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1. Introduction and Statement of the Main Results

The concept of a Hopf algebra grew out of the study of the homology and cohomology rings of H-spaces. Hopf algebras are now considered in many other situations but it remains an interesting problem to determine which Hopf algebras actually occur in one of the situations where they were first considered. In this paper we consider Hopf algebras which as algebras are generated by primitive elements and investigate which of these can arise as the cohomology rings of H-spaces.

The use of terms in the literature is not entirely consistent and so before stating our results we give the main definitions, especially where there is any possibility of confusion.

Let X be a connected complex with finite skeletons which supports an H-structure, that is, there exists a continuous multiplication $m: X\times X\to X$ with two sided homotopy unit. Generally we shall just say that X is an H-space and assume that a particular multiplication is given, or more briefly that $(X, m)\in H$. If R is a field or a ring for which $H^*(X, R)$ is torsion free, there is a canonical isomorphism,

$$H^*(X \times X, R) \cong H^*(X, R) \otimes H^*(X, R)$$

and so the multiplication on X induces a comultiplication m^* on $H^*(X, R)$ giving it the structure of a connected, associative, commutative, graded Hopf algebra over R. Let $\pi_i: X\times X\to X$, $i=1$ or 2, be the projection onto the i-th factor. The graded submodule of primitive elements of $H^*(X, R)$, $P\{H^*(X, R)\}$, is defined to be the kernel of the homomorphism

$$m^* - \pi_1^* - \pi_2^*: H^*(X, R) \to H^*(X \times X, R).$$

The quotient module of indecomposable elements of $H^*(X, R)$, $Q\{H^*(X, R)\}$, is defined to the quotient of the ring $\bar{H}^*(X, R)$ by the ideal $\bar{H}^*(X, R)\cdot\bar{H}^*(X, R)$. As usual an element of $H^*(X, R)$ is called indecomposable if its image in $Q\{H^*(X, R)\}$ is non zero; otherwise it is decomposable. Let

$$\alpha: P\{H^*(X, R)\} \to Q\{H^*(X, R)\} \tag{1.1}$$

be the canonical homomorphism of graded modules. We say that $H^*(X, R)$ is primitively generated with respect to m^*, or m^*-primitive, if α is surjective.

In addition $R[2]$ will stand for a graded polynomial algebra over R all of whose generators have dimension 2 and we shall write E for an exterior algebra on odd dimensional generators over a ring determined by the context.

THEOREM 1.1. *Let $(X, m)\in H$.*

(a) *Suppose that $H^*(X, Z)$ is torsion-free and m^*-primitive; then the ring $H^*(X, Z) \cong Z[2]\otimes E$.*

(b) *Suppose that $H^*(X, Z)$ has no p-torsion where p is an odd prime and that $H^*(X, Z_p)$ is m^*-primitive; then the ring $H^*(X, Z_p)\cong Z_p[2]\otimes E$.*

This is our main result. It follows from Theorem 1.1 that a primitively generated Hopf polynomial algebra in cohomology over Z or Z_p (p odd) must have all its generators in dimension 2, as the torsion conditions are automatically satisfied. Indeed in the former case we have the corollary,

COROLLARY 1.2. *Let $(X, m)\in H$ and suppose that $H^*(X, Z)$ is m^*-primitive and is a polynomial algebra; then X is an Eilenberg-Maclane complex of type $K(Z+Z+\cdots+Z, 2)$.*

Theorem 1.1 (b) gives an interesting contrast between finite and infinite dimensional H-complexes. For all known examples of connected finite complexes which are H-spaces, and for all odd primes p, the following two statements are equivalent,

$$H^*(X, Z) \quad \text{has no } p\text{-torsion,} \tag{1.2}$$

$$H^*(X, Z_p) \quad \text{is } m^*\text{-primitive for some } m. \tag{1.3}$$

If we consider topological groups, in [9] these have been shown to be equivalent. When X has infinite cohomological dimension, there are many examples of H-spaces which satisfy (1.3) but not (1.2), such as the Eilenberg-Maclane complexes $K(Z_p, n)$. Theorem 1.1 (b) implies that in these cases the primitivity of $H^*(X, Z_p)$ requires the existence of torsion in the integral cohomology, provided that we exclude certain simple ring structures.

Theorem 1.1 can be combined with general Hopf algebra theorems to deduce two corollaries which give algebraic characterizations for the cohomology rings of certain H-spaces to be primitively generated.

Let $(X, m)\in HA$ if the multiplication m is homotopy associative and let $(X, m) \in HAC$ if m is both homotopy associative and homotopy commutative. (In fact, it is sufficient to require that the comultiplication induced upon the rational cohomology ring by m is coassociative or coassociative, cocommutative respectively.)

COROLLARY 1.3. *Let X be 1-connected and $(X, m)\in HAC$.*

(a) *Suppose that $H^*(X, Z)$ is torsion-free; then $H^*(X, Z)$ is m^*-primitive if and only if the ring $H^*(X, Z)\cong Z[2]\otimes E$.*

(b) *Suppose that $H^*(X, Z)$ has no p-torsion where p is an odd prime; then $H^*(X, Z_p)$ is m^*-primitive if and only if the ring $H^*(X, Z_p)\cong Z_p[2]\otimes E$.*

COROLLARY 1.4. *Let X be 2-connected and $(X, m) \in HA$.*

(a) *Suppose that $H^*(X, Z)$ is torsion free; then $H^*(X, Z)$ is m^*-primitive if and only if the ring $H^*(X, Z) \cong E$.*

(b) *Suppose that $H^*(X, Z)$ has no p-torsion where p is an odd prime; then $H^*(X, Z_p)$ is m^*-primitive if and only if the ring $H^*(X, Z_p) \cong E$.*

It should be noted that Theorem 1.1 (b) is false without the restriction that p is an odd prime, for $\Omega' SO$, the loop space on the infinite special orthogonal group gives a counter example when $p=2$. Our methods of proof do however yield the following,

THEOREM 1.5. *Let $(X, m) \in H$. Suppose that $H^*(X, Z)$ has no 2-torsion and that $H^*(X, Z_2)$ is m^*-primitive; then*

(a) *the ring $H^*(X, Z_2) \cong Z_2[4k+2] \otimes E$, where $Z_2[4k+2]$ is a polynomial algebra on generators whose dimensions are congruent to 2 mod4, and*

(b) *the Steenrod square restricted to the indecomposable quotient*

$$Sq^{4k}: Q\{H^*(X, Z_2)\}_{4k+2} \to Q\{H^*(X, Z_2)\}_{8k+2}$$

is a monomorphism for each k.

Results somewhat similar to Theorems 1.1 and 1.5 were first considered in [12], see especially Theorem 1; related results can be found in [8], [9] and [23].

This paper is one of several in which complex K-theory is used to investigate the cohomology of H-spaces, under torsion free conditions, see [14], [15], [16], [17] and [18]. More precisely we make considerable use of the generalized cohomology operations of [14] and [16]. These are homomorphisms defined on the cohomology groups of certain complexes with suitable coefficients, which are closely related to higher order cohomology operations. In our context they have the great advantage over the latter that under torsion free assumptions we can perform algebraic arguments with them more simply. One of the most powerful tools in the study of the cohomology of an H-space is the Bockstein spectral sequence as developed by W. Browder and others. However to obtain significant results by these methods, the existence of torsion in the integral cohomology of an H-space can be as essential as is the lack of torsion in the methods we are describing. Interesting results can arise by combining these techniques, see [15] and [17]. This justifies our continuing to develop our methods.

Now Theorem 1.1 (a) of this paper can be proved more directly than seems possible for either Theorem 1.1 (b) or Theorem 1.5. The essential differences in the proofs occur because the hypotheses of Theorem 1.1 (a) imply that the comultiplication m^* induced on the rational cohomology ring is both coassociative and cocommutative, whereas it is not clear that this is true in the other two cases. However we shall deduce Theorem 1.1 (a) as a corollary to Theorem 1.1 (b) to avoid repetition and because Theorem 1.1 (a) is a simple corollary of a rather deeper result which the author hopes to prove on another occasion. In section 2 we shall develop the properties of the multiplicative ψ^k-modules over Q_p introduced in [14] and [16]. The emphasis here is on the case when p is an odd prime as many of the corresponding results for $p=2$ were proved in [16]. Section 3 contains two short technical lemmas which are used in section 4 where the key theorems are proved. The proofs of the results of this introduction are completed in section 5.

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2. Multiplicative ψ^k-Modules over Q_p

This section is concerned with establishing some properties of the cohomology and unitary K-theory of finite complexes under torsion-free assumptions. We use an axiomatic approach as in [16] and [17] and consider certain "Multiplicative ψ^k-modules over Q_p". We establish the existence and some properties of homomorphisms S^i and Q^i in torsion-free cohomology rings, with suitable coefficients. The S^i are related in a simple manner to the cyclic reduced powers, while the Q^i are essentially the higher order cohomology operations studied by Maunder in [21]; both are defined using the Adams operators in complex K-theory (or equivalently, using the Chern character). It will be shown that the S^i and Q^i satisfy generalized Adem type relations, Corollary 2.12, and generalized naturality properties, Corollary 2.28. We could also establish a generalized Cartan formula, but as we have no use for it here, we give a simplified version, Corollary 2.20. In the case of the prime 2 much of this was considered in section 2 of [16] and we shall follow this closely for general primes. The main differences occur because we use a "Splitting Theorem" of Adams and the proof of this theorem, which we only sketch, leads to improvements in some of the proofs of [16]. Where this is the case, we provide details; otherwise, when a proof is a simple extension of that given in [16] for the corresponding result with $p=2$, the extension is left to the reader.

The material of this section has been used in more than one context and has undergone numerous revisions during the last few years. It now incorporates several ideas and suggestions due to J. F. Adams, in addition to the "Splitting theorem" and is now very different from the first version. I am most grateful to Professor Adams for the generous guidance he gave while I was writing the material for this section.

Let Q_p be the subring of rationals with denominators not divisible by the prime p. M will be a finitely generated Q_p-module equipped with linear maps $\psi^k: M \to M$, for each integer k, and filtered by submodules

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_g \supset M_{g+1} = 0$$

such that $\psi^k(M_i) \subset M_i$ for each i.

Let $N=\sum_{0\leqslant i\leqslant g} N_i$ be the associated graded module, $N_i=M_i/M_{i+1}$.

The module M and the maps ψ^k will always satisfy six axioms; those concerned with the additive structure are A1, A2 and A3.

A1. *The Splitting Condition.* N is a free graded Q_p-module.

A2. *The Commuting Condition.* $\psi^k\psi^l=\psi^l\psi^k$.

A3. *The Integrality Condition.* Let $u\in M_n$; then there exists a finite set of elements $u_i\in M_{n+i(p-1)}$, $0\leqslant i\leqslant h$, such that $w=\sum_{0\leqslant i\leqslant h} p^{-i}u_i$ satisfies $\psi^k(w)=k^n w$ in $M\otimes Q$ for each k, and $u-u_0\in M_{n+1}$.

M is called a ψ^k-module over Q_p if it satisfies A1, A2 and A3. As indicated above, the reason for considering such modules is that if X is a finite complex whose integral homology is free of p-torsion, then $M=K(X, Q_p)$ is a ψ^k-module over Q_p, where $K(\ , Q_p)$ is the tensor product of the unitary K-theory of [7] with Q_p. For the Atiyah-Hirzebruch spectral sequency with Q_p-coefficients collapses, and therefore

$$H^{2n}(X, Q_p)\cong K_{2n}(X, Q_p)/K_{2n+1}(X, Q_p),$$

Let $M_i=K_{2i}(X, Q_p)=K_{2i-1}(X, Q_p)$ and let ψ^k be the Adams operators of [2]. Then $N_i=M_i/M_{i+1}\cong H^{2i}(X, Q_p)$. The verification that M satisfies all the properties of a ψ^k-module is straightforward; the integrality condition is proved using the "Integrality theorem on the Chern character" of [1]. This proof is thus a simple generalization of that given as Lemma 2.1 of [16] in the case $p=2$. Thus for X as above,

LEMMA 2.1. *$M=K(X, Q_p)$ is a ψ^k-module over Q_p.*

We return to the general case.

