On the Homology Theory of Central Group Extensions II. The Exact Sequence in the General Case

Autor(en): **Eckmann, Beno / Hilton, Peter J. / Stammbach, Urs**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **47 (1972)**

PDF erstellt am: **27.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-36357

On the Homology Theory of Central Group Extensions II. The Exact Sequence in the General Case

by BENO ECKMANN, PETER HILTON, and URS STAMMBACH

Dedicated to the Memory of Tudor Ganea (1922–1971)

1. Introduction

In [2], a natural exact homology sequence

$$H_3G \xrightarrow{\varepsilon} H_3Q \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2G \xrightarrow{\beta} H_2Q \xrightarrow{\beta} N \xrightarrow{\gamma} G_{ab} \to Q_{ab} \to 0 \tag{1.1}$$

was associated with those central group extensions

$$N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q \tag{1.2}$$

for which $\mu: N \otimes N \to G_{ab} \otimes N$ is the zero-map. Such extensions were called *weak stem-extensions* in [2]; they include, of course, the case of stem-extensions, i.e., central extensions (1.2) with $N \subset [G, G]$. We write here ε (or μ) for any homomorphism induced by ε (or μ). The maps δ and β are "boundary" homomorphisms, and χ is a "commutator map", c.f. [2].

The purpose of the present paper, which is a supplement to [2], is to replace (1.1) by a general sequence valid for *all central extensions* (1.2). This general sequence differs from (1.1) only in that the term $G_{ab} \otimes N$ is replaced by a certain quotient $(G_{ab} \otimes N)/U$, where U is in the kernel of χ. In fact, the portion

$$G_{ab} \otimes N \to \cdots \to Q_{ab} \to 0 \tag{1.3}$$

of (1.1) was established in [2] for any central extension. In order to describe the subgroup U, we apply (1.3) to the extension

$$N \overset{\bar{\mu}}{\rightarrowtail} N \overset{\bar{\varepsilon}}{\twoheadrightarrow} 1$$

and get $N \otimes N \xrightarrow{\bar{\chi}} H_2N \to 0 \to N \to N \to 0$. The map of extensions given by the commutative diagram

$$\begin{array}{ccccc} N & \rightarrowtail & N & \twoheadrightarrow & 1 \\ = \downarrow & & \downarrow \mu & & \downarrow \\ N & \rightarrowtail & G & \twoheadrightarrow & Q \end{array}$$

induces a map of sequences

$$\begin{tikzcd}
N \otimes N \arrow[r, tail, "\bar{\chi}"] \arrow[d, "\mu"] & H_2 N \arrow[r] \arrow[d, "\mu"] & 0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
G_{ab} \otimes N \arrow[r, "\chi"] & H_2 G \arrow[r] & H_2 Q \arrow[r] & N \arrow[r] & G_{ab} \arrow[r] & Q_{ab} \arrow[r] & 0 .
\end{tikzcd}$$

Hence $\chi(\mu \ker \bar{\chi})=0$, and we take

$$U = \mu \ker \bar{\chi} \subset G_{ab} \otimes N . \tag{1.4}$$

Of course $U=0$ in the case of a weak stem-extension.

Our task is then to define $\delta: H_3 Q \to (G_{ab} \otimes N)/U$ and prove the exactness of

$$H_3 G \xrightarrow{\varepsilon} H_3 Q \xrightarrow{\delta} (G_{ab} \otimes N)/U \xrightarrow{\chi} H_2 G , \tag{1.5}$$

where, of course, χ denotes the map induced by the commutator map χ of [2]. This will be done in Section 2 by using the elementary techniques of [2]. We base ourselves on the free presentation

$$\begin{tikzcd}
 & & N \arrow[d, tail, "\mu"] \\
R \arrow[r, tail] \arrow[d, tail] & F \arrow[r, two heads] \arrow[d, "="] & G \arrow[d, two heads, "\varepsilon"] \\
S \arrow[r, tail] \arrow[d, two heads] & F \arrow[r, two heads] & Q \\
N & &
\end{tikzcd} \tag{1.6}$$

