Generalized Antipodes and the Borsuk Antipode Theorem

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The fact that one has the notion of antipodal points on the n-sphere, S^n, is often very useful. The Borsuk-Ulam theorem, which asserts that if $f: S^n \to R^n$ thenthere exists $x \in S^n$ with $f(x) = f(-x)$, is a particularly nice result having many interesting and amusing corollaries. The object of this paper is to define a generalized notion of when two points on an arbitrary Riemannian manifold are antipodal and then to exploit this concept to prove a generalization of the Borsuk antipode theorem as stated in [2, § 33]. It would seem that antipodal points as defined here should prove useful in other areas of differential geometry and topology. At any rate, it opens up a new direction for further theorems of the type just mentioned. Some of the interesting problems that arose in this context are listed in § 3.

§ 1. Generalized Antipodes

All the manifolds considered in this section will be closed and Riemannian. Recall that a Riemannian manifold is a pair (M^n, d), where M is a C^∞-manifold and d is a Riemannian metric on its tangent bundle. We shall assume basic facts from differential geometry which can be found in [3]. For example, d induces a metric on M which we shall also denote by d. In addition, all the necessary geodesics which we require will exist because of compactness.

Note. The metric d will always be fixed, and so we shall omit any reference to it except in isolated instances where there might be some confusion otherwise.

Let X be a closed subset of M.

DEFINITION. Let $y \in M$. We say that Γ is a minimal geodesic between y and X if Γ is a geodesic of M connecting y to a point $x \in X$ whose length is equal to the distance from y to X.

DEFINITION. $A(X) \equiv A(X, d) \equiv \{y \in M \mid$ there are at least two distinct minimal geodesics from y to $X\}$.

Elements of $A(X)$ are called the antipodal points, or antipodes, of X. This is in direct analogy with S^n, in case X is a point. We shall say that two points $x, y \in M$ are antipodal if $x \in A(y)$.

Note. Clearly, if $x \in A(y)$, then $y \in A(x)$, so that we do not have to distinguish between x being antipodal to y and y being antipodal to x.

There are easy examples to show that $A(x)$ need not be closed; however, we have the following lemma:

LEMMA 1. *The closure of $A(x)$ is the set of cut points of x. (See* [3, VIII. 7] *for the definition of a cut point.)*

Proof. We shall only outline the proof which was suggested by F. W. Warner. Let K be the closure of $A(x)$, and let $C(x)$ be the set of cut points of x. It is seen easily from the definitions that $A(x) \subseteq C(x)$. Since $C(x)$ is closed (see [3]), it follows that $K \subseteq C(x)$. Therefore, to prove the lemma, it suffices to show that $A(x)$ is dense in $C(x)$.

Now it is also shown in [3] that if $y \in C(x)$, then either $y \in A(x)$ or y is a conjugate point of x. But conjugate points of x (considered as a subset of the tangent space of M at x) were studied in [5]. There it was proved that the conjugate locus splits into a regular and a singular part and that the regular conjugate locus is dense in the conjugate locus and is a submanifold on the tangent space of codimension one. Let $y \in C(x)$ be a conjugate point. One proves that y belongs to the closure of $A(x)$ as follows: First, one may assume that y belongs to the regular part of the conjugate locus where one knows something about the behavior of the exponential map. Finally, one considers the two cases where the order of the conjugate point y is greater than 1, so that the kernel of the exponential map is tangent to the conjugate locus, and where the order of the conjugate point is equal to 1.

Remark. Not much seems to be known about the structure of the set of cut points. $C(x)$ is a strong deformation retract of $M - \{x\}$, and $M - C(x)$ is an open ball.

DEFINITION. $A_M \equiv \{(x, y) \in M \times M \mid y \in A(x)\}$.

§ 2. The Generalized Borsuk-Ulam Theorem

This section is devoted entirely to proving the main result of this paper, namely Theorem 1, which is a generalized version of the Borsuk-Ulam Theorem. In what follows, M^n will denote as closed manifold with a fixed Riemannian metric d and W^m will be an arbitrary differentiable manifold, not necessarily compact or without boundary.