LEMMA 2.2. *Let $M=M^1\oplus M^2$ be a ψ^k-module over Q_p with $\psi^k(M^1)\subset M^1$ for each k; then the quotient module M^2 is a ψ^k-module over Q_p.*

The proof of Lemma 2.2 is clear. Another simple consequence of the axioms is Lemma 2.3.

LEMMA 2.3. *Let $u\in M_n$; then $\psi^k(u)=k^n u \bmod M_{n+1}$, for each k.*

Proof. This follows from A3, see Lemma 2.4 of [16].

These last two lemmas will be used in the proof of the "Splitting theorem" which we now state.

THEOREM 2.4 (Adams). The Splitting Theorem.

Let M be a ψ^k-module over Q_p; then there is a canonical direct sum splitting $M=\sum_\alpha M^\alpha$ where

(a) *α runs over the residue classes* $\bmod(p-1)$,

(b) *each M^α is a ψ^k-module over Q_p,*

(c) $$(N^\alpha)_i=\begin{cases}0, & i\notin\alpha\\ N_i, & i\in\alpha.\end{cases}$$

This theorem is a weaker version of Corollary 8 of [4] and as our application is to take $M=K(X, Q_p)$, the reader can refer to this for a proof. In fact as far as this paper is concerned we can do without the splitting being canonical and this is how the author originally proceeded (a remnant of this approach can be seen in the Appendix to [17]), but the method was less illuminating. Therefore we shall outline a proof of the theorem in our notations. It is similar to the proof of Corollary 8 of [4] and is also due to Adams.

Suppose that $M_{g+1}=0$ and that $u\in M_i$. Let $k_i, k_{i+1},\dots, k_g$ be any $g-i+1$ integers, then it follows from Lemma 2.3 that

$$\prod_{i\leqslant j\leqslant g} (\psi^{k_j}-(k_j)^j)(u)=0. \tag{2.1}$$

Now take a fixed integer k of modulus greater than one and write ψ for ψ^k. Let I and I_p be the ideals generated by $\prod_{0\leqslant i\leqslant g}(\psi-k^i)$ in $Q[\psi]$ and $Q_p[\psi]$ respectively, and form the quotient modules $R=Q[\psi]/I$ and $R_p=Q_p[\psi]/I_p$. It follows from (2.1) that R and R_p act on $M\otimes Q$ and M. We use Lagrange's interpolation formula to define

$$e_i=\prod_{\substack{0\leqslant j\leqslant g\\ j\neq i}}(\psi-k^j)/(k^i-k^j).$$

Then $\sum_{0\leqslant i\leqslant g} e_i=1$ in $Q[\psi]$ and $e_i^2=e_i$, $e_ie_j=0$ $(i\neq j)$ in R, and so these idempotents give a splitting

$$M\otimes Q=\sum_{0\leqslant i\leqslant g} H_i, \quad \text{where} \quad H_i=e_i(M\otimes Q). \tag{2.2}$$

Further

$$\psi^t(u)=t^iu, \quad \text{if} \quad u\in H_i, \tag{2.3}$$

for any ψ^t. It follows from (2.3) that H_i is independent of k and in the particular case of a ψ^k-module considered in Lemma 2.1, if we identify $M\otimes Q$ and $H^{\text{even}}(X, Q)$ by means of the Chern character, the e_i are precisely the idempotents considered before Theorem 5 of [4]. (See section 2 of [5].)

Now we further restrict k to be a primitive root mod p, and let $E_\alpha=\sum_{i\in\alpha} e_i$, where α is a residue class mod $(p-1)$. The central point of the proof of Theorem 2.4 is to show that the idempotents E_α lie in $Q_p[\psi]$. We sketch the proof.

Let $P_\alpha=\prod_{\substack{0\leqslant i\leqslant g\\ i\in\alpha}}(\psi-k^i)\in Z[\psi]$, and so $E_\alpha=A_\alpha\prod_{\beta\neq\alpha}P_\beta$, where $A_\alpha\in Q[\psi]$ and the degree of A_α is less than that of P_α.

Therefore

$$1=\sum_\alpha A_\alpha\prod_{\beta\neq\alpha}P_\beta. \tag{2.4}$$

The coefficients of each A_α lie in Q and so suppose that the lowest power of p which occurs in any A_α is p^f with $f<0$. We reduce (2.4) mod p^{f+1} and obtain an identity

$$0=\sum_\alpha A'_\alpha\prod_{\beta\neq\alpha}P'_\beta \quad \text{in} \quad Z_p[\psi],$$

where P'_β is P_β reduced mod$p$ and at least one A'_α is non zero. The integer k was chosen to be primitive mod$p$ and so the P'_β are coprime. But P'_α divides each summand except $A'_\alpha \prod_{\beta \neq \alpha} P'_\beta$ and so it divides A'_α, which is impossible considering degrees. Thus $f \geqslant 0$ and so each coefficient of A_α lies in Q_p.

The idempotents E_α enable us to write $M = \sum M^\alpha$, where $M^\alpha = E_\alpha(M)$. Condition A2 ensures that for any ψ^t, $\psi^t(M^\alpha) \subset M^\alpha$. One now checks that the splitting is essentially independent of g.

It follows from A1 that

$$M^\alpha = M_\cap (\sum_{j \in \alpha} H_j), \tag{2.5}$$

which implies that the splitting is independent of the particular primitive root used in its construction. We also establish that

$$M_i = M_\cap (\sum_{i \leqslant j \leqslant g} H_j). \tag{2.6}$$

The filtration on M^α is defined by setting $(M^\alpha)_i = M^\alpha \cap M_i$. Then

$$M_i = \sum_\alpha (M^\alpha)_i \quad \text{and} \quad (N^\alpha)_i = \begin{cases} 0, & i \notin \alpha \\ N_i, & i \in \alpha \end{cases}.$$

It is now a simple matter to verify that M^α is a ψ^k-module over Q_p. The truth of A1 follows from A1 for M and (2.6), A2 is immediate and A3 follows from Lemma 2.2. This completes the proof of Theorem 2.4.

We now proceed as in [16]. Let $I_s: M_s \to N_s$ be the quotient map. We filter $N = \sum_{0 \leqslant i \leqslant g} N_i$ with the obvious decreasing filtration $\sum_{j \leqslant i \leqslant g} N_i$ for each j.

LEMMA 2.5. *There exists an isomorphism $J: N \to M$ of filtered Q_p-modules such that $J(N_s) \subset M_s$ and the composition of $J: N_s \to M_s$ with $I_s: M_s \to N_s$ is the identity map on N_s.*

Proof. See Lemma 2.5 of [16].

It is convenient to consider a particular type of splitting $J: N \to M$. Take the canonical direct sum decomposition of M given by Theorem 2.4 (a) and let $J^\alpha: N^\alpha \to M^\alpha$ be a splitting of M^α as given by Lemma 2.5, then we take $J: N \to M$ to be of the form $\sum_\alpha J^\alpha$. In the type of argument that we shall be using later, it is often simpler to work in a graded module rather than a filtered module. This is especially true when we introduce ring structures where the relations among generators in the filtered ring are more complicated than in the associated graded ring, and examples in the theory of Hopf algebras where this is the case do indeed occur, though they will not be considered in this paper. Thus we use the splitting just constructed to lift ψ^k to an endomorphism of N.

Define the linear map $\Phi_J^k: N \to N$ by requiring that the following diagram is strictly commutative,

$$\begin{array}{ccc} M & \xrightarrow{J} & N \\ {\scriptstyle\psi^k}\downarrow & & \downarrow{\scriptstyle\Phi_J^k} \\ M & \xrightarrow[J]{} & N \end{array} \qquad (2.7)$$

An immediate consequence of the definitions and A2 is,

LEMMA 2.6. $\Phi_J^k \Phi_J^l = \Phi_J^l \Phi_J^k$.

We also record for later use, the form which Lemma 2.3 now takes.

LEMMA 2.7. Let $x \in N_n$; *then* $\Phi_J^k(x) = k^n x \bmod \sum_{i>n} N_i$.

The restriction of the choice of splitting to the form $\sum_\alpha J^\alpha$ has ensured that Φ_J^k possess a very convenient property.

PROPOSITION 2.8. $\Phi_J^k(N_n) \subset \sum_{i\geqslant 0} N_{n+i(p-1)}$, *for all k and n.*

Proof. This follows from Theorem 2.4 (c).

It remains to interpret A3 in terms of the Φ_J^k.

THEOREM 2.9. (a) *Let* $x_0 \in N_n$; *then there exists a finite set of elements* $x_i = x_i(J, x_0) \in N_{n+i(p-1)}$, $1 \leqslant i \leqslant t$, *such that* $x = \sum_{0\leqslant i\leqslant t} p^{-i} x_i$ *satisfies* $\Phi_J^k(x) = k^n x$ *in* $N \otimes Q$.

(b) $x_i = x_i(J, x_0)$ *is unique for* $i \geqslant 0$.

Proof. The proof is a simple generalization of that given for Theorem 2.8 of [16], using Proposition 2.8.

This result enables us to define the S^i and Q^i mentioned at the beginning of the section. In fact what we do is to break up Φ_J^k to demonstrate the precise way in which it depends upon the integer k.

Define homomorphisms $S_J^q : N_n \to N_{n+q(p-1)}$ by setting $S_J^q x_0 = x_q$. Let $S_J(t) = \sum_{q\geqslant 0} S_J^q t^q$, and let $Q_J(t) = \sum_{q\geqslant 0} Q_J^q t^q$ be its formal inverse. Finally let $R_J(t; k) = \sum_{q\geqslant 0} R_J^q(k)\, t^q = S_J(p^{-1}t)\, Q_J(k^{p-1}p^{-1}t)$.

When there can be no confusion, the suffix J will be omitted from these notations.

LEMMA 2.10. *Restricting to* N_n, $\Phi^k = k^n R(1; k) = k^n \sum_{q\geqslant 0} R^q(k)$.

Proof. See Lemma 2.9 of [16].

COROLLARY 2.11. $k^n R^q(k)\,(N_n) \subset N_{n+q(p-1)}$.

If $x \in N_n$, Lemma 2.10 implies that the component of $\Phi^k(x)$ in dimension $n+q(p-1)$ is

$$k^n p^{-q}(S^q + k^{p-1}S^{q-1}Q^1 + k^{2(p-1)}S^{q-2}Q^2 + \cdots + k^{q(p-1)}Q^q)\,(x). \qquad (2.8)$$

Therefore we set $T_J^q(k)=p^q R_J^q(k)$. It follows from the definition of Q^q that

$$T^q(k) = (1 - k^{q(p-1)})\, S^q + (k^{(p-1)} - k^{q(p-1)})\, S^{q-1}Q^1 + \cdots + (k^{(q-1)(p-1)} - k^{q(p-1)})\, S^1 Q^{q-1}. \tag{2.9}$$

Corollary 2.11 implies that

COROLLARY 2.12. *Generalized Adem relations.* Let $(k, p)=1$, *i.e. let* k *and* p *be coprime; then* $T^q(k)=0 \bmod p^q$.

The relationship that Corollary 2.12 has when $p=2$ with the Adem relations in the mod 2 Steenrod algebra was partially explained in [16]. For the purposes of this paper we shall require just one relation mod p, but before deriving it, we wish to show that Corollary 2.12 is the best possible results which can be deduced from the axioms.

The S^q have been constructed from the Φ^k, but Lemma 2.10 enables us to reverse the process; given homomorphisms $S^q\colon N_n \to N_{n+q(p-1)}$ from then can be constructed Φ^k.

THEOREM 2.13. *M is a ψ^k-module over Q_p if and only if $k^n R^q(k)\,(N_n) \subset N_{n+q(p-1)}$, for all n, q and k.*

Proof. See Theorem 2.10 of [16].

Now we return to the derivation of the Adem relation mentioned earlier. The following identity is extremely useful and will be used in the proof.

LEMMA 2.14. $T^q(k)=(1-k^{q(p-1)})\, S^q - \sum_{1 \leqslant i \leqslant q-1} k^{i(p-1)} T^{q-i}(k)\, S^i$.

Proof. See the proof of Lemma 2.12 of [16].