of (1.2). Then, as in [2], a partial resolution of $\mathbf{Z}$ over Q is given by

$$\begin{tikzcd}
 & JF \otimes_F \mathbf{Z}Q \arrow[r] \arrow[dr] & \mathbf{Z}Q \arrow[r, two heads] & \mathbf{Z} \\
S_{ab} \arrow[ur, tail] & & JQ \arrow[u, tail] &
\end{tikzcd} \quad ; \tag{1.7}$$

and, similarly, a partial resolution of $\mathbf{Z}$ over G is given by

$$\begin{tikzcd}
 & JF \otimes_F \mathbf{Z}G \arrow[r] \arrow[dr] & \mathbf{Z}G \arrow[r, two heads] & \mathbf{Z} \\
R_{ab} \arrow[ur, tail] & & JG \arrow[u, tail] &
\end{tikzcd} \tag{1.8}$$

Exploiting, as in [2], the reduction theorem, we will henceforth identify, naturally, $H_3 G$ with $H_1(G; R_{ab})$ and $H_3 Q$ with $H_1(Q; S_{ab})$. Thus we may state our result in the following more technical form.

THEOREM. *Given the central extension $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$, there exists a natural homo-*

morphism $\delta: H_1(Q; S_{ab}) \to (G_{ab} \otimes N)/U$ *such that the sequence*

$$H_1(G; R_{ab}) \xrightarrow{\varepsilon} H_1(Q; S_{ab}) \xrightarrow{\delta} (G_{ab} \otimes N)/U \xrightarrow{\chi} H_2 G$$

is exact.

Section 3 contains a number of remarks on various aspects of our result. The most important one concerns the relation between the above Theorem and the approach used in a previous paper [1] by Eckmann and Hilton. In [1], a certain homomorphism σ

$$\sigma: H_4(N, 2) \to G_{ab} \otimes N \tag{1.9}$$

of the Eilenberg-MacLane group $H_4(N, 2) = H_4(K(N, 2))$ into $G_{ab} \otimes N$ was functorially associated with the (arbitrary) central extension (1.2); its definition appears in the Serre spectral sequence of a suitable fibering. From that spectral sequence, a natural exact sequence

$$\begin{aligned} &H_4 Q \to \ker \sigma \to H_3 G/\varrho(H_2 G \otimes N \oplus \operatorname{Tor}(G_{ab}, N)) \\ &\quad \to H_3 Q \to \operatorname{coker} \sigma \xrightarrow{\varrho} H_2 G \to H_2 Q \to N \to G_{ab} \to Q_{ab} \to 0 \end{aligned} \tag{1.10}$$

was obtained in [1], where ϱ is induced by the multiplication map $m: G \times N \to G$. Moreover, it was effectively shown in [2] that $\varrho: G_{ab} \otimes N \to H_2 G$ coincides with $\chi: G_{ab} \otimes N \to H_2 G$ up to sign. We show in Section 3 that

$$U = \sigma H_4(N, 2). \tag{1.11}$$

Thus the relevant portion of (1.10) provides a different (less elementary and algebraic) proof of our Theorem.

Another remark in Section 3 is concerned with an example in which $U \neq 0$ so that the modification introduced in (1.5), when compared with (1.1), is seen to be essential when one goes beyond weak stem-extensions.

2. Proof of the Theorem

We follow the procedure in [2; Theorem 4.3] insofar as it is valid for arbitrary central extensions. Thus we factorize $\varepsilon: H_1(G; R_{ab}) \to H_1(Q; S_{ab})$ as

$$H_1(G; R_{ab}) \xrightarrow{\varepsilon_1} H_1((Q; R_{ab})_N) \xrightarrow{\varphi''} H_1(Q; R/[S, S]) \xrightarrow{\varphi'} H_1(Q; S/[S, S]), \tag{2.1}$$

where ε_1 is the change-of-rings homomorphism. As shown in [2] – it is in any case well-known – ε_1 is surjective, so that it is sufficient to define δ and prove the exactness of

$$H_1(Q; R/[S, R]) \xrightarrow{\varphi'\varphi''} H_1(Q; S_{ab}) \xrightarrow{\delta} (G_{ab} \otimes N)/U \xrightarrow{\chi} H_2 G. \tag{2.2}$$