DEFINITION. Given any map $f: M^n \to W^m$ we let

$$A(f) \equiv A(f, d) \equiv \{(x, y) \in A_M \mid f(x) = f(y)\}.$$

THEOREM 1. a. *If $f: M^n \to W^m$ where $n > m$, then* $\dim A(f) \geqslant n - m$.
b. *If $f: M^n \to W^n$ has $f^*: H^n(W^n; Z_2) \to H^n(M^n; Z_2)$ trivial, then $A(f) \neq \emptyset$.*

The proof of Theorem 1 is an adaptation of the argument given in [2, § 33]. For simplicity, we shall assume that M and W are connected. We begin with a technical lemma which we shall need to be able to perform various constructions.

If $Y \subseteq M$, $\mathrm{int}(Y)$ will denote the interior of Y.

Let $S = S_M: M \times M \to M \times M$ be the involution $S(x, y) = (y, x)$.

LEMMA 2. *Let $x_0 \in M$. For every open neighborhood O of $A_M \subseteq M \times M$ there is a tubular neighborhood U of the diagonal Δ_M in $M \times M$, a closed n-disk D in M with $x_0 \in \mathrm{int}(D)$, and a closed manifold $N^n \subseteq O - U$ satisfying*

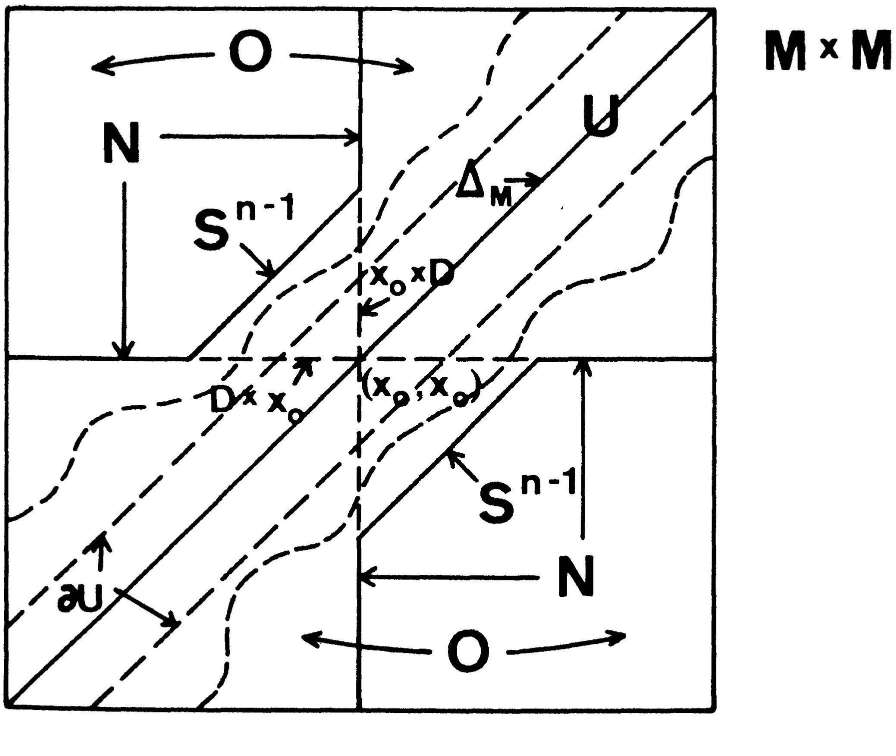
(a) $x_0 \times \partial D \cup \partial D \times x_0 \subseteq O - A_M$;

(b) $S(U) = U$, $S(\partial U) = \partial U$, $S(N) = N$;

(c) $x_0 \times (M - \mathrm{int} D) \cup (M - \mathrm{int} D) \times x_0 \subseteq N$; *and*

(d) *we can identify* $N - (x_0 \times (M - \mathrm{int} D) \cup (M - \mathrm{int} D) \times x_0)$ *with* $S^{n-1} \times [-1, 1]$ *in such a way that* $S(x, t) = (-x, -t)$ *in an open neighborhood of* $S^{n-1} \times 0$ *in* $S^{n-1} \times \times [-1, 1]$. *(We shall identify* $S^{n-1} \times 0$ *with* S^{n-1}.*)*

Proof. The picture below may clarify this lemma.



