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Foliations and Compact Lie Group Actions

by J. S. PASTERNACK [1])

§ 1. Introduction

This paper is about smooth foliations and contains as an application of the main result a theorem on the existence of almost free compact Lie group actions.

Let M be a smooth (i.e., C^{∞}) manifold admitting a smooth foliation. Let $T(M)$ be the tangent bundle of M, let E be the sub-bundle of $T(M)$ consisting of tangents to the leaves of the foliation and let ν be the normal bundle to the leaves $\nu = T(M)/E$. The Bott integrability criterion [5] gives that

$$\mathrm{Pont}^{(r)}(\nu; R) = 0 \quad \text{for} \quad r > 2\cdot\dim(\nu) \tag{1.1}$$

where $\mathrm{Pont}^{(r)}(\nu; R)$ contained in $H^*(M; R)$ is the real Pontryagin ring generated by the real Pontryagin class of ν.

The main result of this paper, Theorem I (properly stated in Section 2), is that for the special case when the foliation admits an appropriate Riemannian structure

$$\mathrm{Pont}^{(r)}(\nu; R) = 0 \quad \text{for} \quad r > \dim(\nu). \tag{1.2}$$

Furthermore, in this special case if ν is an orientable bundle, then

$$\mathrm{Pont}_{\chi}^{(r)}(\nu; R) = 0 \quad \text{for} \quad r > \dim(\nu), \tag{1.3}$$

where $\mathrm{Pont}_{\chi}^{*}(\nu; R)$ is $\mathrm{Pont}^{*}(\nu; R)$ with the real Euler class $\chi(\nu)$ adjoined. In other words, we make an additional differo-geometric assumption on the foliation and prove a stronger result for the normal bundle. It is known that (1.2) is not true for an arbitrary foliation, but to the best of the author's knowledge it is unknown whether or not (1.1) is a best possible result on the rational characteristic classes of the normal bundle.

A Lie group acting smoothly on a manifold generates a smooth foliation of the manifold whenever all of the orbits of the group action are of the same dimension. The leaves of the foliation are the orbits. In case the Lie group is compact the foliation generated by the action will be shown to satisfy the hypothesis of Theorem I and in Section 5 we prove as a Corollary to Theorem I the following result on almost-free compact Lie group actions. (An action is almost free if all the isotropy groups are discrete.)

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COROLLARY 1. *Let M be a smooth n-manifold admitting an almost free action of a compact k-dimensional Lie group. Then letting $T=T(M)$*

$$\mathrm{Pont}^{(r)}(T; R) = 0 \quad \textit{for} \quad r > n-k.$$

It is interesting to note that Corollary 1 and also Theorem I are false for the integral Pontryagin rings. In Section 6 an example is given of a specific 4-manifold admitting an almost free action of the circle group, but $p_1(T) \neq 0$ where $p_1(T) \in H^4(M; \mathbf{Z})$ is the first integral Pontryagin class.

Corollary 1 can be extended without difficulty following a general idea of Bott [5], (cf., p. 92 in [12]).

COROLLARY 2. *Suppose M is an n-dimensional manifold admitting an action of a k-dimensional compact Lie group the action being almost free off a singular set Σ. Then letting $T=T(M)$*

$$\mathrm{Pont}^{(r)}(T; R) \subset j^{(r)}(H^{(r)}(M, M-\Sigma; R)) \quad \textit{for} \quad r > n-k$$

where $j^{(r)}: H^{(r)}(M, M-\Sigma; R) \to H^{(r)}(M; R)$ is inclusion.

It would be interesting to have for each $\omega \in \mathrm{Pont}^{(r)}(T)$, $r>n-k$, a recipe for $\eta \in H^{(r)}(M, M-\Sigma; R)$ satisfying $\omega = j^{(r)}(\eta)$ in terms of the local invariants of Σ and the behavior of the action near Σ. This program has been carried out in case $k=1$ and M orientable by Bott [4], and, Baum and Cheeger [2]. The general case is an open problem.