Note that $(R_{ab})_N = R/[S, R]$; φ'' participates in the exact sequence induced by the coefficient sequence

$$[S, S]/[S, R] \rightarrowtail R/[S, R] \twoheadrightarrow R/[S, S]; \tag{2.3}$$

and φ' participates in the exact sequence induced by the coefficient sequence

$$R/[S, S] \rightarrowtail S/[S, S] \twoheadrightarrow N. \tag{2.4}$$

Note also that $[S, S]/[S, R]=H_2N$ and that Q operates trivially on H_2N (so also, of course, does G). Our main diagram is the following.

$$\begin{tikzcd}
 & & & & H_1(Q; R/[S, R]) \arrow[d, "\varphi''"] & \\
 & & & & H_1(Q; R/[S, S]) \arrow[r, "\varphi'"] \arrow[d, "\delta''"] & H_1(Q; S_{ab}) \\
 & & N \otimes_Q S_{ab} \arrow[r, two heads, "\delta'"] \arrow[d, "\mu"] & N \otimes N \arrow[r, two heads, "\bar{\chi}"] \arrow[d, "\mu"] & {[S, S]/[S, R]} \arrow[d, "\mu"] & \\
H_1(G; R/[S, R]) \arrow[r, "\varphi''"] \arrow[d, two heads, "\varepsilon"] & H_1(G; R/[S, S]) \arrow[r, "\varphi'"] \arrow[d, two heads, "\varepsilon"] & H_1(G; S_{ab}) \arrow[r, "\delta'"] \arrow[d, two heads, "\varepsilon"] \arrow[rru, dashed, "\theta"] & G_{ab} \otimes N \arrow[r, "\chi"] \arrow[d, two heads, "\varepsilon"] \arrow[rd, "\chi'"] & R/[F, R] \arrow[d, two heads, "\varepsilon"] & \\
H_1(Q; R/[S, R]) \arrow[r, "\varphi''"] & H_1(Q; R/[S, S]) \arrow[r, "\varphi'"] & H_1(Q; S_{ab}) \arrow[r, "\delta'"] & Q_{ab} \otimes N \arrow[r, "\chi'"] & R/[F, R]\,[S, S]. &
\end{tikzcd} \tag{2.5}$$

We recall that all arrows labelled $\mu(\varepsilon)$ are induced by $\mu(\varepsilon)$ in (1.2). The sequences $(\varphi', \delta', \chi')$, for G or Q, are exact, being induced by (2.4); and the vertical sequence $(\varphi'', \delta'', \mu, \varepsilon)$ is exact, being induced by (2.3). All the maps χ, χ', $\bar{\chi}$ are "commutator maps", in the obvious sense [2].

LEMMA. *There is a homomorphism* $\theta: H_1(G; S_{ab}) \to [S, S]/[S, R]$ *such that*

(i) $\mu\theta=\chi\delta'$;
(ii) $\theta\mu=\bar{\chi}\delta'$;
(iii) $\theta\varphi'=\delta''\varepsilon$.

Proof of Lemma. We introduced in [2] the commutator map $\theta: JF \otimes_F S_{ab} \to [F, S]/[S, R]$, given by

$$\theta(x - e \otimes_F s[S, S]) = [x, s]\,[S, R]. \tag{2.6}$$

Using the resolution (1.12), θ may be intrerpeted as a homomorphism $\theta: C_1(G; S_{ab}) \to$ $\to [F, S]/[S, R]$. Now $\sum_\iota (x_\iota - e) \otimes_F s_\iota [S, S] \in Z_1(G; S_{ab})$ if and only if $\prod_i [x_\iota, s_\iota] \in$ $\in [S, S]$, so that θ restricts to