Let $O_\varepsilon = \{(x, y) \in M \times M \mid \mathrm{dist}(y, A(x)) < \varepsilon\}$. Then O_ε is an open neighborhood of A_M and $O_\varepsilon \subseteq O$ for some sufficiently small $\varepsilon > 0$ which we fix. One can easily find a tubular neighborhood U of Δ_M such that $U \cap O_\varepsilon = \emptyset$, $S(U) = U$, and $S(\partial U) = \partial U$.

Next, let $\delta = \varepsilon/2$ and let D be an n-disk in $M - A(x_0)$ which contains x_0 in its interior and such that $x_0 \times (M - \mathrm{int} D) \cup (M - \mathrm{int} D) \times x_0 \subseteq O_\delta$. Consider $D \times D \subseteq \subseteq M \times M$ and identify D with D^n so that x_0 corresponds to the origin and geodesics correspond to straight lines. (This identification of D with D^n does not necessarily preserve the metric, however.) Define

$$\alpha : S^{n-1} \times [-1, 1] \to D \times D$$

by $\alpha(x, t) = ((\tfrac{1}{2})(t-1)x, (\tfrac{1}{2})(t+1)x)$. It is easily checked that α is an imbedding and $S(\alpha(S^{n-1}\times[-1,1])) = \alpha(S^{n-1}\times[-1,1])$. Note that $S(\alpha(x,t)) = \alpha(-x,-t)$. Now $\alpha(S^{n-1}\times\{-1,1\}) \subseteq O_\delta$. Suppose $\alpha(S^{n-1}\times(-1,1)) \nsubseteq O_\varepsilon$. The idea will be to push $\alpha(S^{n-1}\times(-1,1))$ into O_ε in a symmetric manner.

Let $(x, t) \in S^{n-1}\times[-1,1]$ and define

$$\beta(x, t) : [-1, 1] \to [-1, 1]x \times [-1, 1]x$$

by $\beta(x,t)(s) = (((\tfrac{1}{2})(t-1) - s(1+t)/2)x, \ ((\tfrac{1}{2})(t+1) + s(1-t)/2)x)$ for $s \geqslant 0$ and $\beta(x,t)(s) = (((\tfrac{1}{2})(t-1) - s(t-1)/2)x, ((\tfrac{1}{2})(t+1) + s(t+1)/2)x)$ for $s \leqslant 0$. Next, given $\mu : S^{n-1}\times[-1,1] \to [-1,1]$, define

$$\alpha_\mu : S^{n-1} \times [-1, 1] \to D \times D$$

by $\alpha_\mu(x,t) = \beta(x,t)(\mu(x,t))$.

CLAIM. *We can find a differentiable μ with the property that*
(1) $\mu(x, 1) = 0 = \mu(x, -1)$, *for* $x \in S^{n-1}$*; and*
(2) $\alpha_\mu(S^{n-1}\times[-1,1]) \subseteq O_\varepsilon$.

