The Spectra of Hyponormal Integral Operators

The Spectra of Hyponormal Integral Operators[1)]

K. F. Clancey and C. R. Putnam

1. Recall that a bounded operator $T=H+iJ$ on a Hilbert space $\mathfrak{H}$ is said to be hyponormal if

$$T^*T - TT^* = D \geqq 0, \text{ that is, } HJ - JH = -iC, C = \tfrac{1}{2}D \geqq 0. \tag{1.1}$$

It is known that such operators behave to some extent like normal operators; in particular, $\operatorname{sp}(H)$ and $\operatorname{sp}(J)$ are just the (real) projections of $\operatorname{sp}(T)$ onto the real and imaginary axes; see Putnam [5b], p. 46.

Let H have the spectral resolution

$$H = \int \lambda \, dE_\lambda, \tag{1.2}$$

and let $E(\Delta)$ be the projection operator associated with an open interval Δ. For any bounded operator T (hyponormal or not), let $T_\Delta = E(\Delta)TE(\Delta)$, regarded as an operator on $E(\Delta)\mathfrak{H}$ and with spectrum $\operatorname{sp}(T_\Delta)$. Since $H_\Delta J_\Delta - J_\Delta H_\Delta = -iC_\Delta$, it is seen that T_Δ is hyponormal on $E(\Delta)\mathfrak{H}$ whenever T is hyponormal on $\mathfrak{H}$. It was shown in [5d] that if T is hyponormal, then

$$\operatorname{sp}(T_\Delta) \subset \operatorname{sp}(T). \tag{1.3}$$

In case the self-commutator D of T in (1.1) is compact, the relation (1.3) was proved by Clancey [2a].

A refinement of (1.3) was proved in [5f] to the following

$$\operatorname{sp}(T_\Delta) \cap \{z : \operatorname{Re}(z) \in \Delta\} = \operatorname{sp}(T) \cap \{z : \operatorname{Re}(z) \in \Delta\}, \tag{1.4}$$

Δ being any open interval. In view of the projection properties mentioned above, the real part of $\operatorname{sp}(T_\Delta)$ lies in the closure of Δ. It was noted in [5f] that, as a consequence of (1.4),

$$\operatorname{Im}[\operatorname{sp}(T) \cap \{z : \operatorname{Re}(z) = s\}] = \bigcap_\Delta \operatorname{sp}(E(\Delta)JE(\Delta)), \quad s \in \Delta, \tag{1.5}$$

the intersection being over all open intervals Δ containing s. This relation will be used below to determine the spectra of certain singular integral operators.

Suppose that

$$a(x), b(x) \in L^\infty(E), a(x) \text{ real}, \quad b(x) \neq 0 \text{ a.e. on } E, \tag{1.6}$$

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where E is a bounded set of positive measure on the real line. Let $T_0 = H_0 + iJ_0$ denote the bounded operator on $L^2(E)$ defined by

$$(H_0 f)(x) = xf(x) \quad \text{and} \quad (J_0 f)(x) = -\left[a(x) f(x) + \frac{b(x)}{i\pi} \int_E \frac{\bar{b}(t)}{t-x} f(t)\, dt\right], \tag{1.7}$$

where the integral is interpreted as a Cauchy principal value. It is easily verified that

$$H_0 J_0 - J_0 H_0 = -iC_0, \quad C_0 f = \pi^{-1}(f, b)\, b, \tag{1.8}$$

so that $C_0 \geqq 0$ and hence T_0 is hyponormal. It is seen that the range of C_0 is spanned by the vector $b \in L^2(E)$ and that $H_0 = x$ has simple spectrum and that the vectors $\{H_0^n b\}$, $n = 0, 1, 2, \ldots$. span $L^2(E)$.

Conversely, if $T = H + iJ$ is any hyponormal operator on H satisfying

$$T^*T - TT^* = D \geqq 0 \quad \text{and } D \text{ has rank one} \tag{1.9}$$

and

$$D = (\ \ , z)\, z \text{ and } \{H^n z\}, \quad n = 0, 1, 2, \ldots, \text{span } H, \tag{1.10}$$

then T is unitarily equivalent to a singular integral operator $T_0 = H_0 + iJ_0$ defined by (1.7). This result was first proved by Xa Dao-xeng [7]; a simpler proof using a result in [5a] was given by Rosenblum [6], p. 326.

It may be noted that the operator T_0 above is irreducible by virtue of the condition that $b(x) \neq 0$ a.e. on E. To see this, note that if $\Omega \neq 0$ reduces T_0, then Ω reduces both H_0 and J_0. If $f \in \Omega$, $f \neq 0$ (that is, $f(x) \not\equiv 0$ a.e.) and if $(f, b) \neq 0$, then $(C_0 f)(x) = \pi^{-1}(f, b)\, b(x) \neq 0$ a.e. on E, and hence $\{(H_0^n C_0 f)(x)\}$, $n = 0, 1, 2, \ldots$, span the space $L^2(E)$, that is, $\Omega = L^2(E)$. If $(f, b) = 0$, then, since $f \neq 0$, $C_0 H_0^N f \neq 0$ for some positive integer N. Otherwise, by Weierstrass' theorem, $f(x)\, b(x) \equiv 0$ a.e. and hence, $f(x) = 0$ a.e., a contradiction. Thus, if $g = H_0^N f \neq 0$, one can proceed as above to show that $\Omega = L^2(E)$.

