

On Factorization into Prime Ideals

Autor(en): **Gilmer, Robert**

Objektyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **47 (1972)**

PDF erstellt am: **27.05.2024**

Persistenter Link: <https://doi.org/10.5169/seals-36351>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

On Factorization into Prime Ideals

ROBERT GILMER¹⁾

Let r be a regular element of the commutative ring R . It is well known that if r can be written as a finite product of prime elements of R , then this representation is unique. We consider here the corresponding question for ideals:

If A is a regular ideal of R such that A can be represented as a finite product of prime ideals of R , is this representation unique?

We begin by listing some observations concerning this question.

(1) Without the assumption that A is regular, the answer to the question is negative, even if R is Noetherian with identity. For example, (0) is prime in R for any integral domain R , and yet $(0) = [(0)]^n$ for each positive integer n . If R is the direct sum of two fields F_1 and F_2 , then $P = F_1 \oplus (0)$ is maximal in R , and $P = P^n$ for each positive integer n .

(2) Even with the assumption that A is regular, the answer to the question is negative, even if R is an integral domain with identity. For instance, $P_1 = P_1 P_2$ for any prime ideals P_1, P_2 of a valuation ring R with $P_1 \subset P_2$; more generally the equality $P_1 = P_1 P_2$ holds for any prime ideals P_1, P_2 of a Prüfer domain with $P_1 \subset P_2$ [1, Theorem 19.3].

(3) The following result appears as Theorem 30.13 of [1]:

Let A be a nonzero ideal of a Noetherian domain D such that A can be expressed as a finite product of prime ideals of D . Then this representation is unique if D contains no identity, and is unique to within factors of D if D contains an identity.

(4) By examining the proof of Theorem 30.13, we can see that the following result, which we label as (*), is valid.

() Let A be a regular ideal of a commutative ring R such that A can be expressed as a finite product of finitely generated prime ideals of R . Then this representation is unique if R contains no identity, and is unique to within factors of R if R contains an identity.*

In this paper we extend (*) to the case where A is finitely generated, but the prime factors of A need not be finitely generated (Theorems 1 and 2). In Proposition 1, we prove that our results are stronger than (*) by proving that for any positive integers

¹⁾ During the writing of this paper, the author received partial support from National Science Foundation Grant GP-19406.

k and n , there is an integral domain D_k with identity containing prime ideals P_1, \dots, P_n such that $P_1^{e_1} P_2^{e_2} \dots P_n^{e_n}$ is finitely generated if and only if $e_1 + \dots + e_n \geq k$. Our proofs of Theorems 1 and 2 are independent of (*) and the result cited in (3). Moreover, our proofs are more elementary than the proof of (30.13) in [1]; these proofs rest on the following facts.

OBSERVATION 1. *If $\{A_i\}_1^n$ is a finite family of ideals of the commutative ring R , then $A_1 A_2 \dots A_n$ is regular if and only if each A_i is regular.*

OBSERVATION 2. *If A is a regular ideal of the commutative ring R and if N is a multiplicative system in R , then the extension of A to the ring of quotients R_N is regular in R_N .*

RESULT 1. [1, Corollary 5.2] *If A and B are ideals of the commutative ring R such that $AB=B$, where B is finitely generated, then there is an element x of A such that $xb=b$ for each b in B : if B is regular, then R has an identity element and $A=R$.*

LEMMA 1. *Assume that A and B are ideals of the commutative ring R , where B is proper, finitely generated, and regular. Moreover, assume that $\{P_1, \dots, P_m\}$ and $\{Q_1, \dots, Q_n\}$ are two families of proper prime ideals of R such that $B = AP_1^{s_1} \dots P_m^{s_m} = AQ_1^{t_1} \dots Q_n^{t_n}$, where each s_i and each t_i is positive. Then each minimal element of the set $\{P_1, \dots, P_m, Q_1, \dots, Q_n\}$ occurs both as a P_i and as a Q_j , and the corresponding exponents s_i and t_j are equal.*

Proof. We assume that the labeling is such that P_1 is a minimal element of $\{P_i\}_1^m \cup \{Q_j\}_1^n$. If ' e ' denotes extension of ideals with respect to the quotient ring R_{P_1} , then

$$B^e = A^e (P_1^e)^{s_1} = A^e (Q_j^e)^{wt_j},$$

where $w=0$ if $P_1 \notin \{Q_j\}_1^n$, while $w=1$ and $P_1=Q_j$ otherwise. The assumption $w=0$ would lead to the equation

$$B^e = B^e (P_1^e)^{s_1},$$

where B^e is finitely generated, regular, and proper, and $(P_1^e)^{s_1}$ is proper in R_{P_1} , in contradiction to Result 1. Hence $w=1$ and

$$B^e = A^e (P_1^e)^{s_1} = A^e (P_1^e)^{t_j}.$$

Again, if $t_j > s_1$, we obtain a contradiction, as above, from the equation

$$B^e = B^e (P_1^e)^{t_j - s_1}.$$

Therefore $s_1 = t_j$, and our proof is complete.