This claim is proved by studying the pairs of antipodal points in the 'plane' $[-1,1]x \times [-1,1]x \subseteq D \times D$. It will be useful to look at the function

$$\theta(x, t) : [-1, 1] \to R$$

given by $\theta(x,t)(s) = \mathrm{dist}(\beta(x,t)(s), A_M)$. To be precise, we are interested in $\theta(x,t)^{-1}[0,\delta)$. Our choice of D enables us to define μ by

$$\begin{aligned} \mu(x,t) &= 0, \text{ if } 0 \in \theta(x,t)^{-1}[0,\delta), \\ &= \inf[0,1] \cap \theta(x,t)^{-1}[0,\delta), \text{ if the length (in } M\text{) of the} \\ &\qquad \text{arc}\,[(\tfrac{1}{2})(t-1), (\tfrac{1}{2})(t+1)]x \text{ is equal to } d((\tfrac{1}{2})(t-1)x, (\tfrac{1}{2})(t+1)x), \\ &= \sup[-1,0] \cap \theta(x,t)^{-1}[0,\delta), \text{ otherwise}. \end{aligned}$$

It is easy to see that μ is well defined and has the desired properties. Furthermore, one can check that α_μ is an imbedding and $\alpha_\mu(S^{n-1}\times[-1,1])$ has the same symmetry properties with respect to S as did $\alpha(S^{n-1}\times[-1,1])$.

Finally, let $N = x_0 \times (M-D) \cup (M-D) \times x_0 \cup \alpha_\mu(S^{n-1}\times[-1,1])$. N will be a differentiable n-manifold (after we round off corners) with $N \subseteq O_\varepsilon$. Conditions (a)–(d) in the lemma are readily checked. This proves Lemma 2.

Proof of Theorem 1.a. *Case* 1. Assume W^m is closed and $n > m$: Let S, O, U, and N be as in Lemma 2. Let B be the double of $B_0 = M \times M - \operatorname{int} U$ and set $X = B \times W \times W$.

Define fixed point free involutions $R: B \to B$ and $T: X \to X$ by $T(b, u, v) = (R(b), v, u)$, where R is the double of $S \mid B_0$. There are natural inclusions $S^{n-1} \subseteq N \subseteq B_0 \subseteq B$, and by Lemma 2, S^{n-1}, N and B_0 are invariant under R. Therefore, X/T, B/R, N/R, and $P^{n-1} = S^{n-1}/R$ are closed manifolds of dimension $2(n+m)$, $2n$, n, $n-1$, respectively.

Now let $\varphi_m \in H^m(X/T; Z_2)$ be the dual of $i_* \mu \in H_{2n+m}(X/T; Z_2)$, where $i: B/R \times \times \Delta_W \to X/T$ is the natural inclusion and μ is the fundamental class of $B/R \times \Delta_W$.

We easily obtain the following fact:

$$\left.\begin{array}{l}\varphi_m \text{ belongs to the kernel of } H^m(X/T; Z_2) \to H^m(X/T - V; Z_2) \text{ for every} \\ \text{open set } V \text{ in } X/T \text{ containing } B/R \times \Delta_W.\end{array}\right\} \qquad \text{(I)}$$

Next, we observe that we can identify N/R with $P^n \# M^n$ in a natural way. In fact, let $S^{n-1} \times [-\varepsilon, \varepsilon]$, $\varepsilon > 0$, be a collar of S^{n-1} in N with the property that $R(x, t) = (-x, -t)$, $x \in S^{n-1}$, $t \in [-\varepsilon, \varepsilon]$. Such a collar exists for some ε by Lemma 2.d. If we think of S^n as $S^{n-1} \times [-\varepsilon, \varepsilon]$ with $S^{n-1} \times \varepsilon$ and $S^{n-1} \times (-\varepsilon)$ collapsed to the north and south pole, respectively, then we get a natural map $\lambda: N \to S^n$ which we can also assume to commute with R and the standard antipodal map on S^n. λ therefore induces a map $\lambda_1: N/R = P^n \# M^n \to P^n$ which essentially collapses the part from M^n to a point. Similarly, we get a map $\lambda_2: N/R = P^n \# M^n \to M^n$ which collapses the part from P^n to a point. (The M^n in N/R comes from Lemma 2.c.)