$$\theta: Z_1(G; S_{ab}) \to [S, S]/[S, R].$$

The group of boundaries $B_1(G; S_{ab})$ is generated (see (1.8)) by elements $r - e \otimes_F s[S, S]$, $r \in R$. Thus θ vanishes on $B_1(G; S_{ab})$ and so induces a homomorphism, which we also designate θ,

$$\theta: H_1(G; S_{ab}) \to [S, S]/[S, R].$$

The commutativity relations (i), (ii), (iii) are now easy consequences of the fact that $\bar{\chi}$, χ and δ'' are all given by "commutator maps"; it is only necessary to add that $\mu: N\otimes_Q S_{ab} \to H_1(G; S_{ab})$ – which is part of the 5-term homology sequence with coefficients in S_{ab} induced by (1.2) – is given by

$$\mu(sR \otimes_Q s'[S,S]) = \{s - e \otimes_F s'[S,S]\},\ s, s' \in S.$$

The proof of the theorem is now formal. Given $a \in H_1(G; S_{ab})$, choose $b \in N\otimes N$ with $\theta a = \bar{\chi} b$. Then b is determined modulo $\ker\bar{\chi}$. We set $\bar{\delta} a = \delta' a - \mu b \bmod \mu \ker\bar{\chi}$, so that $\bar{\delta}$ is a homomorphism

$$\bar{\delta}: H_1(G; S_{ab}) \to (G_{ab}\otimes N)/U.$$

Now $\chi\delta' a = \mu\theta a = \mu\bar{\chi} b = \chi\mu b$, so that $\bar{\delta}$ maps $H_1(G; S_{ab})$ to $\ker\chi/\mu\ker\bar{\chi}$. We show that $\bar{\delta}$ is onto $\ker\chi/\mu\ker\bar{\chi}$. For if $\chi' x = 0$, $x \in G_{ab}\otimes N$, then $x = \delta' a$, $a \in H_1(G; S_{ab})$. But then $\mu\theta a = \chi\delta' a = 0$, so that $\theta a = \delta'' u$, $u \in H_1(Q; R/[S,S])$. Let $u = \varepsilon v$, $v \in H_1(G; R/[S,S])$. Then $\theta a = \delta''\varepsilon v = \theta\varphi' v$. Thus if $\tilde{a} = a - \varphi' v$, then $\delta'\tilde{a} = x$, $\theta\tilde{a} = 0$, so that $\bar{\delta}\tilde{a} = x \bmod \mu\ker\bar{\chi}$.

We have thus established the exactness of

$$H_1(G; S_{ab}) \xrightarrow{\bar{\delta}} (G_{ab}\otimes N)/U \xrightarrow{\chi} H_2 G. \tag{2.7}$$

Now let $c \in N\otimes_Q S_{ab}$. Then $\theta\mu c = \bar{\chi}\delta' c$, so that $\bar{\delta}\mu c = \delta'\mu c - \mu\delta' c = 0 \bmod \mu\ker\bar{\chi}$. Thus $\bar{\delta}$ induces

$$\delta: H_1(Q; S_{ab}) \to (G_{ab}\otimes N)/U$$

with

$$\delta\varepsilon = \bar{\delta},$$

and

$$H_1(Q; S_{ab}) \xrightarrow{\delta} (G_{ab}\otimes N)/U \xrightarrow{\chi} H_2 G \tag{2.8}$$

is exact.

It remains to prove the exactness of

$$H_1(Q; R/[S,R]) \xrightarrow{\varphi'\varphi''} H_1(Q; S_{ab}) \xrightarrow{\delta} (G_{ab}\otimes N)/U. \tag{2.9}$$