This paper will assume a knowledge of the theory of characteristic classes and of the Chern-Weil theory although essential results of the latter will be reviewed. The author wishes to thank Professor Raoul Bott for his help and encouragement in this research, and Professor André Haefliger for his suggestions which considerably simplified the definition of an R-foliation and the proof of Theorem I.

§ 2. R-Foliations

The manifolds considered in this paper are smooth finite dimensional paracompact Hausdorf spaces. On a manifold M an *R-foliation*[2]) of codimension q is given by the following data:

(1) An auxiliary q-dimensional Riemannian manifold B.

(2) An open covering $\{U_i\}_{i \in I}$ of M for I some indexing set and for each i a smooth submersion $f_i: U_i \to B$.

(3) For $x \in U_i \cap U_j$ there is an isometry γ_{ji}^x from a neighborhood of $f_i(x)$ onto a neighborhood of $f_j(x)$ satisfying $f_j = \gamma_{ji}^x \circ f_i$ on a neighborhood of x.

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[2]) R-foliations have previously been studied from a different point of view by B. Reinhart [13].

Notice that if one drops the condition that B is Riemannian and that the γ_{ji}^x are local isometries then one recovers a definition equivalent to the usual definition of a foliation (compare [8]). Not every foliation admits a structure as an R-foliation (cf. [13] and [12]); for example the Reeb foliation of S^3 is not an R-foliation.

Given an R-foliation if E is the sub-bundle of $T(M)$ satisfying $E_x = \ker(df_i|_x)$ for $x \in U_i$ then E is tangent to leaves of the foliation and $v = T(M)/E$ is the normal bundle. The main theorem of this paper is the following.

THEOREM I. *For v the normal bundle to an R-foliation of a manifold*

$$\mathrm{Pont}^{(r)}(v; R) = 0 \quad \textit{for} \quad r > \dim(v).$$

Moreover, if v is orientable then

$$\mathrm{Pont}_{\chi}^{(r)}(v; R) = 0 \quad \textit{for} \quad r > \dim(v).$$

The proof of this theorem given in Section 4 is based on Chern-Weil construction of the characteristic classes from the curvature of a connection.

Remark. One can define a pseudo-R-foliation by requiring in the definition that B be a pseudo-Riemannian manifold (i.e., a manifold with a symmetric non-degenerate smooth bilinear form on the tangent bundle). Except for the results on the Euler class Theorem I is true for pseudo-R-foliations, the proof being essentially the same as the proof to be given for Theorem I.

§ 3. Review of Connections and the Chern-Weil Construction

3.1. *Connections*

Let M be a smooth manifold with cotangent bundle T^* and let V be a smooth q-dimensional vector bundle over M. Let $\Gamma(\cdot)$ denote the functor associating to a vector bundle its vector space of smooth sections. A smooth connection on V is an operator $D: \Gamma(V) \to \Gamma(T^* \otimes V)$ satisfying

$$\left.\begin{array}{l} \text{(i) } D(s_1 + s_2) = Ds_1 + Ds_2 \quad \textit{for} \quad s_1, s_2 \in \Gamma(V) \\ \text{(ii) } D(fs) = df \otimes s + fDs \quad \textit{for} \quad s \in \Gamma(V), \ f \text{ a smooth} \\ \quad\ \text{function on } M \text{ and } d \text{ the usual exterior derivative.} \end{array}\right\} \qquad (3.1)$$

Given U an open set of M and a framing $\{s_1, s_2, \ldots, s_q\}$ of V over U then a connection D defines over U a matrix of 1-forms $\|\theta_{ij}\|$ satisfying

$$Ds_i = \sum_{j=1}^{q} \theta_{ij} \otimes s_j.$$

The curvature of D, denoted $K(D)$, is the global section of $\Lambda^2(T^*) \otimes \mathrm{End}(V)$ ($\mathrm{End}(V)$ is the endomorphism bundle of V) which with respect to the framing $\{s_1, s_2, \ldots, s_q\}$

over U satisfies

$$K(D)\,|\,U = \left\| d\theta_{ij} - \sum_{k=1}^{q} \theta_{ik} \wedge \theta_{kj} \right\|. \tag{3.2}$$