Consider the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p_1"'] & B \arrow[r, phantom, "\supseteq"] \arrow[d, "p_2"] & N \arrow[r, phantom, "\supseteq"] \arrow[d] & S^{n-1} \arrow[r, phantom, "\subseteq"] \arrow[d] & S^n \arrow[d] \\
X/T \arrow[r, "\pi"'] & B/R \arrow[r, phantom, "\supseteq_{i_1}"] & N/R \arrow[r, phantom, "\supseteq_{i_2}"] \arrow[d, "\lambda_2"'] \arrow[rr, bend right, "\lambda_1"'] & P^{n-1} \arrow[r, phantom, "\subseteq_{i_3}"] & P^n \\
 & & M^n & &
\end{tikzcd} \qquad \text{(II)}$$

where the i_j are the natural inclusions and π, p_j are the projections. ($P^{n-1} = S^{n-1}/R = S^{n-1}/S$ by Lemma 2.d.) Let $c \in H^1(B/R; Z_2)$, $c_1 \in H^1(P^n; Z_2)$, $c_2 \in H^1(N/R; Z_2)$, and $c_3 \in H^1(P^{n-1}; Z_2)$ be the characteristic classes of the involutions on B, S^n, N, and S^{n-1}, respectively (see [2, p. 60] for a definition of the c's). By naturality and the commutativity of (II), we have that $i_1^*(c) = c_2$, $i_2^*(c_2) = c_3$, $\lambda_1^*(c_1) = c_2$, and $i_3^*(c_1) = c_3$.

Define

$$\psi: H^i(P^n; Z_2) \oplus H^i(M; Z_2) \to H^i(N/R; Z_2)$$

by $\psi(u+v) = \lambda_1^*(u) + \lambda_2^*(v)$ for $u \in H^i(P^n; Z_2)$, $v \in H^i(M; Z_2)$. The following fact is easily checked:

ψ is an isomorphism for $i \neq 0$, n and onto for $i = 0$, n. (III)

Finally, assume $f: M^n \to W^m$ is a map. Define a cross-section

$$\sigma : B/R \to X/T$$

by $\sigma(p_2(x, y)) = p_1(x, y, f(x), f(y))$, i.e., $\pi\sigma = 1$.

LEMMA 3. *$i_1^* \sigma^*(\varphi_m) = c_2^m + \lambda_2^*(v) \in H^m(N/R; Z_2)$, for some $v \in H^m(M; Z_2)$.*

Proof. By (III), we can write $i_1^* \sigma^*(\varphi_m) = \psi(u + v)$ for some $u \in H^m(P^n; Z_2)$ and $v \in H^m(M; Z_2)$. Since $\sigma^*(\varphi_m)$ depends only on the homotopy class of f, we may assume that $f(D) = y_0 \in W$ and $\sigma(p_2(z)) = p_1(z, y_0, y_0)$ for $z \in S^{n-1} \subseteq N \subseteq B$. In this way we get a commutative diagram

$$\begin{tikzcd} P^{n-1} \arrow[r, "\sigma_1"] \arrow[d, "i_1 i_2"'] & B/R \times \Delta_W \arrow[d, "i"] \\ B/R \arrow[r, "\sigma"'] & X/T \end{tikzcd}$$

where $\sigma_1(z) = (z, y_0, y_0)$ for $z \in P^{n-1}$. Then it follows just as in the proof of Theorem 33.1 in [2] that $i_2^* i_1^* \sigma^*(\varphi_m) = c_3^m$. But $i_2^* \lambda_2^* = 0$ and $n > m$ imply that $u = c_1^m$. This proves the lemma.