THEOREM 1. *Let $T_0 = H_0 + iJ_0$ be the hyponormal operator on $L^2(E)$ defined by* (1.6) *and* (1.7). *Then* $\text{sp}(T_0)$ *is the set of numbers $z = s + it$ (s, t real) for which*

$$\text{meas}_1 \{x \in E \cap \Delta : -a(x) - |b(x)|^2 - \varepsilon < t < -a(x) + |b(x)|^2 + \varepsilon\} > 0 \tag{1.11}$$

holds for every $\varepsilon > 0$ and for every open interval Δ containing s.

THEOREM 2. *Let T_0 be defined as in Theorem* 1. *Then for almost all points $x \in E$, there exists some vertical segment $\{x + iy : a_x \leqq y \leqq b_x\}$, where $a_x < b_x$, belonging to the spectrum of T_0. In particular,* $\text{sp}(T_0)$ *cannot be totally disconnected.*

Theorem 1 generalizes results of Clancey [2a], Theorem 1 and Putnam [5c]. Its proof will be given in section 2. In a formulation involving a "determining set" or "determining function", Theorem 1 is contained in Clancey [2b] and Pincus [3c]. All of these proofs, including the one of the present paper, use results of either Pincus [3a] or Rosenblum [6] together with the relation (1.4) (or (1.5)) established in [5f]. It may also be noted that in [3c], the operator D of (1.9) is assumed only to be of trace class, rather than of rank one, and that $\mathfrak{H}$ is the least subspace reducing T and containing the range of D.

A hyponormal operator T is said to be completely hyponormal on $\mathfrak{H}$ if there is no non-trivial subspace of $\mathfrak{H}$ which reduces T and on which T is normal. A set S of the complex plane is said to have positive density if for every open disk N,

$$\text{meas}_2\,(S \cap N) > 0 \quad \text{whenever } S \cap N \neq \emptyset. \tag{1.12}$$

It was shown in [5d] that if T is completely hyponormal then its spectrum has positive density. The converse question of whether every compact set S is the spectrum of some completely hyponormal operator is unsettled, although some partial results have been obtained; see [5g], also Theorem 3 below and the remarks in sectoin 4.

For any set S, let S^- denote its closure and $\text{int}\,(S)$ its interior. There will be proved the following

THEOREM 3. *If S is any compact set for which*

$$S = (\text{int}\,(S))^- \tag{1.13}$$

(so that, in particular, S has positive density), then there exists a singular integral operator $T_0 = H_0 + iJ_0$ defined by (1.6) *and* (1.7) *for which*

$$\text{sp}\,(T_0) = S. \tag{1.14}$$

2. *Proof of Theorem* 1. It follows from Pincus [3a], p. 375, that $t \in \text{sp}\,(J_0)$, where J_0 is defined by (1.7), if and only if

$$\text{meas}_1\,\{x \in E: -a(x) - |b(x)|^2 - \varepsilon < t < -a(x) + |b(x)|^2 + \varepsilon\} > 0$$

for every $\varepsilon > 0$. (In this connection, see also Rosenblum [6], p. 323; also the remarks in Pincus and Rovnyak [4], p. 620.) If the multiplication operator $H = H_0 = x$ of (1.7) has the spectral resolution (1.2) then for any open interval Δ (for which $E \cap \Delta \neq \emptyset$), $E(\Delta)\, J_0 E(\Delta)$ is simply the integral operator J_0 restricted to $E \cap \Delta$. It follows that the condition $t \in \text{sp}\,(E(\Delta)\, J_0 E(\Delta))$ reduces to (1.11), and Theorem 1 now follows from (1.5).

Proof of Theorem 2. Since $b(x) \neq 0$ a.e. on E, then

$$E = \bigcup_{n=1}^{\infty} E_n, \text{ a.e., where } E_n = \{x \in E: |b(x)|^2 > 1/n\} \quad \text{for } n = 1, 2, \ldots.$$

Hence, $E_1 \subset E_2 \subset \cdots$ and $\text{meas}_1(E - E_n) \to 0$ as $n \to \infty$. Choose N so large that $\text{meas}_1(E_n) > 0$ for $n \geqq N$. Thus, at almost all $x \in E_n$, where $n \geqq N$, E_n has metric density 1. For such an x, let $L = \text{ess lim sup}\, a(t)$, where $t \to x$ and t is restricted to E_n. Then, in every open interval containing x and for every $\varepsilon > 0$, there exists a subset of E of positive measure for which $|a(x) - L| < \varepsilon$ and $|b(x)|^2 > 1/N$. It follows from the criterion of (1.11) that the segment $x + iy$, where $L - 1/N \leqq y \leqq L + 1/N$, belongs to the spectrum of T_0.