Suppose now that N is a manifold and $f: N \to M$ is a smooth map. The connection D on V pulls back to a connection $f^{-1}(D)$ on the pull back bundle $f^{-1}(V)$; locally over the open set $f^{-1}(U)$, $\{f^{-1}(s_1), \ldots, f^{-1}(s_q)\}$ frames $f^{-1}(V)$ and

$$f^{-1}(D)(f^{-1}(s_i)) = \sum_{k=1}^{q} f^{(1)}(\theta_{ik}) \otimes f^{-1}(s_k) \tag{3.3}$$

where $f^{(1)}$ is the induced map on 1-forms.

By naturality of the exterior derivative, (3.2) and (3.3) can be combined to yield

$$K(f^{-1}(D))\,|\,f^{-1}(U) = f^{(2)}(K(D)\,|\,U), \tag{3.4}$$

where $f^{(2)}$ is the natural induced map on 2-forms.

3.2. *Chern-Weil Construction*

The Chern-Weil theory exploits the fact that the de Rham cohomology of a manifold is isomorphic to the singular cohomology with real coefficients. The idea is to construct from the curvature of a connection on a vector bundle closed differential forms which represent the real characteristic classes of the bundle. Good references are [6], [7] and [14], here we briefly describe the results necessary for our purposes.

Let $gl(q, R)$ be the linear space of $q \times q$ real matrices the Lie algebra of $GL(q, R)$. Suppose ϕ is a symmetric, multilinear real valued map of degree j on $gl(q, R)$,

$$\phi: \underbrace{gl(q, R) \times \cdots \times gl(q, R)}_{j \text{ times}} \to R.$$

The map ϕ is said to be invariant over $GL(q, R)$ if and only if

$$\phi(\chi A_1 \chi^{-1}, \chi A_2 \chi^{-1}, \ldots, \chi A_j \chi^{-1}) = \phi(A_1, A_2, \ldots, A_j) \tag{3.5}$$

whenever $\chi \in GL(q, R)$ and $A_i \in gl(q, R)$. The symmetric multilinear maps on $gl(q, R)$ invariant over $GL(q, R)$ form a graded ring called the characteristic ring of $gl(q, R)$ over $GL(q, R)$. Elements of this ring are called characteristic maps.

Suppose V is a vector bundle over a manifold M and D is a connection on V with curvature $K(D)$. The characteristic maps of $gl(q, R)$ can be extended to even dimensional forms with values in the endomorphism bundle and the fundamental fact of the Chern-Weil theory is the following. For ϕ any characteristic map $\phi(K(D))$ is a closed form where $\phi(K(D)) = \phi(K(D), \ldots, K(D))$ and for each $w \in \mathrm{Pont}^{(r)}(V; R)$ there exists a characteristic map ϕ of degree $r/2$ satisfying

$$w = \left[\phi\left(\frac{1}{2\pi} K(D)\right)\right] \tag{3.6}$$

where $[\cdot]$ denotes cohomology class in $H^*(M; R)$. In fact the total real Pontryagin class $P(V)$ is given

$$P(V) = \left[\det\left(1 + \frac{1}{2\pi} K(D)\right)\right] \tag{3.7}$$

where $\det(\cdot)$ is determinant. Notice (3.6) implies that the cohomology class of $\phi((1/2\pi)\, K(D))$ is independent of the connection D.