Next, define

$$A_0(f) = \{(x, y) \in O \cap B_0 \mid f(x) = f(y)\},$$
$$A_1(f) = p_2(A_0(f)) \subseteq B/R.$$

Again, an argument as in (33.2) of [2] establishes the following:

$$\left.\begin{array}{l} \text{For every open neighborhood } V \text{ of } A_1(f),\ c_2^m + \lambda_2^*(v) \text{ belongs to the} \\ \text{kernel of } H^m(N/R; Z_2) \to H^m(N/R - V; Z_2). \text{ In particular, } c_2^m + \lambda_2^*(v) \\ \text{can be represented by a cocycle } \alpha_m \text{ with support in } V \cap N/R. \end{array}\right\} \quad \text{(IV)}$$

LEMMA 4. *$0 \neq c_2^n \in H^n(N/R; Z_2)$.*

Proof. Clearly it suffices to show that the map $\lambda_1^* : H^n(P^n; Z_2) = Z_2 \to H^n(N/R; Z_2) = Z_2$ is not trivial. But this follows from the fact that $(\lambda_1)_* : H_n(N/R; Z_2) \to \to H_n(P^n; Z_2)$ is an isomorphism.

Now, consider the diagram

$$\begin{tikzcd} c_2^{n-m} \in H^{n-m}(N/R; Z_2) \arrow[r, "j_1^*"] & H^{n-m}(N/R \cap A_1(f); Z_2) \\ c^{n-m} \in H^{n-m}(B/R; Z_2) \arrow[u, "i_1^*"] \arrow[r, "j_2^*"'] & H^{n-m}(A_1(f); Z_2), \arrow[u] \end{tikzcd}$$

where j_1^* and j_2^* are induced by the natural inclusions. Suppose that $j_2^*(c^{n-m}) = 0$.

Consideration of the support of representative cocycles for $c_2^m+\lambda_2^*(v)$ and c_2^{n-m}, together with (IV), would show that $c_2^n=(c_2^m+\lambda_2^*(v))\cdot c_2^{n-m}=0$. But this contradicts Lemma 4.

It follows that

$$j_2^*(c^{n-m}) \neq 0. \tag{V}$$

Observe that $A(f)\subseteq B_0\subseteq B$. Let $A'(f)=p_2(A(f))\subseteq A_1(f)$. There is a commutative triangle of inclusion maps

$$\begin{tikzcd}
c^{n-m} \in H^{n-m}(B/R;Z_2) \arrow[rr, "j_2^*"] \arrow[dr, "j^*"'] & & H^{n-m}(A_1(f);Z_2) \arrow[dl] \\
& H^{n-m}(A'(f);Z_2). &
\end{tikzcd}$$

Assume that $j^*(c^{n-m})=0$. Then there is an open neighborhood of the closure of $A'(f)$ such that the support of c^{n-m} lies outside of this neighborhood. In fact, it is easy to see that there is some O and U in Lemma 2 so that the support of c^{n-m} will lie outside some open neighborhood of the corresponding $A_1(f)$. Therefore, $i_2^*(c^{n-m})=0$. Since this contradicts (V), we have proved that $j^*(c^{n-m})\neq 0$, i.e., $i^*\neq 0$. It follows that $\dim A'(f)\geqslant n-m$ and $\dim A(f)\geqslant n-m$. This finishes the proof of Case 1.

Case 2. W^m arbitrary and $n>m$: This case follows from Case 1 as in [2, 33].

Proof of Theorem 1.b. Suppose $f: M^n\to W^n$ and $f^*: H^n(W;Z_2)\to H^n(M;Z_2)$ is trivial. As before, the case of general W reduces to the case W is closed. Therefore, from now on we assume that W is closed, and we shall keep the same notation as in the proof of Theorem 1.a.

Define $F: B\to W\times W$ by $F(x,y)=(f(x),f(y))$. We may assume that $f(D)=y_0$ and $F(z)=(y_0,y_0)$ for $z\in S^{n-1}\times[-1,1]\subseteq N$, since everything will only depend on the homotopy classes of f and F. Then $F\,|\,N$ is an equivariant map with respect to R and S_W. In fact, $F(N)\subseteq W\vee W=W\times y_0\cup y_0\times W$. F induces a map

$$F_1: N/R = P^n \# M^n \to W \vee W/S_W = W.$$

Let $N_1=(x_0\times(M-\operatorname{int}D))\cup(S^{n-1}\times[0,1])\subseteq N$ and consider the commutative diagram

$$\begin{tikzcd}
& & H^n(W,y_0;Z_2) \arrow[dll, "f^*"'] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr, "F_1^*"] & & \\
H^n(M;Z_2) \arrow[r, phantom, "\approx"] & H^n(M,D;Z_2) \arrow[r, phantom, "\approx"] & H^n(N_1,S^{n-1};Z_2) \arrow[r, phantom, "\approx"] & H^n(N/R,P^{n-1};Z_2) \arrow[r, phantom, "\approx"] & H^n(N/R;Z_2).
\end{tikzcd}$$