THEOREM 1. *Let B be a proper, finitely generated regular ideal of the commutative ring R such that B is a product of proper prime ideals of R . Then the representation of B as a finite product of proper prime ideals is unique.*

Proof. Let

$$B = P_1^{s_1} \dots P_m^{s_m} \quad \text{and} \quad B = Q_1^{t_1} \dots Q_n^{t_n}$$

be two representations of B as a finite product of proper prime ideals. From the proof of Lemma 1, it follows that the set S of primes P_i such that $P_i = Q_j$ for some j , and $s_i = t_j$, is nonempty. We assume that $S = \{P_1, \dots, P_r\}$, where $r \leq m$ and where $P_i = Q_i$ for $1 \leq i \leq r$. Setting $A = P_1^{s_1} \dots P_r^{s_r}$, we have

$$B = AP_{r+1}^{s_{r+1}} \dots P_m^{s_m} = AQ_{r+1}^{t_{r+1}} \dots Q_n^{t_n},$$

and the assumption $r < m$ or $r < n$ would lead to a contradiction of Lemma 1. Hence $r = m = n$, and this completes the proof of Theorem 1.

THEOREM 2. *Suppose that R is a commutative ring without identity and that B is a finitely generated regular ideal of R that is representable as a finite product of prime ideals of R . Then this representation is unique.*

Proof. We consider first the case when $B = R^n$ is a power of R . It is clear that R is the only prime factor of B . Hence we need only prove in this case that $R^n = R^m$ implies that $m = n$. Since R is a ring without identity, this follows immediately from Result 1.

If B is not a power of R , then we write

$$B = R^s P_1^{s_1} \dots P_m^{s_m} = R^t Q_1^{t_1} \dots Q_n^{t_n},$$

where $\{P_i\}_1^m$ and $\{Q_i\}_1^n$ are sets of m and n proper prime ideals of R , where s_i and t_i are positive, and s and t are nonnegative ($R^0 U$, for U an ideal of R , is defined to be U). If $N = R - [(\cup_1^m P_i) \cup (\cup_1^n Q_i)]$, and if ' e ' denotes extension of ideals of R to the quotient ring R_N , then

$$B^e = (P_1^e)^{s_1} \dots (P_m^e)^{s_m} = (Q_1^e)^{t_1} \dots (Q_n^e)^{t_n}$$

in R_N . By Theorem 1, $m = n$ and, by proper labeling, $P_i^e = Q_i^e$ and $s_i = t_i$ for $1 \leq i \leq m$. It follows that $P_i = Q_i$ for $1 \leq i \leq n$, and we have

$$B = R^s P_1^{s_1} \dots P_m^{s_m} = R^t P_1^{s_1} \dots P_m^{s_m}.$$

As before, the assumption $s > t$ would lead to the equation $R^{s-t} B = B$, and to a contradiction of the assumption that R does not contain an identity.

It is clear that Theorems 1 and 2 imply (*). It is conceivable, however, that Theorems 1 and 2 are not actually stronger than (*). That is, if the following statement (**) were true, then (*) would imply Theorems 1 and 2.

(**) If $\{P_i\}_1^m$ is a finite family of regular prime ideals of the commutative ring R , and if e_1, \dots, e_m are positive integers such that $P_1^{e_1} \dots P_m^{e_m}$ is finitely generated, then each P_i is finitely generated.

We proceed to show that a very strong negation of (**) is, in fact, true.

PROPOSITION 1. Let k and n be positive integers, where $k \geq 1$. There is an integral domain with identity containing prime ideals P_1, P_2, \dots, P_n such that the product $P_1^{e_1} P_2^{e_2} \dots P_n^{e_n}$ is finitely generated if and only if $\sum_{i=1}^n e_i \geq k$.

Proof. Let D be a non-Noetherian domain with identity containing distinct ideals A_1, A_2, \dots, A_n such that $A_{i_1} + A_{i_2} + \dots + A_{i_r}$ is not finitely generated for any nonempty subset $\{i_1, i_2, \dots, i_r\}$ of $\{1, 2, \dots, n\}$ ²⁾. Let t be an indeterminate over D , and let E_k be the subring $D[t^{k+1}, t^{k+2}, \dots, t^{2k+1}]$ of $D[t]$; E_k is a graded ring with gradation $D, Dt^{k+1}, Dt^{k+2}, \dots$. We set

$$A = (t^{k+1}, t^{k+2}, \dots, t^{2k+1}), \quad B = (t^{k+1}, t^{k+2}).$$

It is straightforward to verify that

$$A^n = (t^{n(k+1)}, t^{n(k+1)+1}, \dots, t^{n(k+1)+k})$$

for any positive integer n ,

$$\begin{aligned} A^n &= B^n \quad \text{for } n \geq k, \quad \text{and} \\ B^r &= (t^{r(k+1)}, t^{r(k+1)+1}, \dots, t^{r(k+1)+r}) \quad \text{for } r < k. \end{aligned}$$