If V is orientable choose D so as to preserve a Riemannian metric on V. Then $K(D)$ is skew symmetric in local orthonormal framings of V. Let $so(q)$ be the linear space of $q \times q$ skew symmetric matrices the Lie algebra of $SO(q)$ and by analogy with (4.1) define characteristic maps of $so(q)$ over $SO(q)$. Restricting to orthonormal frames coherent with a prescribed orientation of V characteristic maps of $so(q)$ over $SO(q)$ can be defined on $K(D)$. The Chern-Weil construction gives analogous to (4.3) that for each $w \in \mathrm{Pont}_\chi^{(r)}(V; R)$ there exists a characteristic map of $so(q)$ over $SO(q)$ of degree $r/2$ satisfying

$$w = \left[\phi\left(\frac{1}{2\pi} K(D)\right)\right]. \tag{3.8}$$

Notice from (3.8) that the cohomology class of $\phi((1/2\pi)\, K(D))$ is independent of the Riemannian metric on V.

§ 4. Proof of Theorem I

Using our previous notation let M be the manifold with a given R-foliation of codimension q and let E be the sub-bundle of tangents to the leaves, $E_x = \ker(df_i|_x)$.

Let A^* be the graded subalgebra of $\Gamma(\Lambda^*(T^*(B)))$ consisting of those differential forms on B which are invariant under local isometries. Since the γ_{ji}^x are local isometries A^* pulls back to a subalgebra $\hat{A}^*$ of $\Gamma(\Lambda^*(T^*(M)))$. Notice that $\hat{A}^{(r)}$ vanishes for $r > q$. The next step in the proof is to define on the normal bundle v a connection for which the differential forms representing the elements of the Pontryagin ring are contained in $\hat{A}^*$.

Restricted to any U_i the bundle v is canonically isomorphic to the pull back of $T(B)$ by f_i. For $t_x \in T(M)|_x$ with $[t_x]$ denoting the equivalence class in $v|_x$ the canonical isomorphism is given by

$$[t_x] \mapsto (x, (df_i)_x (t_x)).$$

The unique torsion-free Riemannian connection on $T(B)$ is invariant under local isometries, in particular under the γ_{ij}^{x}, and therefore this connection pulls back to a connection on v. Let D be the Riemannian connection on $T(B)$ and $\hat{D}$ its pull back to v.

Since D is invariant under local isometries for any characteristic map ϕ

$$\phi(K(D)) \in A^*.$$

Furthermore, by (3.4) $\phi(K(\hat{D}))$ is the pull back of $\phi(K(D))$ and therefore

$$\phi(K(\hat{D})) \in \hat{A}^*.$$

By the Chern-Weil construction $\mathrm{Pont}^{(r)}(v; R)=0$ for $r>q$. Moreover, $\hat{D}$ preserves the metric on v pulled back from $T(B)$ and therefore if v is orientable $\mathrm{Pont}_{\chi}^{(r)}(v; R)=0$ for $r>q$.

§ 5. Compact Lie Group Actions

A smooth right action of a Lie group G on a manifold M is given by a smooth map $\mu: M \times G \to M$ satisfying

$$\left.\begin{array}{l} \text{(i) } \mu(m, e) = m \quad \text{for all} \quad m \in M \quad \text{where} \quad e \in G \quad \text{is the identity.} \\ \text{(ii) } \mu(\mu(m, g_1), g_2) = \mu(m, g_1 g_2) \quad \text{for all} \quad g_1, g_2 \in G \quad \text{and} \quad m \in M. \end{array}\right\} \tag{5.1}$$

On a Riemannian manifold $(M, <, >)$ an action is an isometric action if for every $g \in G$ the map $\mu(\cdot, g): M \to M$ is an isometry.

PROPOSITION 5.1. *A Lie group acting by isometries on a Riemannian n-manifold* $(M, <, >)$ *having all orbits of dimension k generates an R-foliation of M with codimension* $n-k$.