Now choose k prime to p and combine Corollary 2.13 and Lemma 2.14 to deduce that $(1-k^{q(p-1)})\, S^q = k^{(p-1)} T^1(k)\, S^{q-1} \bmod p^2$, or $(1-k^{q(p-1)})\, S^q = k^{(p-1)}(1-k^{(p-1)}) \times S^1 S^{q-1} \bmod p^2$. Now for suitable k, for example $k=2$ if p is odd or $k=3$ if $p=2$, or see Lemma 4.3, this implies that

$$(1 + k^{(p-1)} + \cdots + k^{(q-1)(p-1)})\, S^q = k^{(p-1)} S^1 S^{q-1} \bmod p,$$

and since $k^{(p-1)}=1 \bmod p$, this implies that

$$S^1 S^{q-1} = q S^q \bmod p. \tag{2.10}$$

The last result we prove before introducing the multiplicative structure enables us to explain in more detail the relationship between the $S^q \bmod p$ and the reduced powers in cohomology.

LEMMA 2.15. (a) *Let $u \in M_n$; then there exist elements $v_i \in M_{n+i(p-1)}$, $0 \leqslant i \leqslant n$, such that $\psi^p(u)=\sum_{0 \leqslant i \leqslant n} p^{n-i} v_i$ and $u - v_0 \in M_{n+1}$.*

(b) *Let* $x_0 \in N_n$; *then there exist elements* $y_i \in N_{n+i(p-1)}$, $0 \leqslant i \leqslant s$ *for some* s, *such that* $\Phi_J^p(x) = \sum_{0 \leqslant i \leqslant s} p^{n-i} y_i$, *where* $x_0 = y_0$ *and* $p^{n-i} y_i \in N_{n+i(p-1)}$.

(c) $S_J^q(x_0) = y_q \bmod p$, $0 \leqslant i \leqslant s$.

(d) $S_J^q(x_0) = 0 \bmod p$, $q > n$.

(e) If $x_0 = I_n(u)$; *then* $y_i = I_{n+i(p-1)}(v_i) \bmod p$, $0 \leqslant i \leqslant n$.

Proof. Parts (a), (b), (c) and (e) are similar to Lemma 2.13 of [16]. Part (d) follows from (b) and (c).

In particular (c) and (e) imply that S_J^q is independent of the choice of J used in its construction. Also if we consider the particular case described in Lemma 2.1, it follows from the results of [6], see especially Proposition 5.6, that $S_J^q \bmod p$ restricted to $N \otimes Z_p$ is just the cyclic reduced power P^q (Sq^{2q}, if $p=2$). The precise relationship between the Q_J^i and higher order cohomology operations will not concern us here. In fact, only in those results where P^q explicitly occurs do we need the identification on the primary level.

Now we introduce the three further axioms mentioned earlier.

A4. M is a commutative filtered ring, that is, M is a commutative ring and $M_i \cdot M_j \subset M_{i+j}$.

A5. ψ^k is a ring homomorphism for each integer k.

A6. $\psi^p(u) = u^p \bmod p$, for each $u \in M$.

If M satisfies all six axioms, we call it a multiplicative ψ^k-module over Q_p. We write N for associated (strictly commutative) graded ring. The application is of course again that described in Lemma 2.1.

LEMMA 2.16. *$M = K(X, Q_p)$ is a multiplicative ψ^k-module over Q_p.*

The analogue of Lemma 2.2 is,

LEMMA 2.17. *Let $M = M^1 \oplus M^2$ be a multiplicative ψ^k-module over Q_p, where M^1 is an ideal in M such that $\psi^k(M^1) \subset M^1$ for all k; then the quotient module M^2 is a multiplicative ψ^k-module over Q_p.*

Before discussing the relationship between the multiplicative structures of M and N, we mention one other result,

LEMMA 2.18. *Let $x \in N_n$; then* $S^n x = x^p \bmod p$.

Proof. Let $J(x) = u$, then in the notation of Lemma 2.15, $v_n = u^p \bmod p$ by A6, where x is x_0. Thus $y_n = I_{pn}(u^p) \bmod p = x^p \bmod p$, and the result follows from Lemma 2.14 (c).

It is clear that in general it will not be possible to choose a splitting $J: N \to M$ which is a ring isomorphism, though this will be possible in most of the later arguments of this particular paper. The most general multiplicative result involving an arbitrary splitting is the next lemma.

LEMMA 2.19.

(a) $J(S_J(p^{-1})(xy))=J(S_J(p^{-1})x)\cdot J(S_J(p^{-1})y)$ in $M\otimes Q$.

(b) $Q_J(p^{-1})J^{-1}(J(x)\cdot J(y))=(Q_J(p^{-1})x)\cdot(Q_J(p^{-1})y)$ in $N\otimes Q$.

Proof. Let $x\in N_n$, then by (2.6), $J(x)=e_nJ(x) \bmod M_{n+1}\otimes Q$. But $J(x)=J(S_J(p^{-1})x) \bmod M_{n+1}\otimes Q$ and so $J(S_J(p^{-1})x)=e_nJ(x) \bmod M_{n+1}\otimes Q$. Lemma 2.9 (a) implies that $\psi^k(J(S_J(p^{-1})x))=k^nJ(S_J(p^{-1})x)$ and it follows therefore from (2.2) and (2.3) that $J(S_J(p^{-1})x)\in H_n$. Thus

$$J(S_J(p^{-1})x) = e_nJ(x). \tag{2.11}$$

Now let $y\in N_s$ and $z\in N_t$ and so $yz\in N_{s+t}$. Since $J(yz)=J(y)\cdot J(z) \bmod M_{s+t+1}$, $e_{s+t}(J(yz))=e_sJ(y)\cdot e_tJ(z) \bmod M_{s+t+1}\otimes Q$ and so $e_{s+t}(J(yz))=e_s(J(y))\cdot e_t(J(z))$, and part (a) follows from (2.11).

Part (b) now follows precisely as in the proof of Lemma 2.4 of [16].

From this lemma we can deduce the necessary Cartan formulae. As mentioned earlier, these are not the most general results but are sufficient for our immediate purposes.

COROLLARY 2.20. *Cartan Formulae.*

(a) *If J is a ring isomorphism; then* $S_J^q(xy)=\sum_{i+j=q}S_J^i(x)\cdot S_J^j(y)$.

(b) *If $J(xy)=J(x)\cdot J(y)$; then* $Q_J^q(xy)=\sum_{i+j=q}Q_J^i(x)\cdot Q_J^j(y)$.

Making use of the proof of the Splitting theorem, we can now give a simple proof of Lemma 4.3 of [16].

LEMMA 2.21. *$N\otimes Q$ and $M\otimes Q$ are isomorphic as rings.*

Proof. Apply (2.6) and obtain

$$N_i\otimes Q=(M_i/M_{i+1})\otimes Q\cong M_i\otimes Q/M_{i+1}\otimes Q=(\sum_{j\geqslant i}H_j)/(\sum_{j>i}H_j)\cong H_i,$$

and the isomorphisms preserve the ring structures. Thus $N\otimes Q$ and $\sum_{i\geqslant 0}H_i$ are isomorphic as graded rings and the result follows.

The two final results on the multiplicative structures are trivial consequences of the axioms.

LEMMA 2.22. *Let $J:N\to M$ be a ring isomorphism; then Φ_J^k is a ring homomorphism.*

LEMMA 2.23. *Let $J:N\to M$ be a ring isomorphism; then $\Phi_J^p(x)=x^p \bmod p$, for all $x\in N$.*

It remains to consider naturality relations for (multiplicative) ψ^k-modules over Q_p. Let L and M be two (multiplicative) ψ^k-modules, where ψ^k is written for the homomorphism of either. A (ring) homomorphism $f: M \to L$ is called a morphism if it commutes with the ψ^k, for each k.

Let $L = L^1 \oplus L^2$ and $M = M^1 \oplus M^2$, where L^1 and M^1 are stable under the action of the ψ^k and if L and M are both multiplicative ψ^k-modules, then L^1 and M^1 are ideals.

LEMMA 2.24. *Let $f: M \to L$ be a morphism such that $f(M^1) \subset L^1$; then f induces a morphism of quotient modules $f^2: M^2 \to L^2$.*

A morphism $f: M \to L$ commutes with the ψ^k and therefore it commutes with the idempotents e_i used in the proof of Theorem 2.4. Using (2.6) it follows that f is filtration preserving. Thus if N and P are the associated graded (rings) modules corresponding to M and L, then f induces a (ring) homomorphism of graded (rings) modules $f_*: N \to P$.

LEMMA 2.25. *Let f and g be two morphisms from M to L such that $f_* = g_*$; then $f = g$.*

Proof. By linearity it is sufficient to show that if $h: M \to L$ is a morphism with $h_* = 0$, then $h = 0$. The hypotheses imply that if $u \in M_n$, then $h(u) \in L_{n+1}$. We argue by induction in M_n for decreasing n. If $u \in M_g$ where $M_{g+1} = 0$, then $\psi^k(u) = k^g u$ by Lemma 2.3. Thus $\psi^k(h(u)) = k^g h(u)$. But if $h(u) \in L_{g+t}$ where $t > 0$ is chosen as large as possible $\psi^k(h(u)) = k^{g+t} h(u) \bmod L_{g+t+1}$. Therefore $k^g h(u) = k^{g+t} h(u) \bmod L_{g+t+1}$ which implies that $h(u) = 0 \bmod L_{g+t+1}$, and as t was chosen as large as possible, that $h(u) = 0$. Now assume that $h(v) = 0$ for any $v \in M_{n+1}$ and let $u \in M_n$. Then $\psi^k(u) = k^n u + w$ where $w \in M_{n+1}$, by Lemma 2.3. Therefore $\psi^k(h(u)) = k^n h(u)$ and we complete the inductive step as in the particular case when $n = g$. This completes the proof.

Let $J: N \to M$ and $K: P \to L$ be splittings of M and L respectively, then f also induces a linear map $f_{JK}: N \to P$ defined by requiring that (2.12) is strictly commutative,

$$\begin{array}{ccc} N & \xrightarrow{f_{JK}} & P \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle K} \\ M & \xrightarrow[f]{} & L \end{array} . \tag{2.12}$$

When $L = M$ and $J = K$, rather than write f_{JJ}, we shall write f_J.

Now a morphism f will also commute with the idempotents E^α which gave the splitting of Theorem 2.4. Thus $f(M^\alpha) \subset L^\alpha$ for each residue class $\bmod(p-1)$. An immediate consequence is,

LEMMA 2.26. $f_{JK}(N_n) \subset \sum_{i \geq 0} P_{n+i(p-1)}$, *for each n.*

It is convenient to write out $f_{JK} \mid N_n$ explicitly as $\sum_{i \geq 0} f_i(n)$, where $f_i(n): N_n \to P_{n+i(p-1)}$ is linear. In particular $f_0(n) = f_* \mid N_n$.

THEOREM 2.27. *For each s*, $0 \leqslant s \leqslant q$,

$$\sum_{s \leqslant i \leqslant q} p^{q-i} f_{q-i}(n + i(p-1))\, S_J^{i-s} Q_J^s = \sum_{0 \leqslant j \leqslant s} p^j S_K^{q-s} Q_K^{s-j} f_j(n) : N_n \to P_{n+q(p-1)}.$$

Proof. The definitions imply that $f_{JK}\Phi_J^k = \Phi_K^k f_{JK}$. We apply this equality to $x \in N_n$ and write this expression out explicitly in the notations introduced above, using Lemma 2.10. The result follows by considering the coefficient of $k^{n+s(p-1)}$ in dimension $n+q(p-1)$.

In particular if we take the cases $s=0$ and $s=q$ we obtain,

COROLLARY 2.28.

(a) $S_K^q f_* = \sum_{0 \leqslant i \leqslant q} p^{q-i} f_{q-i}(n + i(p-1))\, S_J^i : N_n \to P_{n+q(p-1)}$.

(b) $f_* Q_J^q = \sum_{0 \leqslant i \leqslant q} p^{q-i} Q_K^i f_{q-i}(n) : N_n \to P_{n+q(p-1)}$.