Since $f^*=0$ and all the horizontal maps are isomorphisms, we get

$$F_1^* = 0. \tag{VI}$$

Using the diagram

$$\begin{tikzcd}
Y/T \arrow[r, "i_5"] & X/T \\
N/R \times (y_0 \times y_0) \arrow[u, "i_4"] \arrow[r] & B/R \times \Delta_W, \arrow[u, "i"']
\end{tikzcd}$$

where $Y=N\times(W\vee W)$ and an argument similar to the one for (33.4) in [2], we see that

$$i_4^* i_5^*(\varphi_n) = c_2^n \otimes 1. \tag{VII}$$

Let $\sigma_3=\sigma \mid (N/R): N/R \to Y/T$.

LEMMA 5. $\sigma_3^* i_5^*(\varphi_n) = c_2^n \in H^n(N/R;\, Z_2)$.

Proof. Let $\pi_1=\pi \mid (Y,\, T)$, so that $\pi_1\sigma_3=1$. Set $\gamma_n=\pi_1^*(c_2^n)\in H^n(Y/T;\, Z_2)$. Then $\sigma_3^*(\gamma_n)=\sigma_3^*\pi_1^*(c_2^n)=c_2^n$ and $i_4^*(\gamma_n)=i_4^*\pi_1^*(c_2^n)=c_2^n\otimes 1$. By (VII) we have that $i_4^*(\gamma_n+i_5^*(\varphi_n))=c_2^n\otimes 1+c_2^n\otimes 1=0$. Thus

$$\left.\begin{array}{l} \gamma_n+i_5^*(\varphi_n) \text{ lies in the image of} \\ j_6^*: H^n(Y/T,\, N/R\times(y_0\times y_0);\, Z_2)\to H^n(Y/T;\, Z_2). \end{array}\right\} \tag{VIII}$$

Consider the diagram

$$\begin{tikzcd}
H^n(Y/T, N/R \times (y_0 \times y_0); Z_2) \arrow[r, "j_6^*"'] & H^n(Y/T; Z_2) \arrow[d, "\sigma_3^*"] \\
H^n(W, y_0; Z_2) \arrow[u, "\beta^*"] \arrow[r, "F_1^*"'] & H^n(N/R; Z_2),
\end{tikzcd}$$

where β^* is the following isomorphism: First, the projection $N\times(W\vee W)\to W\vee W$ induces a map $\beta: Y/T=N\times(W\vee W)/T\to W\vee W/S_W=W$ with $\beta(N/R\times(y_0\times y_0))=y_0$. But $Y/T-(N/R\times(y_0\times y_0))=N\times W-(N\times y_0)$; and since the projection $N\times W\to W$ clearly induces an isomorphism $H^n(W, y_0;\, Z_2)\to H^n(N\times W,\, N\times y_0;\, Z_2)$, β^* has the same property.

However, $F_1^*=0$ by (VI) and $\sigma_3^* j_6^* \beta^* = F_1^*$. Therefore, $\sigma_3^* j_6^*=0$. Combining this fact with (VIII), we get that $\sigma_3^*(\gamma_n+i_5^*(\varphi_n))=0$. Consequently, $\sigma_3^* i_5^*(\varphi_n)=\sigma_3^*(\gamma_n)=c_2^n$, and Lemma 5 is proved.

The rest of the proof of Theorem 1.b proceeds just as the proof in Theorem 1.a.