Proof. The fact that the orbits are all of the same dimension gives that M has a foliation with leaves these orbits. We may choose a covering of M by coordinate charts $\{(U_i; x_1^i, \ldots, x_k^i, y_1^i, \ldots, y_{n-k}^i)\}_{i \in I}$ with I some indexing set so that the slice defined by each fixed value of $(y^i, \ldots, y_{n-k}^i)$ is a connected component of an orbit intersected with U_i. Since the action is isometric following Reinhart [13, pp. 119–124] we can choose 1-forms $w_1^i, \ldots, w_k^i$ defined on U_i which are zero on all vectors orthogonal to the orbits and $\{w_1^i, \ldots, w_k^i, dy_1^i, \ldots, dy_{n-k}^i\}$ frames the cotangent space over U_i with

$$<, > \mid U_i = \sum_{\mu, \lambda=1}^{k} g_{\mu\lambda}(x, y)\, w_\mu^i \otimes w_\lambda^i + \sum_{\alpha, \beta=1}^{n-k} g_{\alpha\beta}(y)\, dy_\alpha^i \otimes dy_\beta^i.$$

Let $(R^{n-k})^i$ have Riemannian metric

$$\sum_{\alpha, \beta=1}^{n-k} g_{\alpha\beta}(y)\, dy_\alpha^i \otimes dy_\beta^i$$

and let B be the disjoint union of the $(R^{n-k})^i$. The covering $\{U_i\}_{i\in I}$ with the obvious surjections into B define an R-foliation of M. Q.E.D.

Given a Lie group action μ, $\mu: G\times M\to M$, $d\mu(e, m): T(G)_e \oplus T(M)_m \to T(M)_m$ and the image of $T(G)_e \oplus 0$ under $d\mu(e, m)$ is the subspace of $T(M)_m$ consisting of the tangents to the orbit through m. Let E_μ be the collection of all tangents to the orbits of μ; E_μ is a vector subbundle of $T(M)$ if and only if all the orbits are of the same dimension.

DEFINITION. *The action μ is almost free if for each $m\in M$ the isotropy group of m (subgroup of G fixing m) is a discrete subgroup of G.*

PROPOSITION 5.2. *If μ is an almost free action then E_μ is a trivial sub-bundle of $T(M)$.*

Proof. Since μ is almost free, the map from $TG_e\times M\to E_\mu$ given by $(l, m)\mapsto$ $\mapsto d\mu(e, m)\,(l)$ for $l\in TG_e$ is injective for each m because $d\mu(e, m)\,(l)=0$ would imply that the group elements $\exp(tl)$ fix m. The above map is certainly surjective and trivializes the bundle E_μ. Q.E.D.

Proof of Corollary 1. Let μ be the almost free action of a k-dimensional compact Lie group G on an n-dimensional manifold M. Since G is compact there is a metric on M with respect to which μ is an isometric action. By Proposition 5.1, Theorem I can be applied:

$$\mathrm{Pont}^r(T/E_\mu; R) = 0 \quad \text{for} \quad r > \dim(T/E_\mu) = n-k.$$

Now,

$$T \simeq E_\mu \oplus T/E_\mu.$$

Letting $P(\cdot)$ denote the real total Pontryagin class it follows (cf., Milnor [9])

$$P(T) = P(E_\mu) \cup P(T/E_\mu).$$

But by Proposition 5.2, E_μ is trivial and therefore

$$P(E_\mu) = 1 \in H^0(M; R).$$

Thus $P(T)=P(T/E_\mu)$. Q.E.D.

§ 6. Integral Pontryagin Rings: A Counter-Example

Information on characteristic classes in Theorem I and Corollary 1 has been deduced exclusively by the Chern-Weil theory and as a result, all of these theorems are about the *real* Pontryagin ring. In this section an almost free action of S^1 is constructed on a 4-dimensional compact unorientable manifold whose first integral Pontryagin class does not vanish. The example shows that Theorem I and Corollary 1 are false for the integral Pontryagin ring.

To construct the example we need to use the real blow up. Suppose M is a real n-dimensional smooth manifold. Choose a point p, $p \in M$. There exists a real smooth manifold $\hat{M}$ and a map π, $\pi: \hat{M} \to M$, satisfying

(1) π maps $\hat{M} - \{\pi^{-1}(p)\}$ diffeomorphically onto $M - \{p\}$

(2) $\pi^{-1}(p)$ is a real projective $(n-1)$-space.

The point p in M is blown up into a projective space in $\hat{M}$ by replacing p, by the set of directions through p.