If $f = \text{Identity}: M \to M$, then Corollary 2.28 (a) gives an alternative proof that $S_J^q \bmod p$ is independent of the particular splitting used.

We shall need one more result. The proof is obvious.

LEMMA 2.29. *If $f: L \to M$ is a morphism of multiplicative ψ^k-modules and the splittings $J: L \to M$ and $K: P \to L$ are ring isomorphisms; then $f_{JK}: N \to P$ is a ring homomorphism.*

3.

In this short section we prove two lemmas on the unstable action of the S^i under the assumption that there exists a splitting of multiplicative ψ^k-modules which is a ring isomorphism. The existence of this ring isomorphism is not necessary in most parts of these lemmas; several parts are just concerned with the additive structure, but to prove more general results for the remainder would require considering more general Cartan formulae than we have done here.

Let $D(N) \equiv D^1(N) = \sum_{i>0} N_i$ and $D^1(M) = M_1$. Define $D^j(F)$ inductively for $j > 1$ by $D^j(F) = D^{j-1}(F) \cdot D^1(F)$, where $F = N$ or M, and $Q^j(F) = D^1(F)/D^{j+1}(F)$. We write $Q(F)$ for $Q^1(F)$. Then $Q^j(M)$ is a multiplicative ψ^k-module by Lemma 2.17, and if $J: N \to M$ is a ring isomorphism, $Q^j(J): Q^j(N) \to Q^j(M)$ is a splitting of $Q^j(M)$ which is a ring isomorphism.

Suppose that $J: N \to M$ is a ring isomorphism.

LEMMA 3.1.

(a) $(p=2)$ $(S^{n+1} + 2S^n Q^1)(N_n) = 0 \bmod 4$,

(b) $(p=2)$ $S^{n+2}(N_n) \subset 2D^2(N) \bmod 4$,
(c) $(p=2)$ $S^{n+t}(N_n)=0 \bmod 4$, *for* $t>2$,
(d) $(p>2)$ $S^{n+t}(N_n)=0 \bmod p^2$, *for* $t>0$.

Proof. Let $p=2$, then from Lemma 2.10 and Lemma 2.23, we deduce that

$$2^n 2^{-n-t}(S^{n+t} + 2S^{n+t-1}Q^1 + 4S^{n+t-2}Q^2 + \cdots + 2^{n+t}Q^{n+t})(N_n) \subset 2N_{2n+t},$$

for $t>0$. Thus if $t>0$, $S^{n+t}+2S^{n+t-1}Q^1=0 \bmod 4$. When $t=1$ this gives (a). By Lemma 2.18, $S^{n+1}(Q^1(N_n)) \subset D^2(N)$ and so (b) follows by putting $t=2$. If $t>2$, $S^{n+t-1}(Q^1(N_n))=0 \bmod 2$, by Lemma 2.15 (a), and so $S^{n+t}(N_n)=0 \bmod 4$, which proves (c).

The proof of (d) is similar but rather simpler.

LEMMA 3.2. *Suppose that* $D^{p+1}(M)=0$.
(a) $(p=2)$ $(S^n+2S^{n-1}Q^1)(D^2(N)_n)=0 \bmod 4$,
(b) $(p=2)$ $S^{n-1}(D^2(N)_n)=0 \bmod 2$,
(c) $(p=2)$ $S^{n-2}(D^2(N)_n) \subset \sum_{i+j=n-2} S^i(N_{i+1}) \cdot S^j(N_{j+1}) \bmod 2$,
(d) $(p>2)$ $S^n(D^p(N)_n)=0 \bmod p^2$,
(e) $(p>2)$ $S^{n-1}(D^p(N)_n)=0 \bmod p^2$,
(f) $(p>2)$ $S^{n-2}(D^p(N)_n)=0 \bmod p$.

Proof. (a) Let $x \in N_i$ and $y \in N_j$ where $i>0, j>0$ and $i+j=n$.

$$(S^n + 2S^{n-1}Q^1)(xy) = S^n(xy) + 2S^{n-1}(Q^1x \cdot y) + 2S^{n-1}(x \cdot Q^1y),$$

by Corollary 2.20 (b), which equals

$$\sum_{l+m=n} S^l(x) \cdot S^m(y) + 2 \sum_{l+m=n-1} S^l(Q^1x) \cdot S^m y + 2 \sum_{l+m=n-1} S^l(x) \cdot S^m(Q^1y),$$

by Corollary 2.20 (a). We apply Lemma 2.15 (a) and Lemma 3.1 (b) and (c) to write this expression as

$$\begin{aligned} &S^{i+1}x \cdot S^{j-1}y + S^ix \cdot S^jy + S^{i-1}x \cdot S^{j+1}y + 2S^iQ^1x \cdot S^{j-1}y + 2S^{i-1}x \cdot S^jQ^1y \bmod 4 \\ &\quad = (S^{i+1} + 2S^iQ^1)\, x \cdot S^{j-1}y + S^{i-1}x \cdot (S^{j+1} + 2S^jQ^1)\, y \bmod 4, \end{aligned}$$

by Lemma 2.18, and this is zero by Lemma 3.1 (a). This completes the proof of (a). Part (b) is a consequence of the Cartan formula, Lemma 2.18 and Lemma 2.15 (a). Part (c) follows similarly. The proofs of the corresponding results when p is an odd prime are again similar but are a little simpler.

4. The Comultiplication in a Free Commutative Hopf Algebra

Before giving the technical details of this section we digress to motivate our arguments.

Let $(X, m)\in HAC$ and let $H^*(X, Q_p)$ be torsion free. Suppose that the algebra structure of $H^*(X, Q_p)$ is known and that we wish to determine the coalgebra structure. Then to determine the comultiplication m^* on $H^*(X, Q_p)$ it is sufficient to determine the homomorphism induced in cohomology by the H-space squaring map,

$$(m\Delta)^*: H^*(X, Q_p) \to H^*(X, Q_p),$$

where Δ is the usual diagonal map. We indicate briefly why this is true.

Assume that $\Delta^* m^*: H^*(X, Q_p) \to H^*(X, Q_p)$ is given and that m^* is determined in dimensions less than n. Let $x\in H^n(X, Q_p)$ be indecomposable and $\Delta^* m^*(x) = 2x + w$, where w is decomposable. If there exists $k\in Q_p$ such that kx is decomposable, then $m^*(x)$ is immediately determined by the inductive hypothesis. Otherwise kx is indecomposable for all k, and since $H^*(X, Q)$ is m^*-primitive (see for example, Theorem 7.20 of [22]), there exists a particular $k\in Q_p$ and a decomposable element u such that $(kx+u)\in H^n(X, Q_p)$ is primitive. Thus

$$m^*(x) = x\otimes 1 + 1\otimes x - k^{-1}\{m^*(u) - u\otimes 1 - 1\otimes u\}.$$

But $\Delta^* m^* u = 2u - kw$ and this condition completely determines u, for if u_1 and u_2 are two choices for u, $\Delta^* m^*(u_1 - u_2) = 2(u_1 - u_2)$ from which it is not difficult to deduce that $u_1 - u_2$ is zero or primitive. But is also decomposable and there are no decomposable primitive elements in $H^*(X, Q_p)$. Thus u is well defined and $m^*(u)$ is known by the inductive hypothesis; therefore $m^*(x)$ is determined. More generally, rather than just use the H-space squaring map, we shall use an H-space t-th power map.

$$\begin{aligned} &m(m\times 1)(m\times 1\times 1)\ldots(m\times 1\times\cdots\times 1)(\Delta\times 1\times\cdots\times 1) \\ &\quad \ldots(\Delta\times 1\times 1)(\Delta\times 1)\Delta: X\to X, \end{aligned} \tag{4.1}$$

($2t-2$ factors), whose induced homomorphism in cohomology will also determine m^*, given the ring structure of $H^*(X, Q_p)$.

Of course, if the H-space is not homotopy associative, homotopy commutative, in general m^* is not completely determined by $\Delta^* m^*$ but we do obtain useful information about it, which is sufficient for our purposes.

Thus our tactic is to apply the generalized naturality relations of section 2 (Theorem 2.27) to the maps induced in the cohomology and K-theory by these t-th power maps, f_t say. The essential features of $f_t^*: H^*(X, Q_p) \to H^*(X, Q_p)$ is that modulo decomposable elements it is just multiplication by the integer t. Now if $(X, m)\in H$, $H^*(X, Z)$ has no p-torsion and $H^*(X, Z_p)$ is m^*-primitive, it is not difficult to prove that $H^*(X, Q_p)$ is a free commutative algebra over Q_p (Theorem 5.2). Our results give no information on that part of $H^*(X, Q_p)$ which is an exterior algebra and so we consider the polynomial Hopf algebra, $H^*(X, Q_p)/(H^{\mathrm{odd}}(X, Q_p))$, where $(H^{\mathrm{odd}}(X, Q_p))$ is the Hopf ideal generated by classes with odd dimensions. Further, for notational convenience, we truncate our polynomial algebras at height $p+1$.

There remains one other obvious point which needs to be discussed; the results of sections 2 and 3 were proved for finite complexes whereas if $H^*(X, Q_p)$ is a polynomial algebra, then X is not a finite complex. The simplest way of resolving this difficulty is to replace the H-space X in the arguments which follow by one of its finite skeletons of sufficiently high dimension and, if necessary, to alter f_t up to homotopy to ensure that it is cellular.

Let $A=\sum_{i\geqslant 0} A_i$ be a connected graded ring of finite type over Q_p with strictly commutative (not graded commutative) multiplication. We say that A is special if for each integer s there exists a multiplicative ψ^k-module over Q_p, M say, whose associated graded ring N is isomorphic to A as a graded ring in dimensions $\leqslant s$. This is the algebraic analogue of taking finite skeletons of an infinite complex. A ring homomorphism $g:A\to B$ of special graded rings is a morphism if, for arbitrary s, $g=f_*:N\to P$ in dimensions $\leqslant s$ where $P\cong B$ and $N\cong A$ in dimensions $\leqslant s$, and $f:M\to L$ is a morphism of multiplicative ψ^k-modules over Q_p.

Let A be a truncated polynomial algebra of height $p+1$ over Q_p, that is, A has a homogeneous multiplicative basis $1, x, y, z, \ldots$ such that $x^\alpha y^\beta z^\gamma \ldots=0$ if and only if $(\alpha+\beta+\gamma+\ldots)\geqslant p+1$, and there are no other relations among the generators. Then N is also a truncated polynomial algebra of height $p+1$ over Q_p modulo elements of high filtration (considered as a filtered ring), using Lemma 2.21. It follows that there exists a splitting $J:N\to M$ which is a ring isomorphism, again, modulo elements a high filtration. All splittings considered will have this property. We shall agree to call M and N truncated polynomial algebras, where s has been chosen sufficiently large to perform all arguments, and all statements should be taken as being true modulo elements of high filtration. The symbols A and N will be interchanged without comment.

A morphism $g:A\to A$ is a t-map if $Q(g):Q(A)\to Q(A)$, the homomorphism induced on the indecomposable quotient, is just multiplication by t. We are concerned with finding what restrictions are put upon the structure of A if it supports a t-map. This is precisely what was done in [17] when A was finitely generated, but whereas there we took $t=p$, here we shall choose t to be prime to p. The advantage in doing this is that we are able to give a direct sum decomposition of N and a splitting $J:N\to M$ with respect to which the operations of section 2 are well behaved. This is Proposition 4.6 and the first few lemmas which follow are directed towards giving a proof. We continue to use the notations of section 3.

The first lemma is self evident. F is either M or N.

LEMMA 4.1. *$D^i(F)$ is an ideal and a direct summand of $D^j(F)$, for $i\geqslant j$.*

Since ψ^k is a ring homomorphism, $\psi^k\{D^i(M)\}\subset D^i(M)$ and so Lemma 2.17 and Lemma 4.1 imply that $Q^i(M)$ is a multiplicative ψ^k-module and $Q^i(A)$ a special graded ring. If $J:N\to M$ is a splitting (and a ring isomorphism), then $Q^i(J):Q^i(N)\to$

$\to Q^i(M)$ is a splitting for $Q^i(M)$ and there exists a commutative diagram

$$\begin{array}{ccc} Q^i(N) & \xrightarrow{\sigma} & Q^j(N) \\ {\scriptstyle Q^i(J)}\downarrow & & \downarrow{\scriptstyle Q^i(J)} \\ Q^i(M) & \xrightarrow[\sigma]{} & Q^j(M) \end{array} \quad (i \geqslant j), \tag{4.2}$$

where we write σ for the quotient homomorphisms. Also $\sigma: Q^i(M) \to Q^j(M)$ splits as a homomorphism of filtered Q_p-modules, for $i \geqslant j$.