We set $C_i = B + A_i t^{k+3}$ for $1 \leq i \leq n$. Each C_i is a homogeneous ideal of E_k , and $B \subset C_i \subset A$ for each i . Hence

$$C_1^{e_1} C_2^{e_2} \dots C_n^{e_n} = A^{e_1 + \dots + e_n} = B^{e_1 + \dots + e_n}$$

if $e_1 + \dots + e_n \geq k$, and $C_1^{e_1} C_2^{e_2} \dots C_n^{e_n}$ is finitely generated.

If $e = e_1 + \dots + e_n < k$, then $C_1^{e_1} C_2^{e_2} \dots C_n^{e_n}$ is a homogeneous ideal of E_k , and its homogeneous component in $Dt^{e(k+1)+(e+1)}$ is

$$(A_{i_1} + A_{i_2} + \dots + A_{i_r}) t^{e(k+1)+(e+1)},$$

where $\{i_1, i_2, \dots, i_r\}$ is the set of integers j such that $e_j \neq 0$. Because E_k is a graded ring and $A_{i_1} + A_{i_2} + \dots + A_{i_r}$ is not finitely generated as an ideal of D , it follows that $C_1^{e_1} C_2^{e_2} \dots C_n^{e_n}$ is not finitely generated as an ideal of E_k .

The ideals C_i of E_k are not prime in E_k . To obtain our desired example, we let D_k be the subring of $E = E_k[X_1, \dots, X_n]$ consisting of all polynomials f such that the

²⁾ Take, for example, $D = \mathbb{Z}[\{X_j\}_{j=1}^\infty]$, and for $1 \leq i \leq n$, take $A_i = (\{X_j \mid j \in S_i\})$, where S_1, \dots, S_n are distinct infinite subsets of \mathbb{N} , the set of positive integers.

coefficient of $X_1^{e_1} X_2^{e_2} \dots X_n^{e_n}$ in f is in $A_1^{e_1} A_2^{e_2} \dots A_n^{e_n}$ for each $e_1, e_2, \dots, e_n \geq 0$. Again D_k is a graded ring with gradation

$$E_k, \sum A_i X_i, \sum A_i^{e_i} A_j^{2-e_i} X_i^{e_i} X_j^{2-e_i}, \dots;$$

in fact, D_k is a graded subring of E , where E has the usual gradation by degree. The ideal $X_i E \cap D_k = P_i$ is a homogeneous prime ideal of D_k : in fact,

$$P_i = A_i X_i + A_i A_j X_i X_j + \dots = A_i X_i D_k.$$

Hence

$$P_1^{e_1} P_2^{e_2} \dots P_n^{e_n} = A_1^{e_1} A_2^{e_2} \dots A_n^{e_n} X_1^{e_1} X_2^{e_2} \dots X_n^{e_n} D_k$$

is the set of polynomials f in D_k such that the coefficient of $X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}$ in f is zero if $i_j < e_j$ for some j , and is in $A_1^{i_1} A_2^{i_2} \dots A_n^{i_n}$ otherwise. Hence $P_1^{e_1} P_2^{e_2} \dots P_n^{e_n}$ is finitely generated if and only if $e_1 + \dots + e_n \geq k$.

We remark that the prime ideals P_i of Proposition 1 extend to maximal ideals of the quotient ring $(D_k)_S$, where $S = D_k - (\cup_1^n P_i)$. But $(D_k)_S$ is a quotient ring of $L[X_1, \dots, X_n]$, where L is the quotient field of E_k , and hence $(D_k)_S$ is Noetherian.

We have no counterexample to (**) in the case where the ideals P_i are maximal in R . In particular, we know of no example of a regular maximal ideal M of a commutative ring S with identity such that M is not finitely generated, but some power of M is finitely generated. If such M and S exist, then they also exist with M maximal in a quasi-local ring S .

The author acknowledges several discussions with Tom Parker concerning factorization into prime ideals. These discussions were helpful in the preparation of this paper.

REFERENCES

- [1] GILMER, R., *Multiplicative Ideal Theory* (Queen's University, Kingston, Ontario, 1968).
- [2] RIBENBOIM, P., *Anneaux de Rees int\'egralement clos*, J. Reine Angew. Math. 204 (1960), 99–107.

Florida State University, Tallahassee

Received July 13, 1971