambda)\,g(t)^{p^r} \equiv (1-t^n)\,f(t)^{p^{r+1}} \quad (\operatorname{mod} p), \tag{5.8}$$

where $g(t)=(1-t^{an})/(1-t^a)$.

Let A and B be non-zero non-constant terms modulo p with the minimal degree in the left and right sides of (5.8), respectively. Then A is either $t^{p^r a}=t^m$ or $-t^\lambda$. On the other hand, the degree of B is either lp^{r+1} or n, l being a positive integer. Therefore, there are four possible cases:

(a) $m=lp^{r+1}$, (b) $m=n$, (c) $\lambda=lp^{r+1}$, (d) $\lambda=n$.

However, the first three cases are easily eliminated by the original assumption. Suppose that $\lambda = n$. Then (5.8) becomes

$$g(t)^{p^r} \equiv f(t)^{p^{r+1}} \quad (\mathrm{mod}\, p), \tag{5.9}$$

which is impossible, since $g(t) \not\equiv h(t)^p (\mathrm{mod}\, p)$ as is seen from the expansion of $g(t)$. This proves (5.5) (i).

(ii) Suppose that $K_{m,n}$ has period $p^\alpha q^\beta$. Then from (4.6) we obtain

$$\left.\begin{array}{ll}(1)\ \varrho_\lambda(t)\, \Delta_{m,n}(t) \equiv f(t)^{p^\alpha} & (\mathrm{mod}\, p)\\ (2)\ \varrho_\lambda(t)\, \Delta_{m,n}(t) \equiv g(t)^{q^\beta} & (\mathrm{mod}\, q)\end{array}\right\} \begin{array}{l}\text{for the same integer}\\ \lambda,\ (\lambda, p) = 1,\ (\lambda, q) = 1.\end{array} \tag{5.10}$$

We may assume that $\alpha \leqslant p^r$ and $\beta \leqslant q^s$, for otherwise $K_{m,n}$ would have period p^{r+1} or q^{s+1}.

Now from (5.10) (1), we obtain

$$(1 - t^\lambda) \frac{1 - t^{mn}}{1 - t^n} \equiv (1 - t^m) f(t)^{p^\alpha} \quad (\mathrm{mod}\, p). \tag{5.11}$$

Then, as was done in the proof of (i), compare the non-zero non-constant terms with minimal degree on both sides of (5.11) to obtain $\lambda = m$. On the other hand, in the same way, we can show from (5.10) (2) that $\lambda = n$. Therefore, $m = \lambda = n$, which contradicts our assumption. This proves (5.5) (ii).

As another application of our theorems, we consider the knot in S^3 with two bridges [6]. From a simple observation, we know that any knot K with two bridges has period 2, and its image K_0 under the collapse is unknotted. Thus from Corollary 1, we obtain the following.

COROLLARY 7. *If K is a knot in S^3 with two bridges, then*

$$\Delta(t) \equiv \varrho_\lambda(t) \quad (\mathrm{mod}\, 2)$$

for some odd integer λ.

At the conclusion of this paper, we shall complete the list of periods of certain knots in Reidemeister's table [5]. According to Trotter [8],

$$\begin{array}{ll} (1) & 4_1, 6_2, 7_6, 8_{12}, 9_{42}, 9_{45} \text{ can have no period other than } 2,\\ (2) & 3_1, 6_3, 7_7, 8_{20}, 8_{21}, 9_{48} \text{ can have no period other than } 2, 3 \text{ or } 6,\\ (3) & 9_{44} \text{ can have no period other than } 2 \text{ or } 4. \end{array} \tag{5.12}$$

In the first group of knots, the Alexander polynomials of 9_{42} and 9_{45} are irreducible over $Z[t]$. By Corollary 2, we see that these knots do not have period 2. Thus, they have no period at all. On the other hand, the first four knots have period 2, since these are knots with two bridges.

In the second group, all knots except 3_1 do not have period 3 by Corollary 1, and hence, do not have period 6. Also we can see that all knots except possibly 8_{20} and 9_{48} have period 2. Therefore 6_3, 7_7, and 8_{21} have only period 2, and 3_1 has periods 2 and 3.