LEMMA 4.2. *Let $f: M \to M$ induce a t-map $g = f_*: N \to N$; then $Q(f): Q(M) \to \to Q(M)$ is just multiplication by t.*

Proof. Let $h: Q(M) \to Q(M)$ be multiplication by t. Then h is a morphism of (multiplicative) ψ^k-modules and $h_* = Q(f_*) = Q(f)_*$. Lemma 2.25 implies that $h = Q(f)$.

Next we need some results from number theory. Let G_m be the multiplicative group of units Z_m. If $\lambda \in Q$, define $v_p(\lambda)$ to be the exponent of p when λ is expressed as a product of powers of distinct primes.

LEMMA 4.3. *Assume always that the integer t is not divisible by the prime p.*

If $p = 2$; then $v_2(t^s - 1) > \begin{cases} 1, & \text{if } \ s \neq 0 \bmod 2, \\ 2 + v_2(s), & \text{if } \ s = 0 \bmod 2. \end{cases}$

If $p > 2$; then $v_p(t^s - 1) > \begin{cases} 0, & \text{if } \ s \neq 0 \bmod (p-1), \\ 1 + v_p(s), & \text{if } \ s = 0 \bmod (p-1). \end{cases}$

The equalities are attained in the following cases.

(i) $p=2$, *s is odd and t is a generator of* G_4,

(ii) $p=2$, *s is even and t is a generator of* $G_8/(\pm 1)$.

(iii) $p>2$, *and t is a generator of* G_{p^2}.

Proof. We refer to section 2 of [3] for details of this lemma.

Let $g: A \to A$ be a t-map induced by $f: M \to M$. Define $R(M) \subset Q^{p-1}(M)$ and $R(N) \subset Q^{p-1}(N)$ to be the submodules where $Q^{p-1}(f)$ and $Q^{p-1}(f)_*$ are just multiplication by t.

LEMMA 4.4. *(p odd) Let $f: M \to M$ induce a t-map, where t is a generator of G_{p^2}; then $\sigma \mid R(M) \to Q(M)$ $(\sigma \mid R(N) \to Q(N))$ is an isomorphism of filtered (graded) Q_p-modules.*

Proof. We shall give the proof involving $R(M)$. The proof for $R(N)$ is formally almost identical.

First we show that σ is surjective. Let $u \in Q(M)_n$ and choose $v \in Q^{p-1}(M)_n$ such that $\sigma(v) = u$; then $(Q^{p-1}(f)\, v - tv) \in D^2(M)$ by Lemma 4.2. (We write $D^i(M)$ for

$D^i(M)/D^p(M)$ throughout this proof.) Therefore as inductive hypothesis suppose that we have found $v \in Q^{p-1}(M)_n$ such that $(Q^{p-1}(f)\, v - tv) \in D^i(M)$ for some $i \geqslant 2$ and $\sigma(v) = u$.

Lemma 4.1 enables us to write

$$D^i(M) = B^i \oplus D^{i+1}(M), \tag{4.3}$$

for some B^i, and so let $Q^{p-1}(f)\, v - tv = b + w_1$, where $b \in B^i$ and $w_1 \in D^{i+1}(M)$. Thus

$$Q^{p-1}(f)\,(v + \lambda b) = k(v + \lambda b) + \{\lambda(t^i - t) + 1\}\, b + w_1 + w_2,$$

where $w_2 \in D^{i+1}(M)$. But if t is a generator of G_{p^2}, $v_p(t^i - t) = 0$ by Lemma 4.3, and so we set $\lambda = -(t^i - t)^{-1} \in Q_p$. Replacing $v + \lambda b$ by v', $\sigma(v) = \sigma(v') = u$, $v' \in Q^{p-1}(M)_n$ and $(Q^{p-1}(f)\,(v') - tv') \in D^{i+1}(M)$ as required. Since $D^p(M) = 0$ in $Q^{p-1}(M)$, we deduce the existence of $v \in Q^{p-1}(M)_n$ with $\sigma(v) = u$ and $Q^{p-1}(f)\, v = tv$.

We must also show that $\sigma \mid R(M)$ is injective; that is $R(M) \cap D^2(M) = 0$. Let $w \in D^2(M)$, then by (4.3)

$$D^2(M) = B^2 \oplus B^3 \oplus \cdots \oplus B^{p-1}$$

and so $w = b_2 + b_3 + \cdots + b_{p-1}$. Lemma 4.2 implies that $Q^{p-1}(f)\,(w) = \sum_{2 \leqslant i \leqslant p-1} (t^i b_i + c_i)$, where $c_i = 0$ if $b_i = 0$, and $c_i \in B^{i+1} \oplus \cdots \oplus B^{p-1}$. If b_s is the first non-zero component of w, then the component of $Q^{p-1}(f)\, w$ in B^s is $t^s b_s$. Therefore $w \notin R(M)$. This completes the proof of the lemma.

COROLLARY 4.5. (p *odd*). *Let* $f: M \to M$ *induce a t-map, where t is a generator of* G_{p^2}; *then* $\sigma: Q^{p-1}(M) \to Q(M)$ *splits uniquely as a morphism of* ψ^k*-modules over* Q_p.

Proof. $\psi^k\{R(M)\} \subset R(M)$ for each k; for if $u \in R(M)$, then $f(\psi^k(u)) = \psi^k(f(u)) = \psi^k(tu) = t\psi^k(u)$. Therefore we have a splitting of ψ^k-modules,

$$Q(M) \xrightarrow{(\sigma \mid R(M))^{-1}} Q^{p-1}(M) \xrightarrow{\sigma} Q(M).$$

The splitting is unique since the image of $Q(M)$ for any $i: Q(M) \to Q^{p-1}(M)$ must be contained in $R(M)$.

When $p = 2$, both Lemma 4.4 and its corollary are vacuous.

PROPOSITION 4.6. *Let* $f: M \to M$ *induce a t-map, where if p is odd, t is a generator of* G_{p^2}; *then there exists a splitting* $J: N \to M$ *and a filtration preserving monomorphism* $i: Q(M) \to D(M)$ *such that the composition*

$$Q(M) \xrightarrow{i} D(M) \xrightarrow{\sigma} Q(M)$$

is the identity on $Q(M)$ *and*

$$\Phi_J^k\{i_*(Q(N))\} \subset i_*\{Q(N)\} \oplus D^p(N), \quad \textit{for each } k.$$

Proof. First we construct a multiplication ψ^k-module splitting $K:Q^{p-1}(N)\to$ $\to Q^{p-1}(M)$ so that $K\{R(N)\}=R(M)$. Let $J:Q^{p-1}(N)\to Q^{p-1}(M)$ be any splitting and $x_1, x_2,\ldots$ a homogeneous multiplicative basis for $Q^{p-1}(N)$ contained in $R(N)$. Suppose that $x_i\in Q^{p-1}(N)_t$ and $J(x_i)=u_i+v_i$, where $u_i\in R(M)$ and $v_i\in D^2(M)/D^p(M)$. Set $K(x_i)=u_i$ for each i and extend K to be a ring isomorphism; then $I_t(u_t)=x_i$.

Now consider the diagram

$$\begin{array}{ccc} \sum\limits_{i\geqslant t} N_i & \xrightarrow{\sigma} & \sum\limits_{i\geqslant t} Q^{p-1}(N)_i \\ & & \downarrow K \\ M_t & \xrightarrow[\sigma]{} & Q^{p-1}(M)_t. \end{array}$$

Let $\sigma(v_i)=u_i$, where $v_i\in M_t$; if $y_t=I_t(v_i)$, then $\sigma(y_i)=x_i$. Therefore define a splitting $J:N\to M$ by setting $J(y_i)=v_i$ for each i and extending to a ring isomorphism. Thus $K=Q^{p-1}(J)$. Now define the monomorphism $j_*:Q^{p-1}(N)\to D(N)$ by setting $j_*(x_i)=$ $=y_i$ and extending in the obvious manner. Then j_* is induced from $j:Q^{p-1}(M)\to$ $\to D(M)$ defined by setting $j(u_i)=v_i$ and the composition $\sigma j:Q^{p-1}(M)\to Q^{p-1}(M)$ is the identity. The homomorphism $i:Q(M)\to D(M)$ which we require is the composition

$$Q(M)\xrightarrow{(\sigma\mid R(M))^{-1}} Q^{p-1}(M)\xrightarrow{j} D(M). \tag{4.4}$$

That it satisfies the required properties is clear by construction and Lemma 4.4. This completes the proof.

It is clear from the definition of S^q_J and Proposition 4.6 that

$$S^q_J\{i_*(Q(N))\}\subset i_*\{Q(N)\}\oplus D^p(N).$$

In the light of these last two results, we shall assume that there always exists a morphism $g:A\to A$ which is a t-map, where t is a generator of G_{p^2}, $p\geqslant 2$. We use the t-map to define the graded homomorphism

$$h:Q(N)\otimes Z_p\to D^p(N)\otimes Z_p \tag{4.5}$$

which under certain conditions is an obstruction homomorphism to primitivity, as will be made explicit in section 5. Let $i:Q(M)\to D(M)$ be as in Proposition 4.6, and for $x\in Q(N)$ define $h'(x)=f_*(i_*(x))-ti_*(x)$, where f induces the t-map $g:A\to A$. Lemma 4.4 and (4.4), which gives the definition of i, together imply that $h'(x)\in D^p(N)$ and we set h as the reduction of h' mod$p$.

LEMMA 4.7. *The homomorphism h of* (4.5) *is well defined.*

Proof. Let j and k be two choices for the monomorphism $i:Q(M)\to D(M)$ given as (4.4). Then $f_*(j_*(x))=tj_*(x)+w_1$ and $f_*(k_*(x))=tk_*(x)+w_2$, where w_1 and w_2 belong to $D^p(N)$. But $j_*(x)-k_*(x)=w_3$ say, lies in $D^p(N)$, and so $f_*(j_*(x)-k_*(x))=$

$=t^p w_3$. Thus $w_1 - w_2 = (t^p - t)\, w_3$ and so by Lemma 4.3, $w_1 = w_2 \bmod p$, and h is well defined.

We can now state the two main technical results of this section. As we shall eventually identify $S^q \bmod p$ with the cyclic reduced power P^q as in the remarks which follow Lemma 2.15, we shall write P^q for $S^q \bmod p$.

THEOREM 4.8. *Suppose that* $n \neq 1 \bmod p$ *and that* h *vanishes in dimension* pn; *then* $P^1 : Q(N)_{n-p+1} \otimes Z_p \to Q(N)_n \otimes Z_p$ *is surjective.*

A consequence of Lemma 2.18 is that $P^n : Q(N)_n \otimes Z_p \to D^p(N)_{pn} \otimes Z_p$ is well defined, and is non zero whenever $Q(N)_n \neq 0$.

THEOREM 4.9. $\mathrm{Image}\{P^n \mid \mathrm{Kernel}(P^{n-1} : Q(N)_n \otimes Z_p \to Q(N)_{n+(n-1)(p-1)} \otimes Z_p)\} \subset \mathrm{Image}\, h$.

The proofs of these two theorems will occupy most of the remainder of this section, but before starting them we shall deduce the relevant conclusions.