Topologically the blow-up manifold $\hat{M}$ is the connected sum of M and a real projective n-space, RP_n. Whenever n is even $\hat{M}$ is non-orientable. In general, each Stiefel-Whitney number of $T(\hat{M})$ equals the sum of the corresponding Stiefel-Whitney numbers of $T(M)$ and $T(RP_n)$. For example, the blow-up of one point in the n-sphere yields RP_n – the Stiefel-Whitney numbers of the n-spheres vanish.

Algebraically, the blow-up is described as follows. Let $\mathbf{P}(TM)$ be the bundle of $(n-1)$-dimensional projective spaces derived from TM. Choose a coordinate patch $(U; x_1, x_2, \ldots, x_n)$ with $x_i(p) = 0$, $i = 1, 2, \ldots, n$. Let $(y_1, y_2, \ldots, y_n)$ be the dual basis to

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$$

viewed as homogeneous coordinates for $\mathbf{P}(TM) | U$.

To define $\hat{M}$ and π, it is sufficient to define $\pi^{-1}(U)$.

$$\pi^{-1}(U) \subset \mathbf{P}(TM) \mid U$$
$$\pi^{-1}(U) = \{((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \mid x_i y_j = x_j y_i \text{ for each unordered pair } i, j, \ 1 \leqslant i, j \leqslant n\}.$$

For

$$((x_1, x_2, \ldots, x_n), (y_1, \ldots, y_n)) \in \pi^{-1}(U)$$
$$\pi((x_1, x_2, \ldots, x_n), (y, \ldots, y_n)) = (x_1, x_2, \ldots, x_n).$$

There are n coordinate patches covering $\pi^{-1}(U)$

$$V_i = \{((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in \pi^{-1}(U) \mid y_i \neq 0\}$$

on each V_i there are coordinates v_j^i, $j = 1, \ldots, n$. $v_j^i = x_i$ for $j = i$ and $v_j^i = y_j / y_i$ $j \neq i$.

Lifting an action of S^1 to $\hat{M}$. The following lemma about the real blow-up is of central importance in the construction of the example. Let p be contained in M and $\hat{M}$ the blow-up manifold for p; S^1 is the circle group.

LEMMA 6.1. *Suppose S^1 acts on M and the action is almost free on $M - \{p\}$, then the action lifts to an almost free action on $\hat{M}$.*

Proof of Lemma. Choose $<, >$ to be a Riemannian metric on M with respect to which S^1 acts isometrically. Let μ be the smooth map defining the action, $\mu: S^1 \times M \to M$.

$$S^1 = \{e^{i\theta} \mid \theta \in R\}.$$

Define $\overline{d\mu}(1)$ by

$$\overline{d\mu}(1)|_m = d\mu(e, m)\left(\frac{\partial}{\partial\theta}\right).$$

The vector field $\overline{d\mu}(1)$ is an infinitesimal isometry on $(M, <, >)$. For $\varepsilon > 0$ let

$$U_{p,\varepsilon} = \{m \in M \mid \varrho(p, m) < \varepsilon\}$$

where $\varrho(,)$ is the Riemannian distance on $(M, <, >)$. Since p is a fixed point of the isometric action, $U_{p,\varepsilon}$ is stable under the action. The proof of the lemma is now completed by purely local considerations.

Choose ε small enough so that $U_{p,\varepsilon}$ is diffeomorphic to an open set about $O_p \in TM_p$ the diffeomorphism given by the exponential map relatice to the Riemannian connection.