Now $\delta\varphi'\varphi''\varepsilon = \bar{\delta}\varphi'\varphi''$. Moreover $\delta'\varphi'\varphi'' = 0$ and $\theta\varphi'\varphi'' = \delta''\varepsilon\varphi'' = \delta''\varphi''\varepsilon = 0$, so that $\bar{\delta}\varphi'\varphi'' = 0$. Thus $\delta\varphi'\varphi''\varepsilon = 0$, so that $\delta\varphi'\varphi'' = 0$. Conversely, let $\delta\varepsilon a = 0$, $a \in H_1(G; S_{ab})$. It easily follows that there exists $b \in N\otimes N$ with $\delta' a = \mu b$, $\theta a = \bar{\chi} b$. Set $b = \delta' c$, $c \in N\otimes_Q S_{ab}$. Then $\delta' a = \mu\delta' c = \delta'\mu c$ and $\theta a = \bar{\chi}\delta' c = \theta\mu c$. Now set $\tilde{a} = a - \mu c$. Then

$$\varepsilon a = \varepsilon\tilde{a}, \quad \delta'\tilde{a} = 0, \quad \theta\tilde{a} = 0. \tag{2.10}$$

From (2.10) we infer that $\tilde{a} = \varphi' x$, $x \in H_1(G; R/[S,S])$. Then, again from (2.10),

$0=\theta\varphi'x=\delta''\varepsilon x$, so that $\varepsilon x=\varphi''y$, $y\in H_1(Q;R/[S,R])$. Thus

$$\varepsilon a=\varepsilon\tilde{a}=\varepsilon\varphi'x=\varphi'\varepsilon x=\varphi'\varphi''y,$$

and the theorem is completely proved.

3. Remarks

(i) It is implicit in the proof of the Theorem that, given $x\in H_1(Q;S_{ab})=H_3Q$, there exists $a\in H_1(G;S_{ab})$ with $\varepsilon a=x$, $\theta a=0$, so that $\delta x=\delta' a \bmod \mu \ker\bar{\chi}$.

(ii) For weak stem-extensions (1.2), i.e., for the case when $\mu:N\otimes N\to G_{ab}\otimes N$ is the zero-map, δ is the homomorphism $H_1(Q;S_{ab})\to G_{ab}\otimes N$ defined in [2]. For we have, in that case, $\bar{\delta}=\delta'$, whence $\delta=\varepsilon^{-1}\delta'$ as in [2]. On the other hand, we show by an example that there certainly are examples of central extensions $N\rightarrowtail G\twoheadrightarrow Q$ which have $U\neq 0$, so that the modification introduced in (1.5), compared with (1.1), is, in general, necessary. Of course if $U\neq 0$ the central extension cannot be weak stem.

Let p be a fixed prime, let $r\geqslant s$ be positive integers and let $G=G(p^r,p^s)$ be the group

$$G=\{a,b\mid a^{p^r}=b^{p^s}=a^{-1}b^{-1}ab\}.$$

Then (see [2]) a is of order p^{r+s}, and the center of G is generated by a^{p^s}. For any t with $s\leqslant t\leqslant r+s$, let N_t be the (central) subgroup generated by a^{p^t} and let $N_t\rightarrowtail G\twoheadrightarrow Q_t$ be the associated central extension. Then, as pointed out in [2], this extension is stem iff $t\geqslant r$ and weak stem iff $t\geqslant\frac{1}{2}(r+s)$.

Suppose then that $\frac{1}{2}(r+s)$. One then finds that

$$G_{ab}\otimes N_t=\mathbf{Z}_{p^{r+s-t}}\oplus\mathbf{Z}_{p^s},$$
$$U=\mathbf{Z}_{p^{r+s-2t}}$$

so that

$$(G_{ab}\otimes N_t)/U=\mathbf{Z}_{p^t}\oplus\mathbf{Z}_{p^s}.$$

For such a group G one has $H_2G=0$, so that the important part of (1.5) reads

$$H_3G\to\mathbf{Z}_{p^t}\oplus\mathbf{Z}_{p^s}\oplus\mathbf{Z}_{p^s}\to\mathbf{Z}_{p^t}\oplus\mathbf{Z}_{p^s}\to 0.$$