COROLLARY 4.10. *Suppose that* h *vanishes everywhere; then*

(a) $(p=2)$ *A is a truncated polynomial algebra on odd dimensional generators,*

(b) $(p>2)$ *A is a truncated polynomial algebra on generators of dimension one.*

Proof. Given that h vanishes everywhere, Theorem 4.8 implies that

$$P^1 : Q(N)_{n-p+1} \otimes Z_p \to Q(N)_n \otimes Z_p \quad \text{is surjective,} \quad n \neq 1 \bmod p, \tag{4.6}$$

and Theorem 4.9 implies that

$$P^{n-1} : Q(N)_n \otimes Z_p \to Q(N)_{n+(n-1)(p-1)} \otimes Z_p \quad \text{is injective, for all } n. \tag{4.7}$$

We shall show that these together imply that $Q(N)_n = 0$ for $n \neq 1 \bmod p$ by using the Adem relation given as (2.10). In the calculation which follows, we shall need to write $Q(N)_t \otimes Z_p$ where t is a rather cumbersome expression, so just for this proof we shall write $Q(t)$ for $Q(N)_t \otimes Z_p$.

We argue by contradiction. Suppose that $Q(t) \neq 0$, where $t \neq 1 \bmod p$. Then (4.7) implies that $P^{t-1} : Q(t) \to Q(t+(t-1)(p-1))$ is injective. The Adem relation (2.10) implies that $P^1 P^{t-2} = (t-1) P^{t-1}$ and so $P^1 : Q((t-2)p+2) \to Q((t-1)p+1)$ has non zero image. Consider the sequence of vector spaces and homomorphisms,

$$\begin{aligned} &Q((t-p-1)\,p+p+1) \xrightarrow{P^1} Q((t-p)\,p+p) \xrightarrow{P^1} \cdots \xrightarrow{P^1} Q((t-3)\,p+3) \\ &\xrightarrow{P^1} Q((t-2)\,p+2) \xrightarrow{P^1} Q((t-1)\,p+1). \end{aligned} \tag{4.8}$$

Condition (4.6) implies that each P^1, save the last, is surjective and we know that the last is non zero. But repeated applications of (2.10) give, $P^1 P^1 \ldots P^1 = k!\, P^k$, where there are k factors on the left hand side. There are p maps in (4.8) so that $(P^1)^p=0$, which gives the required contradiction.

When $p=2$ this is the required result, but to prove part (b) we use a subsidiary result on the action of the P^i.

LEMMA 4.11. ($p>2$). *Suppose that P^1 vanishes everywere on $N\otimes Z_p$; then P^i vanishes everywhere for all $i>0$.*

This result can be proved for the cyclic reduced powers using the theory of higher order cohomology operations, as developed in [20]. A more elementary proof is given in [19].

Now we complete the proof of part (b) of Corollary 4.10. If $Q(t)\neq 0$ with $t>1$ and $t\neq 1 \bmod p$; then $P^t: Q(t)\to Q(t+(t-1)(p-1))$ does not vanish. But P^1 vanishes everywhere for dimensional reasons, since $Q(t)=0$ if $t\neq 1 \bmod p$. This contradicts Lemma 4.11 and so $Q(t)=0$ unless $t=1$. This completes the proof of the corollary.

Now we return to the proofs of Theorem 4.8 and 4.9. Suppose that $y_n\in Q(N)_n$ and let $i: Q(M)\to D(M)$ be as in Proposition 4.6. Both results follow from a detailed consideration of the equations,

$$\left.\begin{aligned}
&\text{(a)}\quad (p=2)\quad (S^n+2S^{n-1}Q^1)\,f_*(i_*(y_n)) = f_*(S^n+2S^{n-1}Q^1)\,i_*(y_n) + \\
&\qquad\qquad\qquad + 2f_1(2n-1)\,S^{n-1}i_*(y_n) \bmod 4, \\
&\text{(b)}\quad (p>2)\quad S^n f_*(i_*(y_n)) = f_* S^n(i_*(y_n)) + pf_1(n+(p-1)(n-1)) \\
&\qquad\qquad\qquad \times S^{n-1}i_*(y_n) \bmod p^2,
\end{aligned}\right\}\qquad (4.9)$$

which follow from Theorem 2.27 and its corollary. We shall write z_n for $i_*(y_n)$ and also f_i for $f_i(k)$ and let the context dictate where it acts.

LEMMA 4.12.
(a) ($p=2$) $(S^n+2S^{n-1}Q^1)f_*z_n=t(S^n+2S^{n-1}Q^1)\,z_n \bmod 4$,
(b) ($p>2$) $S^n f_* z_n = tS^n z_n \bmod p^2$.

Proof. Lemma 4.4 implies that $f_*z_n=kz_n+w$, where $w\in D^p(N)$. The results follow from Lemma 3.2 parts (a) and (d).

The proof of Theorem 4.9.

Suppose that $y_n \bmod p$ lies in the kernel of $P^{n-1} \mid Q(N)_n\otimes Z_p$. We shall show that there exists $y_{pn}\in Q(N)_{pn}$ such that $h(y'_{pn})=y_n'^p=P^n y'_n$, where y'_n and y'_{pn} are y_n and y_{pn} reduced mod$p$.

First we note that if $w\in D^p(N)$, then $f_J(w)=t^p w$, by Lemma 2.29 and Lemma 4.2. Thus $f_1(w)=0$. Proposition 4.6 implies that $S^{n-1}z_n\in D^p(N) \bmod p$, and so $f_1 S^{n-1}z_n= =0 \bmod p$. Hence by (4.9)

($p=2$) $\quad (S^n+2S^{n-1}Q^1)f_*z_n=f_*(S^n+2S^{n-1}Q^1)\,z_n \bmod 4$,

($p>2$) $\qquad S^n f_* z_n = f_* S^n z_n \bmod p^2$.

Using Lemma 2.18 we write $z_n^p+pz_{pn}$ for $(S^n+2S^{n-1}Q^1)\,z_n$ when $p=2$ or for S^nz_n when $p>2$. Thus by Lemma 4.12, $t(z_n^p+pz_{pn})=t^pz_n^p+pf_*(z_{pn}) \bmod p^2$, or $f_*(z_{pn})= =tz_{pn}+p^{-1}(t-t^p)\,z_n^p \bmod p$. Since t is a generator of G_{p^2}, Lemma 4.3 says that $v_p(t-t^p)=1$ and so $a=p^{-1}(t-t^p)$ is a unit in Q_p. Therefore if $y_{pn}\in Q(N)_{pn}$ is the image of $a^{-1}z_{pn}$ in the indecomposable quotient, $h(y'_{pn})=y_n'^p$ as required. This completes the proof.

The proof of Theorem 4.8 is not quite as simple. It is here that we see something of the reflection of the difficulties which seem to be intrinsic in calculating with higher order cohomology operations. Thus for example, when we further restrict our choice of the splitting J to ensure that the image of a homomorphism takes a particular form, intuitively what we are doing is to restrict the image of a higher order cohomology operation within its indeterminancy class; in certain circumstances this type of relationship can be made quite precise.

The proof of Theorem 4.8.

Let $y_n\in Q(N)_n$ where $n\neq 1 \bmod p$ and set $z_n=i_*(y_n)$. The conclusion of Theorem 4.9 implies that $P^{n-1}:Q(N)_n\otimes Z_p\to Q(N)_{n+(n-1)(p-1)}\otimes Z_p$ is injective, and so $z_{m+(p-1)}=S^{n-1}z_n$ is indecomposable mod$p$, where $m=n+(n-2)\,(p-1)$. The Adem relation (2.10) implies that

$$S^1S^{n-2}z_n=(n-1)\,S^{n-1}z_n \bmod p,$$

and so $z_m=S^{n-2}z_n$ is indecomposable mod$p$. We have

$$S^1z_m=(n-1)\,z_{m+(p-1)} \bmod p. \tag{4.10}$$

We now assume that (4.11) is true.

$$\left.\begin{aligned}(p=2)\quad & (S^n+2S^{n-1}Q^1)\,z_n=z_n^2 \bmod 4,\\ (p>2)\quad & \qquad\qquad\quad S^nz_n=z_n^p \bmod p^2.\end{aligned}\right\} \tag{4.11}$$

If this is not true originally, we alter the splitting so that it becomes true, without affecting the conclusion of Proposition 4.6. Let $K:N\to M$ be a second splitting, then

$$\left.\begin{aligned}(p=2)\quad & (S_K^n+2S_K^{n-1}Q_K^1)\,z_n=(S_J^n+2S_J^{n-1}Q_J^1)\,z_n+2g_1S_J^{n-1}z_n \bmod 4,\\ (p>2)\quad & S_K^nz_n=S_J^nz_n+pg_1S_J^{n-1}z_n \bmod p^2,\end{aligned}\right\} \tag{4.12}$$

by Theorem 2.27 and its corollary, where $g:M\to M$ is the identity map. Let the first term on the right hand sides of (4.12) equal z_n^p+pv, where $v\in i_*\{Q(N)\}\oplus D^p(N)$. Also let $z_{m+(p-1)}=z_{m+(p-1)}^{(1)}+z_{m+(p-1)}^{(2)}$, where $z_{m+(p-1)}^{(1)}\in i_*\{Q(N)\}$ and $z_{m+(p-1)}^{(2)}\in \in ^pD(N)$, and so $z_{m+(p-1)}^{(1)}$ is indecomposable mod$p$. Now we take a basis for $i_*\{Q(N)\}$ one member of which is $z_{m+(p-1)}^{(1)}$. Let the splitting K coincide with J everywhere on this basis, except that we set $K(z_{m+(p-1)}^{(1)})=J(z_{m+(p-1)}^{(1)}+v)$. Now $g_1(z_{m+(p-1)})= =g_1(z_{m+(p-1)}^{(1)})$, from the definition of K, and $g_1(z_{m+(p-1)}^{(1)})=-v$. Thus the right hand side of either equation (4.12) becomes $z_n^p \bmod p^2$ as required. Since $v \in i_*\{Q(N)\} \oplus \oplus D^p(N)$ the conclusion of Proposition 4.6 remains unaltered.

Therefore from (4.11) and Lemma 4.12 we deduce that for $p \geqslant 2$, $pf_1 S^{n-1} z_n = =(t-t^p)\, z_n^p \bmod p^2$. The integer t is a generator of G_{p^2} and so by Lemma 4.3,

$$f_1(z_{m+(p-1)}) = \alpha z_n^p, \quad \text{where} \quad \alpha \neq 0 \bmod p. \tag{4.13}$$

We now consider the equality

$$\Phi^t\{f_J(z_m)\} = f_J \Phi^t(z_m) \tag{4.14}$$

in dimension $m+2(p-1)$. We need further notations. Let

$$\Phi^t(z_m) = t^m z_m + x_{m+(p-1)} + x_{m+2(p-1)} + \text{higher dimensional elements},$$

where $x_i \in N_i$. In the notation of Lemma 2.10,

$$x_{m+(p-1)} = t^m R^1(t)\, z_m = t^m p^{-1}(1 - t^{p-1})\, S^1 z_m.$$

Thus by (4.10),

$$x_{m+(p-1)} = \beta(n-1)\, z_{m+(p-1)} \bmod p, \quad \text{where} \quad \beta(n-1) \neq 0 \bmod p. \tag{4.15}$$

Also let $x_{m+2(p-1)} = x^{(1)}_{m+2(p-1)} + x^{(2)}_{m+2(p-1)}$ where the first summand lies in $i_*\{Q(N)\}$ and the second in $D^p(N)$. Therefore $f_*(x_{m+2(p-1)}) = f_*(x^{(1)}_{m+2(p-1)}) + t^p x^{(2)}_{m+2(p-1)}$. Finally let $f_*(z_m) = tz_m + w_m$, where $w_m \in D^p(N)$.