Finally, 9_{44} does not have period 2. Therefore, it has no period at all. Besides 9_{42}, 9_{44}, and 9_{45}, we now know that at least the following nine knots have no period at all: 8_{16}, 8_{17}, 9_{22}, 9_{30}, 9_{32}, 9_{33}, 9_{34}, 9_{36}, 9_{43}.

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Verallgemeinerung eines Satzes von H. Samelson

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Einleitung

H. Samelson bewies in [7] folgenden Satz: Sei $G = \Omega X$ der Schleifenraum eines punktierten topologischen Raumes X (allgemeiner kann G eine beliebige Gruppe in der Kategorie der punktierten topologischen Räume modulo Homotopie sein). Für $\alpha \in \pi_p(G)$, $\beta \in \pi_q(G)$, $p \geqslant 1$, $q \geqslant 1$, bezeichne $[\alpha, \beta] \in \pi_{p+q}(G)$ das Samelsonprodukt von α mit β. Dann gilt für den Hurewiczhomorphismus $\varphi_n : \pi_n(G) \to H_n(G)$, $n \geqslant 1$ und $p, q \geqslant 1 : \varphi_{p+q}[\alpha, \beta] = [\varphi_p(\alpha), \varphi_q(\beta)] = \varphi_p(\alpha)\,\varphi_q(\beta) - (-1)^{pq}\,\varphi_q(\beta)\,\varphi_p(\alpha)$. Dabei bezeichnet $\varphi_p(\alpha)\,\varphi_q(\beta)$, $\varphi_q(\beta)\,\varphi_p(\alpha)$ das Pontryagin-Produkt.

Im folgenden wird dieser Satz homotopietheoretisch formuliert und bewiesen und in zwei Richtungen verallgemeinert.

Erstens wird statt der Gruppe G nur eine Loop G vorausgesetzt. Es ergeben sich Resultate über den Kommutator und den Assoziator von Loopelementen sowie über gewisse allgemeinere Loopwörter. Zweitens wird in $\pi_n(G) = \pi(S^n, G)$ die Sphäre $S^n = \sum S^{n-1}$ durch eine Einhängung $\sum Y$ (für gewisse Räume Y) ersetzt und der erwähnte Satz von Samelson auf das allgemeine Samelsonprodukt erweitert (Satz 1.2, Satz 2.2 und Korollar 2.6). Die Beweise werden in der Kanschen Homotopietheorie von simplizialen Mengen geführt (vgl. [3]). Dabei wird das Pontryaginprodukt mit Hilfe eines Smashprodukts definiert. Um zu sehen, daß diese Definition mit der üblichen übereinstimmt, wird in Satz 2.8 ein Isomorphismus zwischen $\pi(\wedge_n \Omega, A)$ und $\pi(\Delta[n]/\dot{\Delta}[n], A)$, A simpliziale abelsche Gruppe, konstruiert. Dieser Isomorphismus ist von einem Isomorphismus in der Homotopiekategorie $\mathscr{H}$ von Gabriel-Zisman [3] VI.3.4. zwischen $\wedge_n \Omega$ und $\Delta[n]/\dot{\Delta}[n]$ induziert. Seine explizite Angabe ist vielleicht auch in anderem Zusammenhang interessant.

Ich danke dem Referenten für nützliche Bemerkungen zu dieser Arbeit.