3. *Proof of Theorem* 3. For any Borel set α of the line, let $S(\alpha)$ denote the set $S(\alpha) = S \cap \{z: \text{Re}(z) \in \alpha\}$. For $k = 1, 2, \ldots$, let Π_k denote a grid of squares in the complex plane with sides parallel to the axes and of length 2^{-k}. We assume that the squares contain their lower and left sides and that $z = 0$ is a lower left corner of some square in each grid. Since S is compact then the projection on the x-axis of S is contained in some interval $[c, d]$. Now choose a disjoint family $\{K_p\}$, $p = 1, 2, \ldots$, of Cantor sets of positive measure in $[c, d]$ so that

$$\text{meas}_1\left(\bigcup_{p=1}^{q} K_p\right) \to d - c \quad \text{as} \quad q \to \infty. \tag{3.1}$$

Denote by $R_1, \ldots, R_{n_1}$ the elements of Π_1 satisfying

$$R_j \subset \text{int}(S) \equiv \Omega_1, \quad j = 1, \ldots, n_1, \tag{3.2}$$

and let $R'_1, \ldots, R'_{n_1}$ be respective smaller concentric closed squares of side 2^{-2}. Then for $j = 1, \ldots, n_1$, let K_{p_j} be the first K_p satisfying

$$\text{meas}_2(S(K_p) \cap R_j) > 0 \quad \text{and} \quad p_j > p_{j-1}. \tag{3.3}$$

Set $A_j = S(K_p) \cap R'_j$ and let D_j be the projection on the x-axis of A_j. Clearly, the set $\Omega_2 = \Omega_1 - \bigcup_{j=1}^{n_1} A_j$ is open. Denote by R_j, for $j = n_1 + 1, \ldots, n_1 + n_2$, the squares in Π_2 satisfying

$$R_j \subset \Omega_2, \quad j = n_1 + 1, \ldots, n_1 + n_2. \tag{3.4}$$

Again, form concentric squares $R'_{n_1+1}, \ldots, R'_{n_1+n_2}$ of side 2^{-3} and, for $j = n_1 + 1, \ldots, n_1 + n_2$, let K_{p_j} be the first K_p satisfying (3.3). Repeat the process of forming A_j and D_j for $j = n_1 + 1, \ldots, n_1 + n_2$ and set $\Omega_3 = \Omega_2 - \bigcup_{j=1}^{n_1+n_2} A_j$. If this process is continued for each q and grid Π_q one obtains a family of closed sets $\{A_j\}$, $j = 1, 2, \ldots$, satisfying

$$\text{closure}\left(\bigcup_{j=1}^{\infty} A_j\right) = S. \tag{3.5}$$

Now define functions $a(x)$ and $b(x)$ on $\bigcup D_j$ by setting

$$\left.\begin{aligned} -a(x) &= (\text{value of } y\text{-coordinate of the center of } R_j') \text{ on } D_j, \\ b(x) &= (\text{one-half the length of the side of } R_j')^{1/2} \text{ on } D_j. \end{aligned}\right\} \quad (3.6)$$

Then if T_0 is the singular integral operator given by (1.7) and (3.6) acting on $L^2(\bigcup D_j)$ it follows from Theorem 1 that relation (1.14) holds.

4. *Remarks.* It was shown in [5g] that there exist irreducible hyponormal operators satisfying (1.9) and having totally disconnected spectra. (An example was also given in [5e].) In view of the last part of Theorem 1, such an operator $T=H+iJ$ cannot be of the type $T_0=H_0+iJ_0$ defined by (1.6) and (1.7). That is, by the result of Xa Dao-xeng, since T satisfies (1.9), then relation (1.10) fails to hold.

It was shown in Theorem 3 that any compact set equal to the closure of its interior is the spectrum of some singular integral operator $T_0=H_0+iJ_0$ defined by (1.6) and (1.7). Of course, the spectrum of a general such operator need not be of this type; indeed, if $a(x)=0$ and if $b(x)$ is the characteristic function of a Cantor set E of positive measure, then (cf. Theorem 1) the spectrum of T_0 is the set $E\times[-1, 1]$.

It is interesting to note that although the spectrum of T_0 cannot be totally disconnected, nevertheless, it may be a Mergelyan Swiss cheese. (Recall that this is a set $X=D-\bigcup_{n=1}^{\infty} D_n$ where D is the closed unit disk and the D_n are open disjoint disks in D with radii r_n satisfying $\sum r_n<\infty$, and for which X is nowhere dense; see Zalcman [8], p. 69.) The proof of this assertion depends upon a result of W. K. Allard (see Brennan [1], p. 13) that almost every cross-section of a Swiss cheese is the union of a finite number of disjoint closed intervals; for details, see Clancey [2b].

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University of California, Los Angeles
Purdue University

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