The left hand side of (4.14) in dimension $m+2(p-1)$ is

$$tx_{m+2(p-1)} + t^m R^2(t)\, w_m + t^{m+(p-1)} R^1(t)\, f_1(z_m) + t^{m+2(p-1)} f_2(z_m)$$

and the right hand side component is

$$t^m f_2(z_m) + f_1(x_{m+(p-1)}) + f_*(x_{m+2(p-1)}).$$

We equate these expressions and after some rearrangement obtain,

$$\begin{aligned}
&(f_*(x^{(1)}_{m+2(p-1)}) - tx^{(1)}_{m+2(p-1)}) + (t^p - t)\, x^{(2)}_{m+2(p-1)} \\
&\quad = t^m(t^{2(p-1)} - 1) f_2(z_m) + t^m R^2(t)\, w_m + t^{m+(p-1)} R^1(t) f_1(z_m) - f_1(x_{m+(p-1)}).
\end{aligned}$$

Now $x^{(1)}_{m+2(p-1)} \in N_{pn}$ and $h \mid Q(N)_{pn} \otimes Z_p$ vanishes. Thus the first term in parentheses on the left hand side vanishes mod p. We reduce mod p and obtain

$$\alpha\beta(n-1)\, z_n^p = t^m R^2(t)\, w_m + t^{m+(p-1)} R^1(t)\, f_1(z_m) \bmod p, \tag{4.16}$$

since

$$\begin{aligned}
f_1(x_{m+(p-1)}) &= \beta(n-1)\, f_1(z_{m+(p-1)}) \bmod p, \quad \text{by (4.15)}, \\
&= \alpha\beta(n-1)\, z_n^p \bmod p, \quad \text{by (4.13)}.
\end{aligned}$$

The proof now breaks up into separate arguments for $p=2$ and $p>2$. The latter is the easier and so we give it first. We need the following lemma,

LEMMA 4.13. $(p>2)$. $f_*(z_m)=tz_m \bmod p$.

Proof. Corollary 2.28 (a) implies that $S^{n-2}f_*(z_n)=f_*S^{n-2}z_n \bmod p$. The conclusion follows from Lemma 3.2 (f).

Thus when p is odd, the first term on the right hand side of (4.16) vanishes. But $t^{m+(p-1)}R^1(t)f_1(z_m)=t^{m+(p-1)}p^{-1}(1-t^{p-1})S^1f_1(z_m)$ which equals $t^{(p-1)}\beta S^1f_1(z_m)$, if β is defined as in (4.15). Therefore by (4.16), $S^1f_1(z_m)=t^{-(p-1)}\alpha(n-1)z_n^p \bmod p$, and so as $f_1(z_m)\in D^p(N)$, there exists $w\in D^p(N)$ such that $S^1w=z_n^p \bmod p$. A simple consequence of the Cartan formula for S^1 in a polynomial algebra is that there exists $z_{n-(p-1)}\in N_{n-(p-1)}$ such that $S^1z_{n-(p-1)}=z_n \bmod p$ and modulo decomposable elements; and in the indecomposable quotient there exists $y_{n-p+1}\in Q(N)_{n-p+1}$ such that $S^1y_{n-p+1}=y_n \bmod p$. Thus $P^1 \mid Q(N)_{n-p+1}\otimes Z_p$ is surjective as required.

LEMMA 4.14. $f_*(z_m)=tz_m+\sum_{i+j=n}S^{i-1}v_i\cdot S^{j-1}v_j \bmod 2$, *where* $v_i\in i_*\{Q(N)\}$.

Proof. Again Corollary 2.28 implies that $S^{n-2}f_*(z_n)=f_*S^{n-2}z_n \bmod 2$. The conclusion now follows from Lemma 3.2 (c).

Thus $w_m=\sum_{i+j=n}S^{i-1}v_i\cdot S^{j-1}v_j$. We wish to consider $R^2(t)\,w_m$, where $R^2(t)= =2^{-2}\{(1-t^2)\,S^2+(t-t^2)\,S^1Q^1\}$. Now $S^2w_m=0 \bmod 2$, by the Cartan formula, and $S^1w_m=\sum_{i+j=n}2v'_{2i}\cdot S^{j-1}v'_j$, where $v'_i\in i_*\{Q(N)\}_i$, by Lemma 2.18, and so $S^1Q^1w_m= =-S^1S^1w_m=-2\sum_{i+j=n}S^1v'_{2i}\cdot S^{j-1}v'_j \bmod 4$. Therefore $R^2(t)\;w_m=\sum_{i+j=n}S^1v'_{2i} \times S^{j-1}v'_j \bmod 2$, by Lemma 4.3 since t is a generator of G_4. The right hand side of (4.16) is hence of the form $\sum v''_i\cdot S^1v''_j$, where v''_i and v''_j both lie in $\sum_{i>0}N_i$, as the second term on the right hand side of (4.16) has the same form as when $p>2$. Therefore $\sum v''_i\cdot S^1v''_j=\alpha\beta(n-1)\,z_n^2 \bmod 2$ and so there exists an element $z_{n-1}\in N_{n-1}$ such that $S^1z_{n-1}=z_n \bmod 2$ and modulo decomposable elements. Equivalently there exists an element $y_{n-1}\in Q(N)_{n-1}$ such that $S^1y_{n-1}=y_n \bmod 2$ and so $S^1 \mid Q(N)_{n-1}\otimes Z_2$ is surjective. This completes the proof of Theorem 4.8.

5.

Let $(X, m)\in H$ and suppose that $H^*(X, Z)$ has no p-torsion. We recall from section 3 of [18] the ring structure of $H^*(X, Q_p)$.

LEMMA 5.1. *As an algebra* $H^*(X, Q_p)$ *is isomorphic to a tensor product of Hopf algebras of the following types,*

W1. *An exterior algebra on a single odd dimensional generator,*

W2. *A polynomial algebra on a single even dimensional generator,*

W3. *An algebra with an infinite sequence of even dimensional generators,* 1, x_1, x_r,

$x_{r+1},\ldots$, where $x_1^{p^{r-1}}=x_r$ for some finite integer r, and $x_{r+i}^p=px_{r+i+1}$ for all $i\geqslant 0$, and there are no other relations among these generators.

THEOREM 5.2. *Suppose that, in addition to the above hypotheses, $H^*(X, Z_p)$ is m^*-primitive; then $H^*(X, Q_p)$ is a free commutative graded algebra.*

Proof. We need to show that if $H^*(X, Z_p)$ is m^*-primitive, then $H^*(X, Q_p)$ has no factors of type W3. Let C and C' be the Hopf ideals in $H^*(X, Q_p)$ and $H^*(X, Z_p)$ generated by $H^{\text{odd}}(X, Q_p)$ and $H^{\text{odd}}(X, Z_p)$ respectively, and consider the Hopf algebras $B=H^*(X, Q_p)/C$ and $B'=H^*(X, Z_p)/C'$. Then B' remains a primitively generated Hopf algebra and as an algebra is a tensor product of algebras of types W2 and W3.

Let $j:B\to B'$ be the quotient map, reduction mod$p$. Suppose that x_1 is the first generator of positive degree (the quasi-generator) in a factor of type W3, where we choose the factor so that x_1 has least possible dimension. We shall show that this leads to a contradiction and thus there are no factors of type W3.

We first show that $j(x_1)$ is primitive in B', for $m^*(x_1)=x_1\otimes 1+1\otimes x_1+\sum x'\otimes x''$ and so (in the notation of Lemma 5.1), $m^*(x_1^{p^r})=x_1^{p^r}\otimes 1+1\otimes x_1^{p^r}+(\sum x'\otimes x'')^{p^r}$ mod$p$, where $x_1^{p^r}=0$ mod$p$. Therefore $(\sum x'\otimes x'')^{p^r}=0$ mod$p$ which is only possible if $\sum x'\otimes x''=0$ mod$p$, for it lies within a polynomial subalgebra of B, since x_1 has least possible dimension for any factor of type W3.

Now choose a homogeneous basis for B' composed of primitive classes $z_1, z_2,\ldots$ which we may assume contains $y_1=j(x_1)$, since is both primitive and indecomposable. The non zero monomials in the z_i form a vector space basis for B' over Z_p and lead to a basis for $B'\otimes B'$. Now

$$m^*(x_1^{p^r})=\sum_{0\leqslant i\leqslant p^r}\binom{p^r}{i}x_1^i\otimes x_1^{p^r-i},$$

and so

$$m^*(x_{r+1})=\sum_{0\leqslant i\leqslant p^r}p^{-1}\binom{p^r}{i}x_1^i\otimes x_1^{p^r-i}.$$

Thus if

$$j(x_{r+1})=y_{r+1},$$
$$m^*(y_{r+1})=y_{r+1}\otimes 1+1\otimes y_{r+1}+\sum_{1\leqslant j\leqslant (p-1)}p^{-1}\binom{p^r}{jp^{r-1}}y_1^{jp^{r-1}}\otimes y_1^{(p-j)p^{r-1}},$$

and all terms in the summation on the right hand side are non zero. Thus y_{r+1} is indecomposable but is not primitive. Therefore there exists a decomposable element $w\in B'$ such that

$$m^*(w)=w\otimes 1+1\otimes w-\sum_{1\leqslant j\leqslant (p-1)}p^{-1}\binom{p^r}{jp^{r-1}}y_1^{jp^{r-1}}\otimes y_1^{(p-j)p^{r-1}},$$

so that $y_{r+1}+w$ is primitive. But if $z_1^\alpha z_2^\beta\ldots$ is any non zero monomial in the z_i with dimension that of y_{r+1}, then

$$m^*(z_1^\alpha z_2^\beta\ldots)=(z_1\otimes 1+1\otimes z_1)^\alpha(z_2\otimes 1+1\otimes z_2)^\beta\ldots$$

and the right hand side is linearly independent of $y_1^{jp^{r-1}}\otimes y_1^{(p-j)p^{r-1}}$, $1\leqslant j\leqslant(p-1)$, unless $z_1=z_2=\cdots=y_1$. But for dimensional reasons the monomial must then be $y_1^{p^r}$ which is zero. This gives the required contradiction and completes the proof.

We suppose therefore that $(X, m)\in H$ and that $H^*(X, Q_p)$ is a free commutative algebra over Q_p. Let C be the Hopf ideal generated by $H^{\text{odd}}(X, Q_p)$ and let B be the Hopf algebra $H^*(X, Q_p)/C$, which is a polynomial algebra. We write $\bar{B}$ for the natural augmentation ideal of the graded ring B and define $A^c=\sum_{i\geqslant 0}A_i^c$ to be graded ring $B/(\bar{B})^{p+1}$. Thus A^c is a truncated polynomial algebra of height $p+1$ on generators which necessarily have even dimensions, and so $A_{2i+1}^c=0$ for each i. Finally let A be the strictly commutative graded ring (not graded commutative) obtained from A^c by setting A_i equal to A_{2i}^c.

THEOREM 5.3. *The ring A is a special truncated polynomial algebra of height $p+1$ over Q_p, in the sense of section 4, which supports a t-map $f: A\to A$, where t is a generator of G_{p^2}.*

Proof. The verification is routine. The details can be found in section 3 of [18], and so we just sketch the arguments here.

First we replace X by a finite skeleton X^n of arbitrarily high dimension and let $M=K(X^n, Q_p)$ and $N=H^{\text{even}}(X^n, Q_p)$, as in Lemma 2.16. The continuous map $f_t: X\to X$ is as defined in (4.1) and we suppose that it is cellular so that $f_t(X^n)\subset X^n$. We now make repeated use of Lemma 2.24. First we divide out by the ideal which consists of those $u\in K(X^n, Q_p)$ for which $u^2=0$, so that the quotient ring of $K(X^n, Q_p)$, modulo elements of high filtration, is a polynomial algebra. We next truncate this polynomial algebra at height $p+1$, and let this be M. The fact that both ψ^k and $f_t^!: K(X^n, Q_p)\to K(X^n, Q_p)$ are ring isomorphisms imply, by Lemma 2.24, that M is a multiplicative ψ^k-module over Q_p. The associated graded ring, which is the corresponding quotient ring of $H^*(X^n, Q_p)$, is then a truncated polynomial algebra of height $p+1$ over Q_p, again modulo elements of high dimension, and the homomorphism induced by f_t is the required t-map. This completes the proof.

Now we introduce the obstruction homomorphism of section 4.