1. Ein Satz über multiplikative Objekte

1.1 *Bezeichnungen*

Wie in [3] sei $\Delta^0\mathbf{C}$ die Kategorie der simplizialen Objekte einer Kategorie $\mathbf{C}$ and Δ die Kategorie mit den geordneten Mengen $[n] = \{0, \ldots. n\}$, $n \geqslant 0$ als Objekten und ordnungserhaltenden Abbildungen als Morphismen. Ein multiplikatives Objekt (M, m) einer Kategorie ist ein Objekt M und ein Morphismus $m : M \times M \to M$. **Ab** bzw. .**S** bezeichnen die abelschen Gruppen bzw. punktierten Mengen. $\overline{\Delta^0.\mathbf{S}} = \pi$ ist die Kategorie der punktierten simplizialen Mengen modulo Homotopie; $\Delta[n]$ sei das n-te Standardsimplex, $\dot{\Delta}[n]$ sein Rand, $I := \Delta[1]$ und $\Omega := I/\dot{I}$ der simpliziale Einheitskreis. Für $X \in \Delta^0.\mathrm{S}$ heißt $\sum X := \Omega \wedge X$ Einhängung von X (s. [3] VI.2.3.). Sei $x = (x_0, \ldots, x_n)$ ein Element aus I_n. Dann existiert i, $-1 \leqslant i \leqslant n$, mit $x_j = 0$ für $j \leqslant i$ und $x_j = 1$ für $j > i$. Sei $e_i^n := e_i := x$. Dieselbe Bezeichnung wird für Elemente aus Ω_n verwendet. Durch $\mu_n : I_n \times I_n \to I_n$, $\mu_n(e_i, e_j) = e_k$, $k := \mathrm{Max}\{i, j\}$ wird eine simpliziale Abbildung μ definiert und es gilt $\mu : Id \sim 0$; sei $rt := \mu_n(r, t)$.

Im folgenden Satz wird μ zur Konstruktion einer Homotopie auf einem Produkt von Einhängungen verwendet. Schließlich werde noch für $Y_1, \ldots, Y_n \in \Delta^0.\mathrm{S}$ und $1 \leqslant l \leqslant n$ $v_l : \prod_{i=1}^n Y_i \to \prod_{i=1}^n Y_i$ durch $v_l(y_1, \ldots, y_n) = (y_1, \ldots, y_{l-1}, *, y_{l+1}, \ldots, y_n)$ mit $y_i \in (Y_i)_p$, $p \geqslant 0$, definiert ($*$ bezeichnet den Grundpunkt).

1.2 SATZ: $(G, \perp)$ *und* $(H, \top)$ *seien multiplikative Objekte in* $\Delta^0.\mathrm{S}$, *sodaß für beliebige* $Z \in \Delta^0.\mathrm{S}$, $g \in \Delta^0.\mathrm{S}(Z, G)$, $h \in \Delta^0.\mathrm{S}(Z, H)$ *gilt:*

(1) $g \perp g \sim 0, \quad h \top h = 0.$

(2) $g \perp 0 \sim g, \quad h \top 0 \sim h.$

Für $1 \leqslant i \leqslant n$ *seien* X_i *punktierte simpliziale Mengen und* $X := \prod_{i=1}^n \sum X_i$. *Dann gilt für* $f \in \Delta^0.\mathrm{S}(G, H)$ *und* $a, b \in \Delta^0.\mathrm{S}(X, G)$ *mit*

(3) $av_l = bv_l$ *für* $1 \leqslant l \leqslant n$:

$f(a \perp b) \sim fa \top fb.$

Beweis: Sei $\tilde{a} := a \prod_{i=1}^n pr_i : \prod_{i=1}^n (I \times X_i) \to G$, wobei $pr_i : I \times X_i \to I/\dot{I} \wedge X_i = \sum X_i$ die natürliche Projektion ist. Ebenso sei $\tilde{b} := b \prod_{i=1}^n pr_i$.

Für $p \geqslant 0$ werde $\phi_p : I_p \times \prod_{i=1}^n (I \times X_i)_p \to H_p$ durch $\phi_p(t, r_1, x_1, \ldots, r_n, x_n) :=$ $:= f_p(\tilde{a}_p(r_1, x_1, \ldots, r_n, x_n) \perp \tilde{b}_p(tr_1, x_1, \ldots, tr_n, x_n)) \top f_p(\tilde{b}_p(r_1, x_1, \ldots, r_n, x_n) \perp$ $\perp \tilde{b}_p(tr_1, x_1, \ldots, tr_n, x_n))$ mit $t, r_i \in I_p$, $x_i \in (X_i)_p$ definiert. $\phi = (\phi_p)_{p \geqslant 0}$ ist eine simpliziale Abbildung, da die Multiplikation $(t, r_i) \mapsto tr_i$ simplizial ist. Sei $k := I \times \prod_{i=1}^n pr_i : I \times$ $\times \prod_{i=1}^n (I \times X_i) \to I \times \prod_{i=1}^n \sum X_i$.