Namely, one has a diagram

$$\begin{tikzcd}
H^0(N/R; Z_2) \arrow[rr, "j_1^*"] & & H^0(N/R \cap A_1(f); Z_2) \\
1 \in H^0(B/R; Z_2) \arrow[u, "i_1^*"] \arrow[rr, "j_2^*"] \arrow[dr, "j^*"'] & & H^0(A_1(f); Z_2) \arrow[u] \arrow[dl] \\
& H^0(A'(f); Z_2) &
\end{tikzcd}$$

and one shows successively that $j_2^*(1) \neq 0$ and $j^*(1) \neq 0$. Otherwise, $1 \in H^0(N/R; Z_2)$ and $c_2^n \in H^n(N/R; Z_2)$ have disjoint support, i.e., $c_2^n = 1 \cdot c_2^n = 0$, which is a contradiction. $j^*(1) \neq 0$ clearly implies that $A(f) \neq \emptyset$. This finishes the proof of Theorem 1.b.

§ 3. Some Applications and Problems

We begin with some amusing corollaries of Theorem 1.

COROLLARY 1. *The antipodal relation on S^n with respect to an arbitrary Riemannian metric d always agrees somewhere with the standard antipodal relation, i.e., for every metric d there exists an $x_d \in S^n$ such that $-x_d \in A(x_d, d)$.*

Proof. Let $T: S^n \to S^n$, $T(x) = -x$, be the standard involution and consider the projection $p: S^n \to P^n = S^n/T$.

Note. If M^n is not a homotopy sphere, then we always have $A(x, d) \cap A(x, d') \neq \emptyset$ for arbitrary Riemannian metrics d, d' on M.

COROLLARY 2. *Let T be an involution of a closed Riemannian manifold M^n such that M/T is a C^∞-manifold. If $p: M \to M/T$ is the projection and if $p^*: H^n(M/T; Z_2) \to$ $\to H^n(M; Z_2)$ is trivial, then $T(x) \in A(x)$ for some $x \in M$, i.e., T sends some point to an antipodal point.*

Proof. This corollary also follows immediately from Theorem 1.b applied to the map p.

Another easy consequence is

COROLLARY 3. *Suppose M^n is a closed Riemannian manifold and M^n, W^n are oriented. If $f: M^n \to W^n$ has even degree, then there is an $x \in M$ and some $y \in A(x)$ such that $f(x) = f(y)$.*

It was pointed out by the reviewer that Corollary 1, 2, and probably 3 can be proved directly quite easily without the use of Theorem 1. Namely, one assumes that they are false and uses the minimal geodesics which then exist between appropriate pairs of points to arrive at a contradiction by constructing a homotopy between the identity map and another map which factors through P^n in the case of Corollary 1 and M/T in the case of Corollary 2. Nevertheless, we decided to list these corollaries here because they fit naturally within the framework of Borsuk-Ulam type theorems. It seems reasonable to expect that in time more substantial applications of Theorem 1 will be found. Perhaps the theorem will first have to be improved and we list some directions which this improvement might take.

(1) It does not appear necessary to have M^n either Riemannian or differentiable. M^n probably only has to be a closed topological manifold with a given metric. Geodesics would then be replaced by rectifiable paths and $A(x)$ would become the set of $y \in M$ which are joined to x by at least two distinct rectifiable paths of minimal length. The proof of Theorem 1 could then be copied.

(2) Is it sufficient to have W^m be a simplicial complex (which is perhaps locally finite and countable so that it imbeds in Euclidean space)? This question is raised in [2] for the special case $M = S^n$.

(3) Can one prove a theorem of the following type: Let $f: M^n \to W^m$ and assume that $f_*: H_i(M; Z_2) \to H_i(W; Z_2)$ is not one-to-one for some i. Then $\dim A(f) \geqslant k$, where k is an integer which depends only on i (and of course n and m). Probably the best way to study this problem which takes into account the connectivity of f is to consider (2) first.

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