LEMMA 5.4. *Suppose that $(X, m)\in H$, $H^*(X, Z)$ has no p-torsion and $H^*(X, Z_p)$ is m^*-primitive; then in the notation of section 4 and Theorem 5.3, $h: Q(N)\otimes Z_p\to$ $\to D^p(N)\otimes Z_p$ vanishes everywhere.*

Proof. Let B, A^c and A be defined as in the paragraph before the statement of the last theorem and let B', $A^{c\prime}$ and A' be the corresponding rings where we have Z_p

not Q_p coefficients. Then B' is a primitively generated Hopf algebra and so we choose a multiplicative basis $y_1, y_2, \ldots$ composed of primitive elements. Thus $(f_t)^* y_i = ty_i$, for each i. In the quotient ring, $y_1, y_2, \ldots$ form a multiplicative basis for $A^{c'}$ or A' and still $(f_t)^* y_i = ty_i$. Now let $i_*: Q(N) \to D(N)$ be as constructed in Proposition 4.6 and let $z_1, z_2, \ldots$ be a basis over Q_p for $i_*\{Q(N)\}$, and therefore a multiplicative basis for N. Then $(f_t^!)_* z_i = (f_t)^* z_i = tz_i + w_i$, where $w_i \in D^p(N)$. Let $j: A \to A'$ be reduction mod p and assume, without loss of generality, that $j(z_i) = y_i + \sum_{2 \leqslant s \leqslant p} u_s$, where u_s is a polynomial in the y_i of total degree s. Now as $H^*(X, Q_p)$ is torsion free, $j(f_t)^* z_i = (f_t)^* (jz_i)$ and so

$$ty_i + \sum_{2 \leqslant s \leqslant p} tu_s + j(w_i) = ty_i + \sum_{2 \leqslant s \leqslant p} t^s u_s, \qquad \text{in } A'.$$

Therefore $j(w_i) = \sum_{2 \leqslant s \leqslant p} (t^s - t)\, u_s$ from which it follows that $u_s = 0$, $2 \leqslant s \leqslant (p-1)$, and $j(w_i) = 0$, using Lemma 4.3. Therefore if $i_*(y_i) = z_i$, where $y_i \in Q(N)$, then $h(j(y_i)) = 0$ for each i, which completes the proof of the lemma.

The proof of Theorem 1.1.

(b) Theorem 5.2 and Theorem 5.3 ensure that the hypotheses of Theorem 4.8 and Theorem 4.9 are satisfied. The additional hypothesis of Corollary 4.10 follows from Lemma 5.4. Therefore Theorem 1.1 (b) follows from Corollary 4.10 (b).

(a) If $H^*(X, Z)$ is torsion free and m^*-primitive, then $H^*(X, Z_p)$ is m^*-primitive. Thus $H^*(X, Q_p)$ is a free commutative algebra for all primes p, including $p=2$, by Theorem 5.2. Therefore $H^*(X, Z)$ is a free commutative algebra over Z, and if p is an odd prime, $H^*(X, Z_p)$ is a free commutative algebra over Z_p. Further the dimensions of the generators of the indecomposable quotient of this latter ring coincide with the dimensions of the generators of the indecomposable quotient of the former. Hence part (a) of Theorem 1.1 follows from part (b).

The proof of Corollary 1.2.

Theorem 1.1 (a) implies that $H^*(X, Z) \cong Z[x_2, y_2, \ldots]$ is a finitely generated polynomial algebra, where the generators all lie in $H^2(X, Z)$. If $i_x, i_y, \ldots : X \to K(Z, 2)$ realize these generators, $i_x \times i_y \times \ldots : X \to K(Z, 2) \times K(Z, 2) \times \ldots$ induces an isomorphism of cohomology rings, and therefore of homology groups in each dimension. A theorem of Whitehead implies that $i_x \times i_y \times \ldots$ is a homotopy equivalence.

The conclusion of this corollary implies in particular that the H-space structure on such an X is well determined up to homotopy, and is both homotopy associative and homotopy commutative.

The proof of Theorem 1.5.

Part (a) is proved in a manner similar to Theorem 1.1 (a), except that we use the weaker result Corollary 4.10 (a) in place of part (b). Part (b) of Theorem 1.5 is an immediate consequence of Theorem 4.9, identifying the P^i of that proposition with Sq^{2i}, as in the remarks which follow Lemma 2.15.

The proof of Corollary 1.3(b)

If $H^*(X, Z_p)$ is a coassociative, cocommutative Hopf algebra, $\alpha: P\{H^*(X, Z_p)\} \to$ $\to Q\{H^*(X, Z_p)\}$ is an isomorphism in all odd dimensions by Proposition 4.23 and the remarks which follow in [22]. Thus if $H^*(X, Z_p)$ is a tensor product of a polynomial algebra with generators of dimension 2 and an exterior algebra, to show that it is primitive we must show that α is an isomorphism in dimension 2. But

$$H^1(X, Z_p) \cong H_1(X, Z_p) \cong H_1(X, Z) \otimes Z_p \cong \pi_1(X) \otimes Z_p = 0,$$

since $H_*(X, Z)$ has no p-torsion and X is simply connected. Thus any element of $H^2(X, Z_p)$ is primitive for dimensional reasons. This completes the proof of the corollary in one direction. The converse restates a particular case of Theorem 1.1 (b).

The proof of Corollary 1.4 (b)

If $H^*(X, Z_p)$ is a coassociative Hopf algebra and as an algebra is an exterior algebra over Z_p (p odd), then by the Samelson-Leray theorem (see for example, Theorem 7.20 of [22]), $H^*(X, Z_p)$ is m^*-primitive. If X is 2-connected, then $H^2(X, Z_p)$ $=0$ by an argument similar to that used in the proof of Corollary 1.3 (b), and so by Theorem 1.1 (b), $H^*(X, Z_p)$ is an exterior algebra if it is m^*-primitive.

The proofs of Corollaries 1.3 (a) *and* 1.4 (a)

One can prove these results using Theorem 1.1 and general Hopf algebra theorems. However the author does not know of explicit references for the Hopf algebra results needed, so instead we prove a lemma which enables us to deduce parts (a) of Corollaries 1.3 and 1.4 from parts (b). This lemma is Hopf algebraic, though we state it for cohomology Hopf algebras.

LEMMA 5.5. *Let* $(X, m) \in H$. *Suppose that* $H^*(X, Z)$ *is torsion free and that* $H^*(X, Z_p)$ *is* m^**-primitive for each prime* p*; then* $H^*(X, Z)$ *is* m^**-primitive.*

Proof. Theorem 5.2 implies that $H^*(X, Q_p)$ is a free commutative algebra over Q_p for each prime p, since $H^*(X, Z_p)$ is m^*-primitive. Therefore $H^*(X, Z)$ is a free commutative algebra over Z.

Suppose that $\alpha: P\{H^*(X, Z)\} \to Q\{H^*(X, Z)\}$ is surjective in dimensions $<n$. Choose primitive representatives in $H^*(X, Z)$ for a basis of the indecomposable quotient in dimensions $<n$, y_i say, so that we may choose monomials in the y_i, u_j say, which form a basis for $H^*(X, Z)$ in dimensions $<n$ and for the well defined direct summand of decomposable elements in $H^n(X, Z)$. The set of elements of the

form $u_s \otimes u_t$ which lie in $\bar{H}^*(X, Z) \otimes \bar{H}^*(X, Z)$ in dimensions $\leqslant n$ form a basis in this range of dimensions. We subdivide the subset of basis elements in dimension n into subsets $\{v_i\}$, $\{v_i^*\}$, $\{v_i^{**}\}, \ldots$, where $u_i \otimes u_j$ and $u_s \otimes u_t$ belong to the same subset if and only if $u_i u_j = \pm u_s u_t$. In particular let the first subset consist of those $u_i \otimes u_j$ for which $u_i u_j = 0$.

Let $u_k = y_1^\alpha y_2^\beta \ldots$ be a decomposable element of the basis in $H^n(X, Z)$. Then

$$m^* u_k = (y_1 \otimes 1 + 1 \otimes y_1)^\alpha (y_2 \otimes 1 + 1 \otimes y_2)^\beta \ldots . \tag{5.1}$$

Therefore $m^* u_k = u_k \otimes 1 + 1 \otimes u_k + \sum b_j^{**\cdots} v_j^{**\cdots}$, where all the $v_j^{**\cdots}$ belong to the same subset of basis elements $\{v_i^{**\cdots}\}$, ($k+1$ asterisks), and the $b_i^{**\cdots}$ corresponding to each member of this subset is non zero. Also it is clear that for all such u_k, $|b_i^{**\cdots}| < K$, where K is some constant depending only upon n.

Let $x \in H^n(X, Z)$ be indecomposable with $m^* x = x \otimes 1 + 1 \otimes x + \sum a_i v_i + \sum a_i^* v_i^* + \cdots$, summing over the different subsets of base elements. Since $H^*(X, Z_p)$ is primitive for each p, there exists a decomposable element w, possibly depending upon p, such that $(x+w) \bmod p$ is primitive in $H^*(X, Z_p)$. Let $w = \sum c_k u_k$, then,

$$a_i^{**\cdots} + c_k b_i^{**\cdots} = 0 \bmod p, \quad (k+1 \text{ asterisks}), \text{ for each } i. \tag{5.2}$$

The first consequence of (5.2) is that $a_i = 0$ for each i, since the corresponding b_i are all zero and we may take p to be arbitrarily large. Otherwise (5.2) implies that $a_i^{**\cdots} b_j^{**\cdots} = a_j^{**\cdots} b_i^{**\cdots} \bmod p$ for all p and each pair of integers i and j. Thus $a_i^{**\cdots} : b_i^{**\cdots} = a_j^{**\cdots} : b_j^{**\cdots}$. There are three cases to consider.

First suppose that $a_i^{**\cdots} = 0$ for some i. Since all the $b_i^{**\cdots}$ are non zero, this implies that $a_i^{**\cdots} = 0$ for all i, and so $\sum a_i^{**\cdots} v_i^{**\cdots} = 0$. Second suppose that $|a_i^{**\cdots}| \geqslant |b_i^{**\cdots}|$, necessarily for each i. We may therefore choose c_k so that $a_i^{**\cdots} + c_k b_i^{**\cdots} = 0$ for each i, which implies that we may choose w so that $\sum (a_i^{**\cdots} + c_k b_i^{**\cdots}) v_i^{**\cdots} = 0$. Finally suppose that $0 < |a_i^{**\cdots}| < |b_i^{**\cdots}|$ for each i. By examining (5.1) one sees that the $b_i^{**\cdots}$ have no common factor of modulus greater than one unless $u_k = (y_s \otimes 1 + 1 \otimes y_s)^{q^r}$ for some y_s and some prime q. Then we must have $q|a_i^{**\cdots}| = |b_i^{**\cdots}|$ for each i, and

$$m^*(u_k) = u_k \otimes 1 + 1 \otimes u_k + q^{-1} \sum_{1 \leqslant i \leqslant q^r - 1} \binom{q^r}{i} y_s^j \otimes y_s^{q^r - i}.$$

But by considering $x \bmod q$ in $H^*(X, Z_q)$ one can show that $H^*(X, Z_q)$ cannot then be primitive (as in the proof of Theorem 5.2). Thus the third possibility cannot in fact arise.

Therefore we can choose w so that $x+w$ is primitive in $H^*(X, Z)$, which completes the inductive step and thereby proves the lemma.

The proofs of Corollary 1.3 (a) and Corollary 1.4 (a) are now easily completed, for the implications which do not arise almost immediately from the theorem follow from Lemma 5.3 and the Hopf algebraic results quoted in Corollaries 1.3 (b) and 1.4 (b), using the additional fact that $H^*(X, Z_2)$ is strictly commutative in the sense of (7.17) of [22].

This completes the proofs of the results of section 1. Finally we mention a companion result to Theorem D of [17], which is implicit in Theorem 5.9.

Let Y be a complex with finite skeletons, whose cohomology ring with Q_p coefficients is a finitely generated polynomial algebra, possibly truncated at height greater than p, with even dimensional generators, y_i say.

A continuous map $f: Y \to Y$ is a t-map if $f^*(y_i) = ty_i$ modulo decomposable elements of the polynomial algebra.

THEOREM 5.6. *Suppose that Y supports a t-map, where t is a generator of G_{p^2}; then each y_i has dimension* 2.

This follows from Theorem 4.9 by choosing $2n$ to be the highest possible dimension for a generator of $H^*(Y, Q_p)$.

In particular this implies that if $(X, m) \in H$, then $H^*(X, Q_p)$ is not a finitely generated polynomial algebra, as does Theorem D of [17]. In fact neither is a particularly strong result, but we shall not extend the results here. Such information has applications in deciding when a map between classifying spaces of Lie groups is homotopic to a map induced from a homomorphism of the Lie groups.

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