Für $p \geqslant 0$ werde $\psi_p : I_p \times \prod_{i=1}^n (\sum X_i)_p \to H_p$ durch $\psi_p(x) := \phi_p(y)$, falls $k_p(y) = x$, definiert. ψ_p ist wohldefiniert, da wegen (1) und (3) $\phi_p(t, r_1, x_1, \ldots, r_n, x_n) = *$ gilt, falls $1 \leqslant i \leqslant n$ existiert mit $r_i = (0, \ldots, 0)$ oder $r_i = (1, \ldots, 1)$ oder $x_i = *$.

Da k und ϕ simpliziale Abbildungen sind, ist auch $\psi = (\psi_p)_{p \geqslant 0}$ simplizial: Bezeichne η_* den von $\eta : [q] \to [p]$ induzierten simplizialen Operator; dann gilt $\eta_* \psi_p(x) =$ $= \eta_* \phi_p(y) = \phi_q(\eta_* y) = \psi_q(k_q(\eta_* y)) = \psi_q(\eta_* k_p(y)) = \psi_q(\eta_* x)$.

Also ist ψ eine punktierte simpliziale Homotopie. ψ an der Stelle $t = (0, \ldots, 0)$ bzw. $t = (1, \ldots, 1)$ werde mit $\psi_{(0)}$ bzw. $\psi_{(1)}$ bezeichnet. Nach Definition gilt

$\psi_{(0)} = f(a \perp 0) \top f(b \perp 0) \sim fa \top fb$ wegen (2) und

$\psi_{(1)} = f(a \perp b) \top f(b \perp b) \sim f(a \perp b) \top 0 \sim f(a \perp b)$ wegen (1) und (2).

Also folgt $fa \top fb \sim f(a \perp b)$.

1.3 BEISPIELE: 1) Multiplikationen $\perp$ mit (1) und (2) ergeben sich auf folgende

Weise. Für eine Gruppe G und $a, b \in G$ sei $a \perp b := ab^{-1}$, oder allgemeiner: $a \perp b$ sei ein Wort in a und b, sodaß $1 =$ Summe der Exponenten von a und $-1 =$ Summe der Exponenten von b gilt, also zum Beispiel $a \perp b = a[a, b]^s b^{-1}$ ($[a, b]$ ist der Kommutator von a, b). Analog läßt sich auf einer Loop in $\Delta^0.\mathrm{S}$ bzw. $\overline{\Delta^0.\mathrm{S}}$ eine Operation $\perp$ mit (1), (2) einführeren. Eine Loop in der Kategorie der Mengen ist eine Menge L mit Multiplikation, neutralem Element und eindeutiger Lösbarkeit der Gleichungen $ax = b$, $ya = b$ für alle $a, b \in L$.

2) Beispiele für $a, b \in \Delta^0.\mathrm{S}(X, G)$ mit (3) erhält man für multiplikative Objekte, die sich wie in 1) aus Loopstrukturen auf G ergeben, wie folgt: Für $x_j \in \Delta^0.\mathrm{S}(\sum X_j, G)$ sei $y_j := x_j pr_j$ in $\Delta^0.\mathrm{S}(\prod_{i=1}^n \sum X_i, G)$. Die Loopmultiplikation von G in $\overline{\Delta^0.\mathrm{S}}$ habe einen Repräsentanten $m: G \times G \to G$, sodaß (G, m) H-Objekt in $\Delta^0.\mathrm{S}$ ist; die von m in den Morphismenmengen induzierte Multiplikation werde mit $\cdot$ bezeichnet. Für $n = 2$ sei $a := y_1 \cdot y_2$, $b := y_2 \cdot y_1$; für $n = 3$ sei $a := (y_1 \cdot y_2) \cdot y_3$, $b := y_1 \cdot (y_2 \cdot y_3)$. Dann ist (3) erfüllt. Ist $\perp$ die Differenz von Loopelementen, so ist $a \perp b$ im Fall $n = 2$ der *Kommutator* von y_1, y_2 und im Fall $n = 3$ der *Assoziator* von y_1, y_2 und y_3. Ähnlich ergeben sich *Distributoren*, die zum Beispiel für eine simpliziale Gruppe G einer Kategorie $\mathbf{C}$ in der simplizialen Gruppe $Hom_{\Delta^0\mathbf{C}}(G, G)$, die außer der Gruppenmultiplikation noch die von der Komposition induzierte Multiplikation besitzt, auftreten.