At the point p, the infinitesimal generator of the action, $\overline{d\mu}(1)$, vanishes and the Lie bracket with respect to $\overline{d\mu}(1)$ defines a linear map $TM_p \to TM_p$. Denote this linear by $L_p(\overline{d\mu}(1))$. In [10], Kobayashi shows that (dim M) is even and there exists non-zero real numbers $a_1, a_2, \ldots, a_{n/2}$ and an orthonormal frame for TM_p, $e_1, \ldots, e_n$ so that relative to this frame of the matrix of $L_p(\overline{d\mu}(1))$ is given by

$$\begin{pmatrix} 0 & -a_1 & 0 & \cdots & & & \\ a_1 & 0 & 0 & \cdots & & & \\ 0 & 0 & 0 & -a_2 & 0 & \cdots & \\ \cdots & & a_2 & 0 & 0 & \cdots & \\ \cdots & & 0 & 0 & \ddots & & \\ \vdots & & & & & & \\ \vdots & & & & & 0 & -a_{n/2} \\ \vdots & & & & & & \\ \cdots & & & & & a_{n/2} & 0 \end{pmatrix} \tag{7.1}$$

Let $x_1, \ldots, x_n$ be the dual basis of $e_1, \ldots, e_n$ on TM_p, restricted to $\exp_p^{-1}(U_{p,\varepsilon})$ and viewed as coordinates on $U_{p,\varepsilon}$. It is further shown in [10] that the action of μ on $U_{p,\varepsilon}$ is given by

$$\begin{pmatrix} \cos a_1\theta & -\sin a_1\theta & 0 & \cdots & \\ \sin a_1\theta & \cos a_1\theta & 0 & \cdots & \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \\ \vdots & \vdots & & \ddots & \\ \vdots & \vdots & & \cos a_{n/2}\theta & -\sin a_{n/2}\theta \\ \vdots & \vdots & & \sin a_{n/2}\theta & \cos a_{n/2}\theta \\ \vdots & \vdots & & & \end{pmatrix} \tag{7.2}$$

The skew-eigenvalues $a_1, a_2, \ldots, a_{n/2}$ are seen to be integers.

Denote by $\mathbf{P}(TM_p)$ the projective space of TM_p. Letting $(y_1, \ldots, y_n)$ be homogeneous coordinates on $\mathbf{P}(TM_p)$ relative to the basis $e_1, \ldots, e_n$, the matrix in Equation (7.2) defines an action, μ', of S^1 on $\mathbf{P}(TM_p)$. Noting the skew-symmetry in matrix (7.1), μ' is almost free. The product action $\mu' \times \mu$ is almost free on $P(TM)|U_{p,\varepsilon}$. Further, $\pi^{-1}(U_{p,\varepsilon})$ is stable under $\mu' \times \mu$ and thereby lifts μ to $\hat{M}$. Q.E.D.

EXAMPLE 7.1. An action of S^1 is first defined on complex projective 2-space CP_2. Denote by $[z_0, z_1, z_2]$ the equivalence class in CP_2 of $(z_0, z_1, z_2) \in \mathbf{C}^3 - \{(0, 0, 0)\}$ and define the action μ by

$$(e^{i\theta}, [z_0, z_1, z_2]) = [z_0, e^{i\theta} z_1, e^{i2\theta} z_2].$$

The points [1, 0, 0], [0, 1, 0] [0, 0, 1] are fixed points of the action μ and μ is almost free on

$$CP_2 - \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}.$$

Blow-up in turn the three fixed points and lift the action, as described in Proposition 7.1, to an almost free action on the resulting blown up manifold. Let M be the blown up manifold; M is the connected sum of CP_2 and three copies of real projective space RP_4.

To complete the counter example we now show that $p_1(T(M)) \neq 0$ where $p_1(T(M)) \in H^4(M; \mathbf{Z})$ is the first integral Pontryagin class. Since M is compact and non-orientable it follows (cf., p. 90 in [3]) that $p_1(T(M)) = (w_2(T(M)))^2$. However, the Stiefel-Whitney numbers of $T(M)$ equal the Stiefel-Whitney numbers of $T(CP_2)$ plus three times the Stiefel-Whitney numbers of $T(RP_4)$, and [11]

$$[w_2(T(RP_4))]^2 = 0$$

and

$$[w_2(T(CP_2))]^2 = \beta^2 \qquad \beta \in H^2(CP_2, \mathbf{Z}_2). \quad \beta \neq 0$$

Thus $p_1(T(M)) \neq 0$.

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