In fact, one may calculate H_3G from the (non-central) extension $N_0\rightarrowtail G\twoheadrightarrow Q_0$ with $N_0=\{a\}$, and one finds

$$H_3G=\begin{cases}\mathbf{Z}_{p^r}\oplus\mathbf{Z}_{p^s}, & p \quad \text{odd}\\ \mathbf{Z}_{2^{r+1}}\oplus\mathbf{Z}_{2^s}, & p=2,\quad r\geqslant 2,\\ \mathbf{Z}_8, & p=2,\quad r=1.\end{cases}$$

(iii) It is, of course, always true that the homomorphisms $\varepsilon: H_n G \to H_n Q$, induced by a group homomorphism $\varepsilon: G \to Q$, embed in an exact sequence in a natural way, since they are defined by means of a chain map $C(G) \to C(Q)$. Thus the exact sequence we have established in this note may be interpreted as providing a calculation of the "relative H_3" for this chain map. We may also give a topological interpretation in which $(G_{ab} \otimes N)/U$ then appears as the third homology group of the Thom complex of the fibration $K(G, 1) \to K(Q, 1)$.

(iv) The naturality of the sequence (1.8) follows easily from the fact that, given a commutative diagram

$$\begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow \\ N' & \rightarrowtail & G' & \twoheadrightarrow & Q' \end{array}$$

we may associated with (2.12) a map of the presentation (1.10) of $N \rightarrowtail G \twoheadrightarrow Q$ to the corresponding presentation of $N' \rightarrowtail G' \twoheadrightarrow Q'$. The details may be left to the reader.

(v) *The relation between σ and μ* $\ker \bar{\chi}$ (see introduction). The homomorphism σ associated in [1] with the extension (1.2) factorizes, by naturality, as $\sigma = \mu\bar{\sigma}$,

$$H_4(N, 2) \xrightarrow{\bar{\sigma}} N \otimes N \xrightarrow{\mu} G_{ab} \otimes N .$$

The map of extensions given by

$$\begin{array}{ccccc} N & \rightarrowtail & N & \twoheadrightarrow & 1 \\ \| & & \downarrow & & \downarrow \\ N & \rightarrowtail & G & \twoheadrightarrow & Q \end{array}$$

induces a map of the sequences (portions of 1.10)

$$\begin{array}{ccccccc} 0 & \to & N \otimes N/\bar{\sigma}H_4(N, 2) & \xrightarrow{\bar{\chi}} & H_2 N & \to & 0 \\ \downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \\ H_3 Q & \to & G_{ab} \otimes N/\sigma H_4(N, 2) & \xrightarrow{\chi} & H_2 G & \to & H_2 Q . \end{array}$$

Thus

$$\bar{\sigma}H_4(N, 2) = \ker \bar{\chi}, \tag{3.1}$$

so that

$$\sigma H_4(N, 2) = \mu \ker \bar{\chi} = U . \tag{3.2}$$

Therefore (1.10) contains a proof of our Theorem (by entirely different methods). On the other hand, the proof given in this paper contains an explicit description of the natural map $\delta: H_3 Q \to (G_{ab} \otimes N)/U$ which was not available from the spectral sequence argument.

(vi) In (1.10) we saw that we may factor $\varrho((H_2G\otimes N)\oplus\mathrm{Tor}(G_{ab}, N))$ out of the first term of (1.5). This, too, is clear on elementary grounds. Now ϱ is induced by the multiplication $m:G\times N\to G$. Let $p:G\times N\to G$ be the projection. Then $\varepsilon m=\varepsilon p:G\times\times N\to Q$, so that the kernel of $\varepsilon_*:H_3G\to H_3Q$ contains the ϱ-image of anything in $H_3(G\times N)$ killed by p_*. Now the kernel of $p_*:H_3(G\times N)\to H_3G$ is

$$H_3N\oplus(H_2G\otimes N)\oplus(H_1G\otimes H_2N)\oplus\mathrm{Tor}(G_{ab}, N), \tag{3.3}$$

so that the ϱ-image of all of (3.3) is certainly in the kernel of $\varepsilon:H_3G\to H_3Q$.

ETH Zürich
Battelle Seattle Research Center

Received January 14, 1972