1.4 *Bemerkung:* Satz 1.2 läßt sich von $\Delta^0.\mathrm{S}$ auf $\Delta^0\mathbf{C}$ für gewisse Kategorien $\mathbf{C}$ verallgemeinern.

1) Sei $\mathbf{C}$ eine Kategorie mit Nullobjekt, (endlichen) Produkten und Koprodukten. Sei $K \in \Delta^0\mathrm{S}$ (mit K_n endlich für alle n) und $X \in \Delta^0\mathbf{C}$. Nach Kan [4] ist ein Produkt $K \otimes X \in \Delta^0\mathbf{C}$ definiert; für $\mathbf{C} = .\mathrm{S}$ ist $K \otimes X = K \times X / K \times \{*\}$. Damit läßt sich auf $\Delta^0\mathbf{C}$ eine Homotopierelation einführen und für $X, Y \in \Delta^0\mathbf{C}$ ist $Hom(X, Y) \in \Delta^0\mathrm{S}$ bzw. $\Delta^0.\mathrm{S}$ wie im Falle $\mathbf{C} = .\mathrm{S}$ erklärt. Zur Definition der Einhängung werde zunächst ein Smashprodukt $K \wedge X$, $K \in \Delta^0.\mathrm{S}$ (mit K_n endlich für alle n), $X \in \Delta^0\mathbf{C}$ eingeführt. Sei $(K \wedge X)_n := \coprod_k X_n^k$, $k \in K_n^* := K_n \setminus \{*\}$ und $X_n^k := X_n$ für $k \in K_n^*$; für $\varphi: [m] \to [n]$ sei $\varphi_*^{K \wedge X}$ durch $\varphi_*^{K \wedge X}\, in_k = in_{k'}\, \varphi_*^X$ für $k \in K_n^*$, $k' = \varphi_*^K(k) \neq *$ und $\varphi_*^{K \wedge X}\, in_k = 0$ für $\varphi_*^K(k) = *$ definiert. $K \wedge X$ ist ein Verallgemeinerung des Smashprodukts in [3]. In natürlicher Weise gilt

$\Delta^0\mathbf{C}(K \wedge X, Y) \cong \Delta^0.\mathrm{S}(K, Hom(X, Y))$ und

$Hom_{\Delta^0\mathbf{C}}(K \wedge X, Y) \cong Hom_{\Delta^0.\mathrm{S}}(K, Hom(X, Y))$.

Die Einhängung $\sum X$ von X sei jetzt wie im Falle $\mathbf{C} = .\mathrm{S}$ in [3] durch $\sum X := \Omega \wedge X$ definiert.

2) Satz 1.2 gilt für die Situation in 1), falls $\mathbf{C}$ folgende Distributivitätsbedingung erfüllt: Für $n \geqslant 1$, endliche Indexmengen $I(k)$, beliebige Objekte $X_{i(1)}, \ldots, X_{i(n)}$ aus $\mathbf{C}$, $i(k) \in I(k)$ und $J = \prod_{k=1}^n I(k)$ ist der Morphismus

$\langle \coprod_{k=1}^n in_{i(k)} \rangle : \coprod_J (\prod_{k=1}^n X_{i(k)}) \to \prod_{k=1}^n (\coprod_{I(k)} X_{i(k)})$ epimorph. $\mathbf{C}$ hat diese Eigenschaft, falls der Funktor $- \times X$, $X \in \mathbf{C}$ mit Koprodukten oder Fasersummen ver-